Author's manuscript accepted to Acta Informatica.
Online available via DOI 10.1007/s00236-014-0202-1

Exploiting a Hypergraph Model for Finding Golomb Rulers

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July 11, 2014

Abstract Golomb rulers are special rulers where for any two marks it holds that the distance between them is unique. They find applications in radio frequency selection, radio astronomy, data encryption, communication networks, and bioinformatics. An important subproblem in constructing "compact" Golomb rulers is GOLOMB SUBRULER (GSR), which asks whether it is possible to make a given ruler Golomb by removing at most k marks. We initiate a study of GSR from a parameterized complexity perspective. In particular, we consider a natural hypergraph characterization of rulers and investigate the construction and structure of the corresponding hypergraphs. We exploit their properties to derive polynomial-time data reduction rules that reduce a given instance of GSR to an equivalent one with $O(k^3)$ marks. Finally, we complement a recent computational complexity study of GSR by providing a simplified reduction that shows NP-hardness even when all integers are bounded by a polynomial in the input length.

Keywords Sidon Set, Hitting Set, NP-Hardness, Parameterized Complexity, Data Reduction, Problem Kernel, Forbidden Subgraph Characterization

1 Introduction

A *ruler* is a finite non-empty subset of the natural numbers $\mathbb{N}$, its elements are called *marks*. A ruler R is called *Golomb ruler* if no two pairs of marks from R have the same distance.

An extended abstract of this work appeared in the Proceedings of the 2nd International Symposium on Combinatorial Optimization (ISCO 2012), April 17-21, 2012, Athens, Greece, volume 7422 of Lecture Notes in Computer Science, pages 368–379, Springer, 2012. Apart from providing missing proofs, the major changes in this article are extended observations on the structure of the considered hypergraphs, a partial improvement in the running time of the data reduction rules (Theorem 4), and a strengthened NP-hardness result (Theorem 5).

Manuel Sorge was supported by Deutsche Forschungsgemeinschaft, projects DARE (NI 369/11) and DAPA (NI 369/12).

Hannes Moser was supported by Deutsche Forschungsgemeinschaft, project AREG (NI 369/9).

Mathias Weller was supported by Deutsche Forschungsgemeinschaft, project DARE (NI 369/11).

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For instance, $\{0, 1, 3, 7\}$ forms a Golomb ruler while $\{0, 1, 3, 5\}$ does not since $5 - 3 = 3 - 1 = 2$, that is, distance 2 appears twice. The number of marks on a ruler is called its *order* n and the distance between its smallest mark (which can be assumed to be zero without loss of generality) and its largest mark is called its *length*. One of the shortest Golomb rulers containing four marks is $\{0, 1, 4, 6\}$. While it is easy to construct Golomb rulers,[1] finding a shortest Golomb ruler for a given order n is assumed to be computationally intractable [11, 24], however, there is no NP-hardness result for this problem so far. Due to the usefulness of short Golomb rulers in a multitude of practical applications, there have been several computational studies based on heuristics and massive parallelism [9, 12, 13, 23, 28, 30, 33]. The applications of Golomb ruler construction include radio frequency selection, radio astronomy, data encryption, communication networks, and bioinformatics [3, 6, 7, 24, 29]. For instance, when placing radio channels in the frequency spectrum, intermodulation interference is introduced by nonlinear behavior in transmitters and receivers: Three channels at frequencies a, b, c may intermodulate and create interference at the frequency $d = a + b - c$. This type of interference is avoided when placing the channels according to marks of a Golomb ruler. Then, there is no channel at frequency d, that is, $\{a, b, c, d\}$ is not part of any Golomb ruler because we have $d - a = b - c$. Currently, shortest Golomb rulers up to order $n = 27$ are known [12].

When constructing short Golomb rulers one often has to place a number of marks within a set of limited possible positions. Meyer and Papakonstantinou [24] formalized this as the Golomb Subruler problem and showed its NP-completeness.[2]

> Golomb Subruler (GSR)
> Input: A finite ruler $R \subseteq \mathbb{N}$ and an integer $k \geq 0$.
> Question: Is there a Golomb ruler $R' \subseteq R$ such that $|R \setminus R'| \leq k$?

