Parameterized Complexity of Candidate Control in Elections and Related Graph Problems

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joint work with

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Voting systems

Election

Set of votes V, set of candidates C.
A vote is a ranking (total order) over all candidates.

$C = \{a, b, c\}$ and V :

$$v_1 : a > b > c$$
$$v_2 : a > c > b$$
$$v_3 : c > a > b$$
$$v_4 : b > a > c$$

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\begin{aligned}
v_1 &: a > b > c \\
v_2 &: a > c > b \\
v_3 &: c > a > b \\
v_4 &: b > a > c
\end{aligned}
$$

Voting system: Rule that determines a winner

- **Plurality**: The candidate that is placed first in the highest number of votes wins.
- **Copeland**: based on points for pairwise comparisons between candidates (defined later)

Control in Elections

Control

External agent influences the outcome of an election in (dis)favor of a distinguished candidate.

10 types studied with respect to classification into P and NP, e.g. [Bartholdi et al., Mathematical and Computer Modelling 1992] [Hemaspaandra et al., Artificial Intelligence 2007], ...

We consider control by **adding or deleting candidates**

NP-hard for Copeland and plurality votings [Hemaspaandra et al., Artificial Intelligence 2007], [Faliszewski et al., AAAI 2007], [Faliszewski et al., AAIM 2008]

Does the hardness hold under realistic assumption like a small number of modifications?

Parameterized Complexity — Introduction I

Given an NP-hard problem with input size n and a parameter k.
Basic idea: Confine the combinatorial explosion to k.

Definition

A problem of size n is called *fixed-parameter tractable* with respect to a parameter k if it can be solved in $f(k) \cdot n^{O(1)}$ time.

Parameterized Complexity — Introduction II

Completeness program developed by Downey and Fellows (1999)

$$\text{FPT} \subseteq \overbrace{\text{W[1]} \subseteq \text{W[2]} \subseteq \ldots \subseteq \text{W[P]}}^{\text{Presumably fixed-parameter intractable}}$$

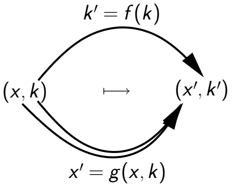
Parameterized Reduction:

Let $L, L' \subseteq \Sigma^* \times \mathbb{N}$ be two parameterized problems.

$L \leq_{par} L'$ if

1.) $(x, k) \in L$ iff $(x', k') \in L'$.

2.)



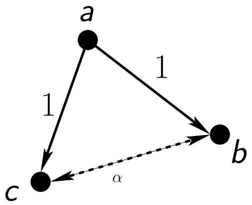
Copeland voting

A candidate wins the **head-to-head contest** against another candidate if he is ranked better in more than half of the votes.

Copeland$^\alpha$-winner

Every candidate gets one point for winning a head-to-head contest.
If two candidates are tied both get α points for $0 \leq \alpha \leq 1$.
The candidate with the highest score is the Copeland$^\alpha$-winner.

$v_1 : a > b > c$
$v_2 : a > c > b$
$v_3 : c > a > b$
$v_4 : b > a > c$



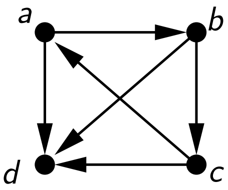
$\text{score}(a) = 2$
$\text{score}(b) = \text{score}(c) = \alpha$

Control in Copeland and Digraph Problems

$a > b > c > d$
$c > a > b > d$
$b > d > c > a$

$\text{score}(a) = \text{score}(b) = \text{score}(c) = 2$

Can candidate a become Copeland$^\alpha$-winner by deleting k candidates?



$d_{out}(a) = d_{out}(b) = d_{out}(c) = 2$

Can vertex a become the only vertex with maximum outdegree (or minimum indegree) by deleting k vertices?

Control in Copeland and Digraph Problems

$a > b > c > d$
$c > a > b > d$
$b > d > c > a$

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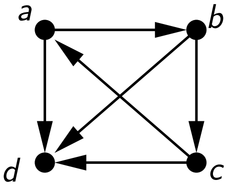


$d_{out}(a) = d_{out}(b) = d_{out}(c) = 2$

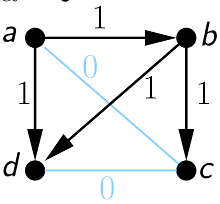
Can vertex a become the only vertex with maximum outdegree (or minimum indegree) by deleting k vertices?

