Matrix Robustness, with an Application to Power System Observability

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2nd Algorithms and Complexity in Durham Workshop
September 2006

Outline

1. Power system observability
2. Complexity of Matrix Robustness
3. Algorithms for Matrix Robustness
   - Mixed-integer program (MIP)
   - Pseudorank-based heuristic
4. Experiments

Power system observability

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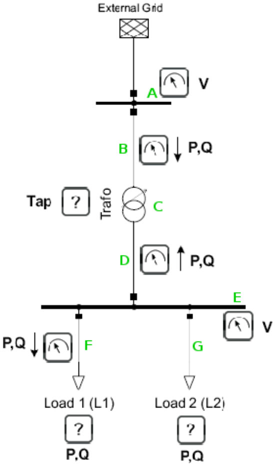
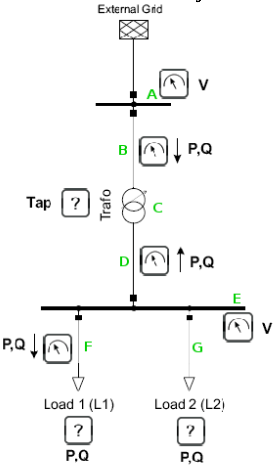
- Placing one measuring device per state is not feasible.
- Often, states can be calculated from measurements at other points, exploiting Kirchhoff's circuit laws and similar rules.
- A power system is called observable if all states are measured or can be calculated.

Measurement Jacobian



Measurement Jacobian

The measurement Jacobian stores the "sensitivity" $\partial y / \partial x$ of a measurement y with respect to a state x.



Measurements	States				
	P(E)	Q(E)	Tap(C)	P(G)	Q(G)
V(A)	0	0	0	0	0
P(B)	1	0	0	1	0
Q(B)	0	1	0	0	1
P(D)	-1	0	0	-1	0
Q(D)	0	-1	0	0	-1
V(E)	0	0	-1	0	0
P(F)	1	0	0	0	0
Q(F)	0	1	0	0	0

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Lemma ([Monticelli&Wu, IEEE Trans. Power Appar. Syst 1985])

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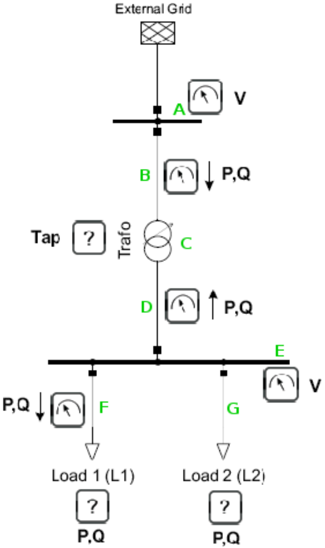
Corollary

One can decide in $O(n^3)$ time whether a power system is observable by Gaussian elimination.

Measurement Jacobian



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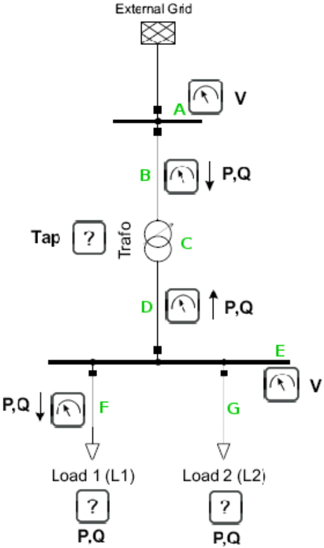
Measurement Jacobian



		States				
		P(E)	Q(E)	Tap(C)	P(G)	Q(G)
Measurements	P(B)	1	0	0	1	0
	Q(B)	0	1	0	0	1
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	P(F)	1	0	0	0	0
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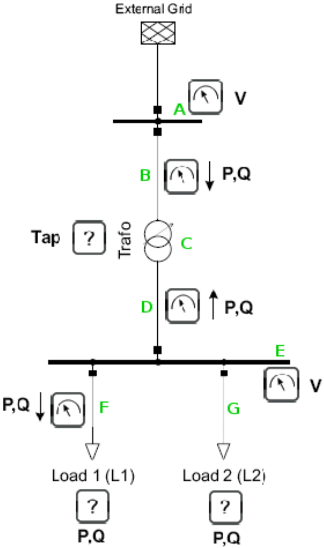
Measurement Jacobian



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Rank 5 $\Rightarrow$ Power system is observable.

Robust observability

Measurements may fail over time or be down due to maintenance.

Definition (Robust Power System Observability)

Instance: An observable network and an integer $k > 0$.
Question: Is the network still observable after the outage of k arbitrary measurements?

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By the main theorem, this is equivalent to:

Definition (Matrix Robustness)

Instance: An $m \times n$ matrix M over an arbitrary field $\mathbb{F}$ with full rank n, $m \geq n$, and an integer $k > 0$.
Question: Is M robust against deletion of k rows, that is, is the rank of M preserved if any k rows are deleted?

Matrix Weakness

For simplicity, we consider the complement MATRIX WEAKNESS.

Definition (MATRIX WEAKNESS)

Instance: An $m \times n$ matrix M over an arbitrary field $\mathbb{F}$ with full rank n, $m \geq n$, and an integer $k > 0$.
Question: Can we find k rows such that M drops in rank when they are deleted?

Generalized Minimum Circuit

Definition (Generalized Minimum Circuit)

Instance: An $m \times n$ matrix M over an arbitrary field and a positive integer k.
Question: Is there a linearly dependent subset of the column vectors of M with at most k elements?