Meyer and Papakonstantinou's hardness reduction creates rulers with marks exponential in the input size and thus leaves open whether there are pseudo-polynomial algorithms for GSR. By pseudo-polynomial algorithms we refer to algorithms with running time polynomial in both the largest integer $\max(R \cup \{k, |R|\})$ and the (binary) encoding length of the input.

Contribution We consider a natural (equivalent) formulation of GSR in which every "conflict" (two pairs of marks of R that measure the same distance) is viewed as a hyperedge of a hypergraph. Removing a set of marks in order to obtain a Golomb ruler is thus equivalent to finding a subset of vertices that intersects every hyperedge. Such a subset of vertices is called "hitting set". We first show that the hypergraphs corresponding to a given instance of GSR can be computed in time cubic in the number of input marks and this is worst-case optimal. Together with a known result on computing hitting sets [14] this implies that GSR can be solved in $O(3.076^k + n^3)$ time on n-mark rulers, that is, GSR is fixed-parameter tractable with respect to k.

We further study the hypergraphs, called "conflict hypergraphs", arising from instances of GSR and give certain forbidden configurations. It is not hard to see that the class of conflict hypergraphs is not closed under taking sub(hyper-)graphs. (Hence it is also not closed under taking minors, where it is allowed to remove vertices or hyperedges and to

[1] The definition of Golomb rulers is equivalent to the one for Sidon sets in the group $(\mathbb{Z}, +)$. Sidon sets in Abelian groups $(G, +)$ are subsets of G such that for any four elements a, b, c, d it holds that $a + b \neq c + d$. Some upper and lower bounds are known for the size of Sidon sets, see the works of Dimitromanolakis [11], Drakakis [16] and references therein.

[2] For brevity we reformulated the problem slightly. The original problem is to find a Golomb subruler containing at least a given number of marks. Clearly, this problem and our reformulation are equivalent under polynomial-time many-one reductions.

contract hyperedges which means to remove it and merge all vertices it contains into one.) In contrast, the class of conflict hypergraphs is closed under taking induced subgraphs. Hence, there is a characterization through (possibly infinitely many) forbidden induced subgraphs. The forbidden configurations we obtain entail that any such characterization is indeed necessarily infinite. We note that the problem of deciding whether a given hypergraph is a conflict hypergraph is in NP. However, whether this problem is NP-hard and the problem of completely characterizing the structure of the conflict hypergraphs are left as important challenges for future research.

Moreover, we use the forbidden configurations to develop efficient data reduction rules for GSR. More specifically, these rules reduce a given instance of GSR to one with $O(k^3)$ marks in $O((n + m) \log n)$ time. Here, m is the number of conflicts. Notably, the instances resulting from the data reduction contain $O(k^3)$ conflicts. This bound on the number of conflicts is unlikely to be achievable for finding hitting sets in *general* hypergraphs [10].

Finally, using the hypergraph notion, we provide a many-one reduction for proving the NP-completeness of GSR. The corresponding construction is simpler than the one given by Meyer and Papakonstantinou [24] and shows hardness even when each mark is bounded by a polynomial in the input size. Thus, there are no pseudo-polynomial algorithms for GSR unless P = NP. This result more closely explains the computational difficulty experienced when solving GSR: Since currently considered instances consist only of "small" marks in the order of hundreds [12], pseudo-polynomial algorithms, if they existed, would likely be efficient in practice.

2 Preliminaries

A central tool for our analysis of GSR are hypergraphs. A hypergraph basically is a system of subsets over some universe. More precisely, a *hypergraph* $H = (V, E)$ consists of the *universe* or *set of vertices* V and the *set of hyperedges* E, where for each hyperedge $e \in E$, we have $e \subseteq V$. If used in context of hypergraphs, we use "edge" as synonym for "hyperedge". In particular, we work with 3,4-hypergraphs, meaning that all hyperedges have cardinality three or four. An edge of cardinality d is sometimes called d-edge. In this work, the vertices of a hypergraph will one-to-one correspond to marks on a ruler and the edges will one-to-one correspond to "conflicts" between marks, which will be defined later. We often use the corresponding terms synonymously. A *sub(hyper-)graph* of a hypergraph $H = (V, E)$ is a hypergraph $H' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$ such that each $e' \in E'$ satisfies $e' \subseteq V'$. For a vertex set W, the *(vertex) induced sub(hyper-)graph* $H[W]$ of H is defined as $(W, \{e \in E \colon e \subseteq W\})$. Two hypergraphs $H = (V, E), H' = (V', E')$ are isomorphic if there is a one-to-one mapping $\phi : V \to V'$ such that $e \in E$ if and only if $\{\phi(w) \colon w \in e\} \in E'$ for all vertex sets $e \subseteq V$. For a given hypergraph, we use n to denote the number of vertices and m to denote the number of hyperedges. With respect to rulers, n denotes the number of marks and m denotes the number of conflicts. If a vertex v is contained in an edge e, then e is said to be *incident* to v.

