A Multivariate Complexity Analysis of Determining Possible Winners Given Incomplete Votes

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21st International Joint Conference on Artificial Intelligence
July 2009

Motivation

Typical voting scenario for joint decision making:

Voters give preferences over a set of candidates as linear orders.

Example: candidates: $C = \{a, b, c, d\}$

profile:	vote 1:	a	>	b	>	c	>	d
	vote 2:	a	>	d	>	c	>	b
	vote 3:	b	>	d	>	c	>	a

Aggregate preferences according to a voting protocol.

Realistic settings: voters may only provide partial information.

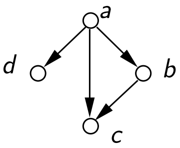
How to deal with partial information?

We consider the question if a distinguished candidate can still win.

Partial orders

A **partial order** is a transitive, reflexive, and antisymmetric relation.

Example: $C = \{a, b, c, d\}$
partial order: $a \succ b, a \succ d, b \succ c$



possible **extensions**:

1. $a > d > b > c$
2. $a > b > d > c$
3. $a > b > c > d$

An extension of a profile of partial votes extends every partial vote.

Computational Problem

Possible Winner

Input: A voting rule r, a set of candidates C, a profile of partial orders $P_O = (O_1, \ldots, O_n)$ on C, and a distinguished candidate $c \in C$.
Question: Is there an extension profile $P = (V_1, \ldots, V_n)$ where V_i extends O_i for all $1 \leq i \leq n$ and c wins according to r?

Necessary Winner: Does a candidate win in every extension?

The computational difficulty of answering this questions clearly depends on the voting rule.

Related problems:

- Manipulation
- Preference Elicitation

Known results

Two studied scenarios for POSSIBLE WINNER:

1. bounded number of candidates [CONITZER ET AL., JACM 2007], [PINI ET AL., IJCAI 2007], [WALSH, AAAI 2007]
   - unweighted voters: always polynomial time
   - weighted voters: NP-complete for Borda, maximim, STV, and plurality-with-runoff rules
2. unbounded number of candidates, unweighted voters
   - in P for Condorcet [KONCZAK AND LANG, 2005]
   - NP-complete for STV [BARTHOLDI AND ORLIN, SCW 1991], [PINI ET AL., IJCAI 2007], and [WALSH, AAAI 2007]
   - NP-complete for Copeland, maximim, Bucklin, ranked pairs, some scoring rules (including Borda) [XIA AND CONITZER, AAAI 2008]
   - NP-complete for k-approval for all $2 \leq k \leq$ "number of candidates" and some further scoring rules [BETZLER AND DORN, MFCS 2009]

Parameterized Complexity

Given an NP-hard problem with input size n and a parameter k
Basic idea: Confine the combinatorial explosion to k

Definition

A problem of size n is called *fixed-parameter tractable* with respect to a parameter k if it can be solved in $f(k) \cdot n^{O(1)}$ time.

Parameterizations

number of candidates

fixed-parameter tractable for all positional scoring rules, Bucklin, Copeland, ranked pairs, and maximin.

Algorithmic idea: Integer Linear Programming + Lenstra's result

number of votes

(discussed in the following)

number of undetermined pairs

1. per vote: NP-hard for a constant number for Borda, Copeland, maximin, and ranked pairs. [Xia and Conitzer, AAAI 2008]
2. in total: fixed-parameter tractable for all positional scoring rules, Bucklin, Copeland, ranked pairs, and maximin. Algorithmic idea: Branching

Parameter "number of votes"

a constant-size coalition/number of partial votes

	MANIPULATION	POSSIBLE WINNER
Copeland	NP-c [1]	NP-c
Maximin	NP-c [2]	NP-c
Ranked Pairs	NP-c [2]	NP-c
Bucklin	P [2]	**NP-c**
k-approval	P	**NP-c**
Borda	?	**NP-c**

[1] [Faliszewski et al., AAMAS 2008] (for specific tie-breaking)

[2] [Xia and Conitzer, AAAI 2008]

k-approval

scoring vector $(1, \ldots, 1, 0, \ldots, 0)$ starting with k ones

Theorem

For k-approval, Possible Winner is NP-complete for a partial profile that consists of two partial orders when k is part of the input.