Delete c !

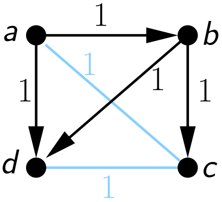
Instances with ties

Points rewarded for a tie: α

$\alpha = 0$



$\alpha = 1$



Max-Outdegree Deletion (MOD)

Input: A digraph $D = (W, A)$, a distinguished vertex a, and an integer $k \geq 1$

Task: Delete at most k vertices such that a becomes the only vertex with maximum outdegree.

Min-Indegree Deletion (MID)

Input: A digraph $D = (W, A)$, a distinguished vertex a, and an integer $k \geq 1$

Task: Delete at most k vertices such that a becomes the only vertex with minimum indegree.

MOD and MID in acyclic graphs

Theorem

In acyclic graphs MID can be solved in polynomial-time whereas MOD is W[2]-complete.

Proof:

MIN-INDEGREE DELETION

Observation: There is always a vertex with indegree zero.

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MAX-OUTDEGREE DELETION

Parameterized Reduction from the W[2]-complete HITTING SET.

Hitting Set

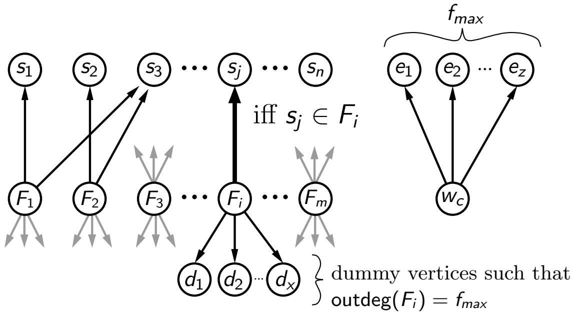
Input: A subset family $\mathcal{F} = \{F_1, F_2, \dots, F_m\}$ of a base set $S = \{s_1, s_2, \dots, s_n\}$ and an integer $k \geq 1$.
Task: Find a subset $S' \subseteq S$ of size at most k such that for every $1 \leq i \leq m$ we have $S' \cap F_i \neq \emptyset$.

W[2]-hardness of MOD in acyclic digraphs

Subsets $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ over elements $S = \{s_1, s_2, \ldots, s_n\}$, e.g., $F_1 = \{s_1, s_3\}, F_2 = \{s_2, s_3\}, \ldots$

Let f_{max} denote the maximum subset size $\max_{i=1}^{m} |F_i|$

$\Longrightarrow$



Vertex w_c can become maximum degree vertex by deleting k vertices iff there is a hitting set of size k.

Overview of the complexity of MOD and MID

maximum degree means: "maximum outdegree" for MOD and "maximum indegree" for MID

	# deleted vert. k		maximum degree d		(k, d)	
	MOD	MID	MOD	MID	MOD	MID
general digraphs	W[2]-c	W[2]-c	NP-c for $d \geq 3$	FPT	FPT	FPT
acyclic digraphs	W[2]-c	P	NP-c for $d \geq 3$	P	FPT	P
tournaments	W[2]-c	W[2]-c	-	-	-	-

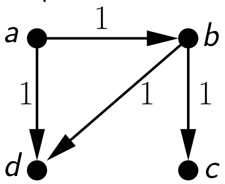
- W[2]-hardness in tournaments: Reduction from DOMINATING SET
- NP-completeness result in acyclic graphs: Reduction from a restricted version of HITTING SET
- FPT-results: Branching strategies

Back to Copeland voting

Standard way to "encode" a digraph into a Copeland election, e.g. [Faliszewski et al., AAAI 2007]

$$\text{arc "}u \to v\text{"} \quad \Rightarrow \begin{cases} u > v > \overrightarrow{C \setminus \{u,v\}} \\ \overleftarrow{C \setminus \{u,v\}} > u > v \end{cases}$$

Example:



$$"a \to b" : \quad \begin{cases} a > b > c > d \\ d > c > a > b \end{cases}$$

$$"a \to d" \ldots$$

Lemma

MOD $\leq_{par}$ Control by Deleting Candidates for Copeland$^{\alpha}$

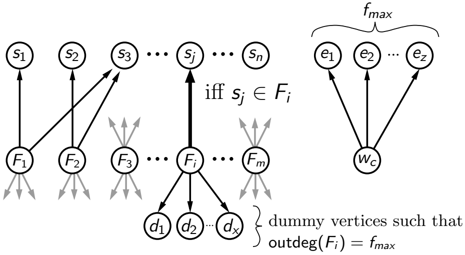
Constant number of votes

Question: [Faliszewski et al., AAIM 2008]
Parameterized complexity with respect to the "number of votes"

Theorem

Control by deleting candidates for Copeland elections in NP-hard for six votes.