An *independent set* $I \subseteq V$ of a hypergraph $H = (V, E)$ is a set of vertices such that no hyperedge $e \in E$ is a subset of I. In contrast, a *hitting set* $C \subseteq V$ of H is a set of vertices that has non-empty intersection with each edge in H. In the Hitting Set problem, a hypergraph H and an integer $\ell \geq 1$ is given and it is asked whether there is a hitting set in H that has cardinality at most ℓ. We will characterize GSR as a special type of Hitting Set on 3,4-hypergraphs.

The *incidence graph* of a given hypergraph $H = (V, E)$ is the bipartite graph $(V \cup E, E')$ with $E' = \{\{u, v\} \mid (u \in E) \wedge (v \in V) \wedge (v \in u)\}$. When referring to hypergraphs in algorithms, we assume them to be represented as adjacency lists of their corresponding incidence graph with a small tweak. Observe that, if the maximum number of vertices in an edge is at most some fixed constant, adjacency lists support addition of a vertex or an edge in $O(1)$ time. To support removal of a vertex and all its incident hyperedges in $O(\deg(v))$ time, where $\deg(v)$ is the number of hyperedges incident to v, and to support removal of a constant-size hyperedge in $O(1)$ time, we modify the adjacency list data structure as follows. We use a doubly linked list ℓ_v for each vertex v in the incidence graph and for each vertex u adjacent to v in the incidence graph, the list ℓ_v contains a tuple of u and a pointer to the occurrence of v in u's adjacency list ℓ_u. It is then easy to achieve the claimed running times.

Besides hypergraph notation, we also use concepts of parameterized complexity analysis [15, 18, 26]. A computational (typically NP-hard) problem is called *fixed-parameter tractable* with respect to a given parameter k (typically a positive integer) if instances of size ℓ can be solved in $f(k) \cdot \ell^{O(1)}$ time. Herein, $f(k)$ is an arbitrary computable function. Note that fixed-parameter tractability is a stronger statement than just "solvable in polynomial time for constant parameter values" since k is not allowed to influence the degree of the polynomial.

An important concept in parameterized complexity is kernelization [8, 20, 22]. Formally, a kernelization of a parameterized problem P is a polynomial-time algorithm that, given an instance (I, k) of P, computes an instance (I', k') of P such that both $|I'|$ and k' are bounded by a function depending only on k and such that (I', k') is a yes-instance if and only if (I, k) is a yes-instance. We call the output (I', k') a *problem kernel*.

3 Hypergraph Characterization

In this section, we formalize the above-mentioned hypergraph characterization of rulers with respect to the Golomb property, consider computing the corresponding hypergraphs, and derive some of their structural properties. The hypergraph characterization serves as basis for the succeeding sections.

Let $R \subseteq \mathbb{N}$ be a ruler. We say that two marks $a, b \in R$ *measure* the *distance* $|a - b|$. We say that the measurements of two pairs of marks a, b and c, d *overlap* if the length of the ruler $\{a, b, c, d\}$ is strictly smaller than $|a - b| + |c - d|$. A *conflict* is an inclusion-wise minimal non-Golomb ruler. That is, a conflict is a set of three or four marks that consists of two distinct unordered pairs of marks that measure the same distance. See also Figure 1. The *conflict hypergraph* of a ruler R is the hypergraph $H_R = (R, E)$, where E is the set of all conflicts contained in R. With respect to rulers and conflict hypergraphs, we synonymously use the terms vertices and marks, as well as edges and conflicts, respectively. Analogously to d-edges (edges of cardinality d), we speak of 3-conflicts and 4-conflicts. The following lemma is obvious.