Proof: Reduction from Independent Set

Independent Set:
Given: An undirected graph $G = (V, E)$ and an integer $t > 0$.
Question: Is there a size-t subset $V' \subseteq V$ such that there is no edge between any two vertices of V'?

Reduction

IS-instance $(G = (V, E), t) \Rightarrow$ Possible Winner-instance:

- distinguished candidate c
- one candidate c_i for every vertex from V (C_V)
- two candidates e_{ij} and e'_{ij} for every edge $\{v_i, v_j\} \in E$ and one candidate e_{ij} if there is no such edge (C_E)
- set of dummy candidates (D)

Reduction

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partial profile:

$p_1 : c \succ D \succ C_V \succ C_E$

$p_2 : c \succ C_V \cup C_E \succ D$ with $\begin{array}{l} c_i \succ e_{ij},\ c_j \succ e'_{ij} \text{ if } \{v_i, v_j\} \in E \\ c_i \succ e_{ij},\ c_j \succ e_{ij} \text{ if } \{v_i, v_j\} \notin E \end{array}$

Adjust k appropriately

Reduction II

partial profile:

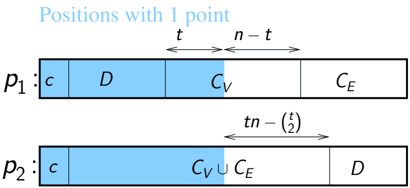
$p_1 : c \succ D \succ C_V \succ C_E$

$p_2 : c \succ C_V \cup C_E \succ D$

with

$c_i \succ e_{ij},\ c_j \succ e'_{ij}$ if $\{v_i, v_j\} \in E$

$c_i \succ e_{ij},\ c_j \succ e_{ij}$ if $\{v_i, v_j\} \notin E$



Observations:

- every candidate except c can make at most one point (unique winner)
- in p_1: t candidates from C_V must make one point
- in p_2: such a candidate must make zero points.

Reduction II

partial profile:

$p_1 : c \succ D \succ C_V \succ C_E$

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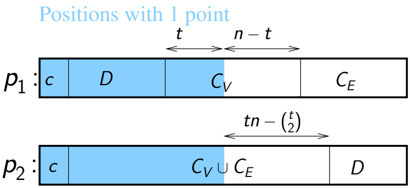

in p_2:

- every candidate from C_V is placed in front of $n-1$ candidates from C_E.
- if the t candidates that must make zero points "push" disjoint candidate sets to the zero positions, then $t \cdot n$ zero positions would be necessary.
- $\binom{t}{2}$ candidates from C_E must be "shared"
- c_i and c_j share a candidate if $\{x_i, x_j\} \notin E$

Reduction II

partial profile:

$p_1 : c \succ D \succ C_V \succ C_E$

$p_2 : c \succ C_V \cup C_E \succ D$

with

$c_i \succ e_{ij},\ c_j \succ e'_{ij}$ if $\{v_i, v_j\} \in E$

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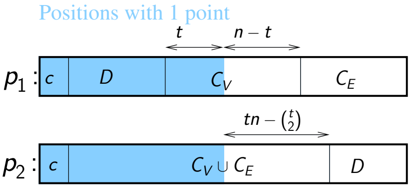

$\Rightarrow$ If c is a possible winner, then G has a size-t independent set.

Overview of results

	# candidates	# votes (total/partial)	k
Copeland	**FPT**	NP-c [1] (x/2)	$\mathbf{2^k}$
Maximin	**FPT**	NP-c [2] (x/2)	$\mathbf{2^k}$
Ranked Pairs	**FPT**	NP-c [2] (x/1)	$\mathbf{2^k}$
Bucklin	**FPT**	**NP-c** (5/2)	$\mathbf{2^k}$
k-approval	**FPT**	**NP-c** (2/2)	$\mathbf{2^k}$
Borda	**FPT**	**NP-c** (6/3)	$\mathbf{1.82^k}$

where k denotes the total number of open pairs.

[1] [Faliszewski et al., AAMAS 2008] (for specific tie-breaking)
[2] [Xia and Conitzer, AAAI 2008]

Open questions

- improve running times: number of candidates
- combined parameters
- restriction to CP-nets (see [Xia and Conitzer, AAAI 2008]) could lead to further parameters
- efficient enumeration of all extensions