On Problem Kernels for Possible Winner Determination Under the k-Approval Protocol

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35th International Symposium on
Mathematical Foundations of Computer Science
August 2010

Motivation

Voting scenarios arise in many situations:

- political elections
- committees: decisions about job applicants
- web site rankings
- recommender systems
- choice of restaurant
- ...

Motivation

Typical voting scenario for joint decision making:

Voters give preferences over a set of candidates as linear orders.

Example: candidates: $C = \{a, b, c, d\}$

profile:

vote 1:	a	>	b	>	c	>	d
vote 2:	a	>	d	>	c	>	b
vote 3:	b	>	d	>	c	>	a

Aggregate preferences according to a voting rule

Partial information

Realistic settings: voters may only provide partial information.

For example:

- not all voters have given their preferences yet
- new candidates are introduced
- a voter cannot compare several candidates because of lack of information

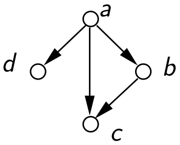
How to deal with partial information?

We consider whether a distinguished candidate can still win.

Partial vote

A **partial vote** is a transitive and antisymmetric relation.

Example: $C = \{a, b, c, d\}$
partial vote: $a \succ b \succ c, a \succ d$



possible **extensions**:

1. $a > d > b > c$
2. $a > b > d > c$
3. $a > b > c > d$

An extension of a profile of partial votes extends every partial vote.

Computational Problem

Possible Winner

Input: A voting rule r, a set of candidates C, a profile of partial votes, and a distinguished candidate c.
Question: Is there an extension profile where c wins according to r?

Considered voting rule:

k-approval

In every vote, the best k candidates get one point each. A candidate with most points in total wins.

Known results for Possible Winner

Results for several voting systems, [Konczak and Lang, 2005], [Pini et al., IJCAI 2007], [Walsh, AAAI 2007], [Xia and Conitzer, AAAI 2008], ...

Results for *k*-approval
(unweighted voters + unbounded number m of candidates)

- k-approval is NP-hard for two (or more) partial votes
  [Betzler, Hemmann, and Niedermeier, IJCAI 2009]
- k-approval is NP-hard for any fixed $k \in \{2, \dots, m-2\}$
  [Betzler and Dorn, JCSS 2010]

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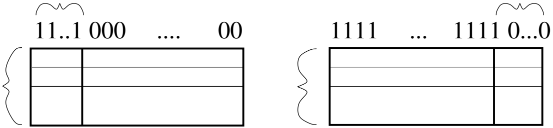
Question of this work

Becomes the problem tractable when the number k of "one-positions" and the number of votes is small?

Combined parameters

2 scenarios:

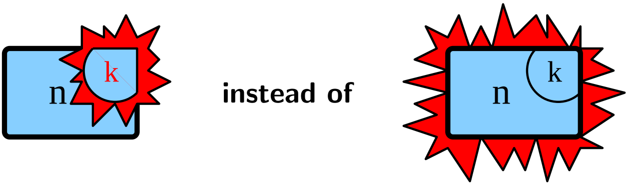
- "number of partial votes" and "number of one-positions" k
- "number of partial votes" and "number of zero-positions" k_0



Motivation: Small committee selects few winners/losers out of a large set of candidates

Parameterized Complexity

Given an NP-hard problem with input size n and a parameter p
Basic idea: Confine the combinatorial explosion to p



Definition

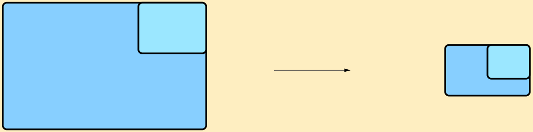
A problem of size n is called *fixed-parameter tractable* with respect to a parameter p if it can be solved exactly in $f(p) \cdot n^{O(1)}$ time.

Parameters: pairs of integers

Problem kernel

Let $L \subseteq \Sigma^* \times \Sigma^*$ be a parameterized problem. An instance of L is denoted by (I, p).

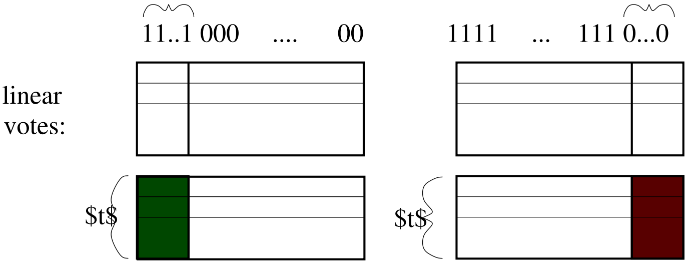
Kernelization



- $(I, p) \in L \iff (I', p') \in L$
- $|p'| \leq |p|$
- $|I'| \leq g(|p'|)$

If g is a polynomial, we say L admits a *polynomial problem kernel*.