Lemma 1 *Let R be a ruler and $H_R = (R, E)$ be its conflict hypergraph. Then R is Golomb if and only if $E = \emptyset$.*

We note that a similar hypergraph characterization has been used to show lower bounds for Sidon sets [4] (a Golomb ruler is a special case of a Sidon set), and to prove that a problem related to Sidon sets is efficiently solvable on parallel random access machines [2]. Furthermore, the notion of conflict is implicit in numerous studies related to Golomb rulers, for example, see Meyer and Papakonstantinou [24]. However, we are not aware of studies related to the structural properties of the conflict hypergraphs and how they can be used to derive useful algorithms for finding Golomb rulers.

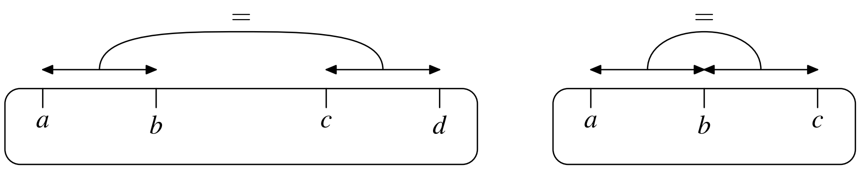

Figure 1: Two rulers with the marks a, b, c, and d, respectively. To the left, we see that the marks a and b measure the same distance as c and d. We consider this to be a conflict with respect to Golomb rulers and model it as an edge $\{a, b, c, d\}$ in the corresponding hypergraph. To the right we see a degenerated form of a conflict which leads to an edge with only three vertices.

Algorithm HypergraphConstruction: Constructing a conflict hypergraph for a given ruler

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Input: A finite ruler $R \subseteq \mathbb{N}$.
Output: A hypergraph $H_R = (R, E)$.

Start with an empty hypergraph H with vertex set R;
Create an empty map M that maps integers to lists;
$\delta_{\max} \leftarrow \max\{x : x \in R\} - \min\{x : x \in R\}$;
for $i \in R$ **do**
    for $j \in R, i < j \leq i + \delta_{max}/2$ **do**
        Add (i, j) to the list mapped to $j - i$ by M;
for $i \in R$ **do**
    for $j \in R, i < j \leq i + \delta_{max}/2$ **do**
        for (k, l) *in the list mapped to* $j - i$ *by* M, $j \leq k$ **do**
            Add the edge $\{i, j, k, l\}$ to H;
return H;

3.1 Hypergraph Construction

We now consider the construction of conflict hypergraphs. It is obvious that they can be constructed in $O(n^4)$ time. We show that this bound can be improved to $O(n^3)$ in the worst case and this is also tight.

Instead of the trivial approach of verifying every possible tuple, one can consider the distances between marks present in the ruler and examine which of them lead to edges in the graph. Algorithm HypergraphConstruction describes such a procedure. In this algorithm we use an auxiliary map M that maps every measurable distance to pairs of marks that measure it. First, we fill M: The first two loops iterate over distances present in R and add every pair of vertices to the entry in M corresponding to their distance. Then, for every short distance in R (every distance at most half the maximum distance in R) M contains a list with all pairs of marks that measure this distance. In the second step, we add the edges to the designated conflict hypergraph H: The last three nested loops again iterate over distances present in the ruler and simply add an edge to H for every pair of marks that measure this distance. To formally prove the correctness of Algorithm HypergraphConstruction we need the following auxiliary lemma.

Lemma 2 *Every edge in a conflict hypergraph is due to two pairs of marks that measure the same distance and the measurements do not overlap.*

Proof Assume that there are four marks $a < b < c < d$ such that the following equation holds

$$c - a = d - b =: \delta.$$

That is, the measurements of the pairs a, c and b, d overlap. Then, the non-overlapping measurements $|b - a|$ and $|d - c|$ also form a conflict, because subtracting the overlap $c - b$ from the distance δ gives $b - a = d - c$. □

From Lemma 2 we also get the following observation.

Observation 1 *Every edge in the conflict hypergraph of the ruler R is due to a distance that is at most half the maximum distance measurable by marks on R.*

Observation 1 allows us to disregard distances measured by marks that are more than half the length of the ruler apart, because measurements of such long distances must overlap. This basically gives the correctness of Algorithm HypergraphConstruction. The running time can be shown to be cubic:

Lemma 3 *Algorithm HypergraphConstruction constructs a conflict hypergraph for its input ruler in* $O(n^3)$ *time.*

Proof The correctness of the algorithm essentially follows from Lemma 2 and Observation 1. It remains to prove the running time bound.