Proof: Reduction from a restricted version of Hitting Set.
Subsets $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ over elements $S = \{s_1, s_2, \ldots, s_n\}$



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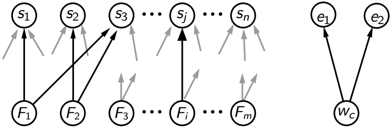
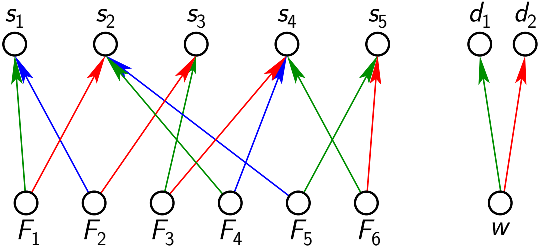


subsets over two elements,
that is $|F_i| = 2$, and
every elements is in 3 subsets

Constant number of votes

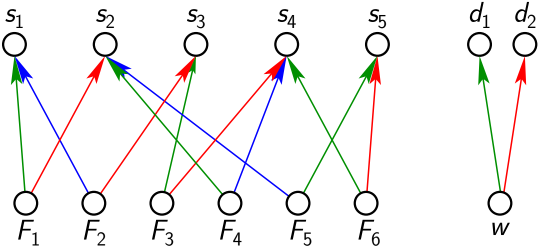


Proper edge-coloring: a vertex cannot have adjacent arcs with the same color

Theorem (König, 1916)

A bipartite graph with maximum degree d can be edge-colored with d colors in polynomial time.

Constant number of votes

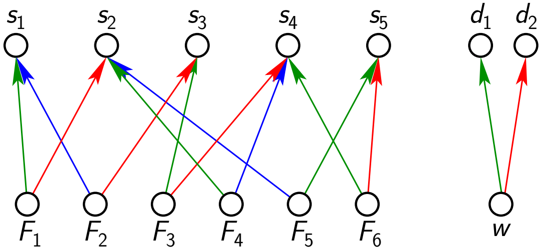


Proper edge-coloring: a vertex cannot have adjacent arcs with the same color

All arcs of one color can be encoded into two votes.
For example:

$\overleftarrow{F_2} > \overleftarrow{s_1} > F_4 > s_4 > F_5 > s_2 > G > R$
$R > G > F_5 > s_2 > F_4 > s_4 > F_2 > s_1$

Constant number of votes



Proper edge-coloring: a vertex cannot have adjacent arcs with the same color

Theorem

Control by deleting/adding candidates in Copeland$^\alpha$ elections with $\alpha \in \{0, 1\}$ is NP-complete for a constant number of votes.

Further results for Control

Parameter: Number of deleted/added vertices

CC: Constructive Control "make a candidate win"

DC: Destructive Control "prevent a candidate from winning"

	Copeland$^\alpha$		Plurality	
	CC	DC	CC	DC
Adding Candidates (AC)	**W[2]-c**	P	W[2]-h	W[2]-h
Deleting Candidates (DC)	**W[2]-c**	P	**W[2]-h**	**W[1]-h**

Bold faced results are new.

- Copeland: W[2]-completeness follows from W[2]-hardness of the corresponding problems in tournaments
- Plurality:
  - Constructive control: Parameterized reduction from MOD
  - Destructive control: Parameterized Reduction from CLIQUE

Conclusion

- Relations between natural digraph problems and candidate control in Copeland votings
- Parameterized complexity for MOD and MID for the parameters "number of deleted vertices" and "maximum degree" in different graph classes
- Parameterized complexity of candidate control in Copeland and plurality votings with respect to the parameters "number of deleted/added candidates"
- NP-hardness for a constant "number of votes" for two important special cases of Copeland votings

Outlook

- Destructive control by deleting candidates for plurality voting: Containment in W[1] ?
- Parameterized complexity of Copeland$^\alpha$ with respect to the "number of votes" for $0 < \alpha < 1$?
- Investigate other parameterizations for MOD/MID
- Parameterized complexity of other kinds of control
- Average-case hardness (or tractability) of the considered problems