Main results



Parameter	(t, k)	(t, k_0)
	FPT (superexponential kernel) **no polynomial kernel** (unless coNP $\subseteq$ NP/poly)	FPT **polynomial kernel**

Polynomial kernel for (t, k_0)

1. Fix the distinguished candidate c as good as possible
2. $z(c') :=$ # zero-positions such that c' is beaten by c
3. **If** $\sum_{c' \in C} z(c') > t \cdot k_0$, **then** return "no"
   else replace "irrelevant" candidates by a bounded number.

Example: $C := \{a, b, c, d_1, \ldots, d_s\}$, $k_0 = 2$

candidate	points in linear votes
d_i	≤ 10
b	11
a	12
c	12

partial votes:

$v_1 : a \succ c, b \succ d_1, d_2 \succ d_3 \succ \cdots \succ d_3$

$v_2 : d_1 \succ d_2 \succ \cdots \succ d_s \succ b \succ c$

$v_3 : d_s \succ c, a \succ d_2$

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$v_2 : d_1 \succ d_2 \succ \dots \succ d_s \succ b \succ c$

$v_3 : d_s \succ c, a \succ d_2, c \succ C \setminus \{c, d_s\} \quad \Rightarrow d_s > c > \dots$

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candidate	points in linear votes	# zero-positions
d_i	≤ 10	0
b	11	1
a	12	2
c	12	—

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$v_1 : a \succ c, b \succ d_1, d_2 \succ d_3, c \succ C \setminus \{c, a\}$

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Example: $C := \{a, b, c, \mathbf{d}, d, \cancel{d_1, d_2, \ldots, d_s}\}$, $k_0 = 2$

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$v_2 : d_1 \succ d_2 \succ \cdots \succ d_s \succ b \succ c \quad \Rightarrow \quad d \succ b \succ c$

$v_3 : d_s \succ c, a \succ d_2, c \succ C \setminus \{c, d_s\} \quad \Rightarrow \quad c \succ a \succ d$

Polynomial kernels

Theorem

For k-approval, Possible Winner with t partial votes admits a polynomial kernel with respect to (t, k_0) with k_0 zero-positions.

Crucial idea: Number of candidates that **have to** take **zero-positions** is bounded in a yes-instance.

Why does this not work for (t, k)?

Polynomial kernels

Theorem

For k-approval, POSSIBLE WINNER with t partial votes admits a polynomial kernel with respect to (t, k_0) with k_0 zero-positions.

Crucial idea: Number of candidates that **have to** take **zero-positions** is bounded in a yes-instance.

Why does this not work for (t, k)?

In a yes-instance, there might be an unbounded number of candidates that **may** take a **one-position**.

Theorem

For k-approval, POSSIBLE WINNER with t partial votes does not admit a polynomial kernel with respect to (t, k) unless coNP $\subseteq$ NP/poly.

Kernelization for (t, k)

Talk

Basic idea for Possible Winner for 2-approval leading to a kernel with $O(t^3)$ candidates for an instance with t partial votes

Paper:

1. "extension" to k-approval
   $\Rightarrow$ superexponential kernel showing FPT wrt. (k, t)
2. "improve" the kernel size for 2-approval from having $O(t^3)$ candidates to $O(t^2)$ candidates based on matching techniques on bipartite graphs

Kernelization for 2-approval

Parameter t: number of partial votes

Phase I:

1. Fix c as good as possible
2. Fix all candidates that not beaten by c when assuming at least one one-position at zero-positions in all votes.
   $\Rightarrow$ remaining candidates can make at least one point

Phase 2:

For every partial vote v_i, compute the set $\text{First}(v)$ of candidates that can take the first position.

Example: $v_i : a \succ b, c \succ b, d \succ e \succ f \Rightarrow \text{First}(v_i) = \{a, c, d\}$

Reduction Rule

If $|\text{First}(v)| \geq 2t$, then arbitrarily choose $2t$ candidates from $\text{First}(v)$ and fix all remaining candidates in v at zero-positions.

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Correctness: Consider an extension profile of unreduced instance such that c wins

Extend all votes except v in the same way in the reduced instance.

At most $2 \cdot (t-1)$ candidates assume a one-position in these votes.

Since $|\text{First}(v)| \geq 2t$, there must be two candidates that still can make one point in v without beating c.

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Since $|\text{First}(v)| \geq 2t$, there must be two candidates that still can make one point in v without beating c.

Remaining case: $|\text{First}(v)| < 2t$ for a vote v

Overview of results

Possible Winner for k-approval with t partial votes

k_0 denotes the number of zero-positions

(t, k_0)	(t, k)
polynomial kernel (FPT)	superexponential kernel (FPT) no polynomial kernel 2-approval : polynomial kernel with $O(t^2)$ candidates
constant k_0: dyn. prog. $2^{O(t)}$ constant t: search tree $2^{O(k_0)}$	? ?

Thank you