For the implementation of the map M, we use red-black trees or any other dictionary data structure that supports inserting and retrieving a value for a particular key in $O(\log |M|)$ time, where $|M|$ is the number of used keys. Thus, we can support the insertion of an integer into a list mapped by M and the retrieval of a mapped list in $O(\log |M|)$ time. Note also that $|M| \in O(n^2)$ at any point of the execution of Algorithm HypergraphConstruction.

Obviously the running time of Algorithm HypergraphConstruction is mainly dependent on the last three nested loops in lines 7–9. The two outer loops each iterate at most n times. Retrieving the list from M in line 9 can be done in time $O(\log(n^2)) = O(\log n)$. The iteration of the innermost loop in line 9 is bounded by a term in $O(n)$, because any fixed distance δ between two marks on the ruler R can occur at most $2n$ times: δ can be measured at most two times by one mark with any other mark. Adding the edge to the hypergraph is possible in $O(1)$ time and, thus, the running time is in $O(n^3)$. □

Note that we only consider short distances in the loop-headers in lines 5 and 8 of Algorithm HypergraphConstruction. However, the omission of long distances does not influence the asymptotic upper bound on the running time. This is a heuristic trick that could prove useful in practice.

Unfortunately, the running time cannot be further improved because there are rulers that contain $\Omega(n^3)$ conflicts.

Lemma 4 *The upper bound on the running time of Algorithm HypergraphConstruction is tight.*

Proof There are conflict hypergraphs that contain $\Omega(|R|^3)$ edges: This holds for graphs constructed from rulers whose marks form an interval in $\mathbb{N}$. Consider the ruler $R = \{1, \ldots, n\}$ where n is even; obviously, every distance from 1 up to $n - 1$ is measured by marks in R. Let $\delta \leq n/2$ be a fixed positive integer. How many possibilities are there to choose two pairs of marks such that they both measure δ and the measurements do not overlap? One can place one pair leftmost on the ruler, count every possible placement of the other pair to the right, then move the first one to the right by one and iterate. Summing over every distance δ up to $n/2$, this gives a lower bound on the number of edges in H_R:

$$\sum_{\delta=1}^{n/2} \sum_{j=1}^{n-2\delta} \sum_{k=j+\delta}^{n-\delta} 1 = \frac{n}{24}(2n^2 - 3n - 2) \in \Omega(n^3)$$

No edge is counted twice here because if there is an edge due to two different distances $1 \leq \delta_1 < \delta_2$, then the measurements of δ_2 overlap: let $a < b$ and $c < d$ both measure δ_1 with $b \leq c$. Where can δ_2 be placed? Certainly not between a, b and the other pair and also not between the pairs a, d and b, c. It can only be measured by both of the pairs a, c and b, d and, thus, the measurements overlap. However, in the above construction we are only counting non-overlapping measurements. □

Algorithm HypergraphConstruction and Lemma 4 now yield the following theorem.

Theorem 1 *There is a hypergraph characterization for rulers such that Golomb rulers one-to-one correspond to hypergraphs without edges. The worst-case time complexity of computing the conflict hypergraph for a ruler with n marks is* $\Theta(n^3)$.

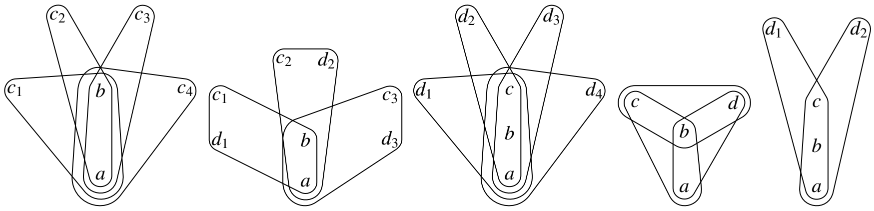
Theorem 1 implies that GSR is fixed-parameter tractable with respect to the parameter "number k of deleted marks": By Lemma 1 Golomb rulers and only these correspond to edge-less conflict hypergraphs. Thus, the task of removing marks to obtain a Golomb subruler reduces to the task of removing vertices from a hypergraph to obtain an edge-less graph. This is exactly the Hitting Set problem and, thus, we can apply algorithms known for this problem to GSR. Hitting Set instances with m edges and at most four vertices per edge can be solved in $O(3.076^{\ell} + m)$ time, where ℓ is the sought hitting set size [14]. This implies that GSR can be solved in $O(3.076^k + n^3)$ time. Notably, the instances created in the reduction sketched above seem rather restricted and this might lead to speedups.