Using matroid theory, one can show:

Theorem

Matrix Weakness *on a field* $\mathbb{F}$ *is many-one equivalent to* Generalized Minimum Circuit *on* $\mathbb{F}$*. The matrices of both problems can be transformed into each other in polynomial time.*

Complexity of Matrix Robustness

GENERALIZED MINIMUM CIRCUIT in turn is equivalent to GENERALIZED MINIMUM DISTANCE from coding theory, which is known to be NP-complete for any finite field [VARDY, IEEE Trans. Inform. Theory '97].

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Matrix Robustness *is coNP-complete for any finite field.*

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Corollary

MATRIX ROBUSTNESS *is coNP-complete for any finite field.*

Complexity for infinite fields (such as $\mathbb{Z}$ for our application) is open.

Mixed-integer formulation

Definition (Most Comprehensive Hyperplane)

Instance: An $m \times n$ matrix M over an arbitrary field $\mathbb{F}$ with full rank n, $m \geq n$ and an integer $k > 0$.
Question: Is there a hyperplane in the vector space $\mathbb{F}^n$ containing at least $n - k$ row vectors of M?

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- Variables:
  - hyperplane H, represented by its normal vector x
  - binary variables d_i with $d_i = 0$ iff y_i lies in the hyperplane H
- Goal: minimize $\sum_i d_i$
- Central constraints:

$$\langle y_i, x \rangle - d_i \leq 0$$
$$-1 \cdot \langle y_i, x \rangle - d_i \leq 0$$

assuming $||y_i|| \leq 1$ and $||x|| \leq 1$.

Pseudorank

Pseudorank is a simplification of the rank concept that considers only *pairwise* linear dependencies.

Definition

The pseudorank is the minimum of the number of rows and the number of columns after exhaustive elimination of pairwise linear dependencies both within rows and within columns.

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Empirical observation

Using pseudorank instead of rank for the observability of power networks is often sufficient (rank often equals pseudorank).

Idea

Use pseudorank robustness as a heuristic for robustness.

Pseudorank-based heuristic

If an $m \times n$-matrix M is to be not robust in terms of pseudorank, then one of three conditions must hold:

1. After deleting k rows, there is a zero column.
2. After deleting k rows and then eliminating pairwise linearly dependent rows, there are less than n rows left.
3. After deleting k rows, there are two dependent columns.

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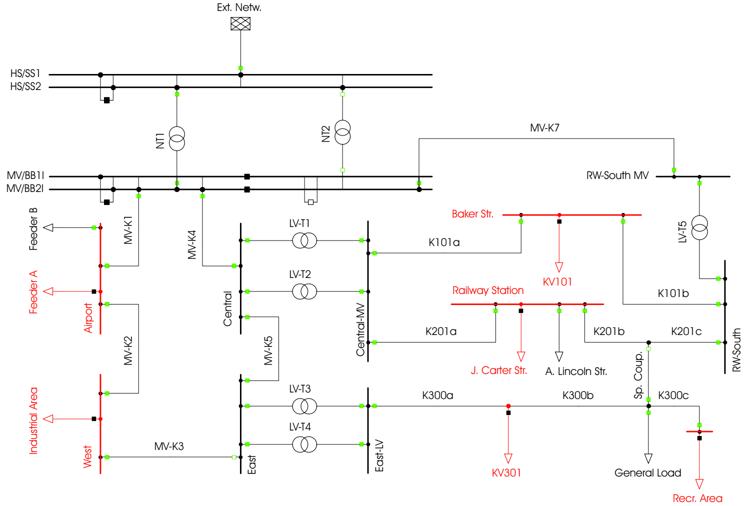
We check condition 3 separately for all pairs (M_i, M_j) of columns, that is, we try to determine a factor c such that $M_j = c \cdot M_i$ after deleting k rows.

Theorem

MATRIX ROBUSTNESS *with respect to the pseudorank can be solved in* $O(s \cdot m \log m)$ *time for an* $m \times n$*-matrix, where* s *is the number of nonzero matrix entries.*

Electrical networks



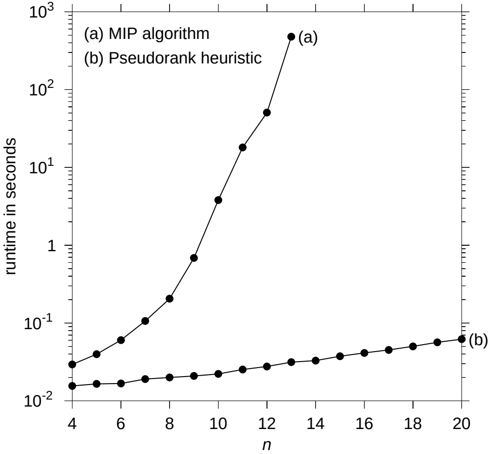
MV/LV

Electrical networks

	Dimension	k	Runtime in seconds	
			MIP	Pseudorank
Treelike	18×8	2	0.05	0.02
MV/LV	78×12	2	0.15	0.04
Nine-Bus	40×12	4	17.61	0.03
IEEE Std 399-1997	150×29	2	1.18	0.15
Namibia	411×164	1	477.09	4.70

Random instances

Random matrices of size $5n \times n$, with entries from $\{-9, \ldots, 9\}$ and 80 % sparsity (each point average over 20 instances)



Summary

- Robust power system observability can be framed as a matrix problem
- A MIP formulation provides optimal solutions
- A heuristic based on pseudoranks does very well in practice

Open questions:

- Is MATRIX ROBUSTNESS also hard for infinite fields?
- Is MATRIX ROBUSTNESS fixed-parameter tractable with respect to the number of deletions?