3.2 Observations on the Structure of Conflict Hypergraphs

We are interested in the structure of the constructed hypergraphs, because we would like to develop efficient algorithms exploiting the specific structure of GSR. This proves successful in that we are able to give forbidden subgraphs that we use in Section 4 to give effective data reduction rules. However, the structure of conflict hypergraphs is also interesting on its own. In this regard, our studies merely form a starting point for further research.

At first, notice that the set of conflict hypergraphs is a strict subset of all hypergraphs with edges of size three and four. This is because the construction algorithm can be carried out using $O(n^3)$ edge additions, n being the number of marks and thus vertices. However, general 3,4-hypergraphs can contain $\binom{n}{4} \in \Omega(n^4)$ edges.

It is interesting to determine which hypergraphs can and which cannot be constructed. For example, this could be done through a forbidden (induced) subgraph characterization: a set F of hypergraphs such that a 3,4-hypergraph H is a conflict hypergraph for a ruler if and only if H does not contain a hypergraph $G \in F$ as (induced) subgraph. Unfortunately, we could not provide a forbidden (induced) subgraph characterization F. However, we make partial progress by providing some forbidden subgraphs and forbidden induced subgraphs (see Figure 2). These might be helpful in research towards a forbidden subgraph characterization of conflict hypergraphs and in deriving more efficient algorithms for GSR.



(a) Small hand (b) Crocus (c) Large hand (d) Rotor (e) Scissors

Figure 2: Forbidden subgraphs (2a, 2c, and 2d) and forbidden induced subgraphs (2b and 2e) of conflict hypergraphs of rulers. Letters (and their indices) represent vertices and closed curves encircling vertices represent hyperedges.

We prove the absence of the subgraphs shown in Figure 2 and a certain kind of hypercycle. This shows that a characterization through a *finite* number of forbidden subgraphs is not possible (see below for details). The observations used to obtain that the subgraphs in Figures 2a and 2b are forbidden are also used in our data reduction rules in Section 4. For the forbidden subgraphs in Figures 2c through 2e, proofs are given in Appendix A.

Proposition 1 (Forbidden subgraph "small hand") *The graph shown in Figure 2a is a forbidden subgraph in a conflict hypergraph.*

Proof In a 3-conflict there is one mark exactly between the other two. Let a, b be two marks on a ruler. Either a, b or a third mark c can be the middle mark of a 3-conflict containing a and b. In either of the three cases, c is uniquely defined and hence there are at most three 3-conflicts that contain both a and b. □

For the forbidden induced subgraph shown in Figure 2b, we first obtain two observations about conflicts that intersect in two marks. We use these observation again in Section 4 to derive effective data reduction rules. They are thus slightly more general than needed here. Fix two distinct marks $a < b$ on a ruler R and let $C = \{a, b, c, d\}$ be a 4-conflict in H_R with $c < d$. We call C to be $\{a, b\}$*-perpendicular* if $a - c = d - b$ and $\{a, b\}$*-parallel* if $a - b = c - d$. For example, the conflict $C = \{a, b, c, d\}$ with $a = 0$, $b = 3$, $c = 1$, $d = 4$ is $\{a, b\}$-parallel, because $a - b = -3 = c - d$, but it is not $\{a, b\}$-perpendicular as $a - c = -1 \neq d - b$. We would like this definition to convey the intuition that, if C is $\{a, b\}$-parallel, then it is due to the distance measured by a and b. If it is $\{a, b\}$-perpendicular, then it results from the distance measured by some other pair of marks. This suggests that C is either $\{a, b\}$-perpendicular or $\{a, b\}$-parallel; let us prove this in a formal way.

Lemma 5 *Let $H_R = (R, E)$ be a conflict hypergraph and $a, b \in R$. Any 4-conflict in H_R that contains both a and b is either $\{a, b\}$-perpendicular or $\{a, b\}$-parallel.*

Proof Let $C = \{a, b, c, d\}$ be a 4-conflict in H_R and without loss of generality assume that $a < b$ and $c < d$. Let us first show that C is not both $\{a, b\}$-perpendicular and $\{a, b\}$-parallel. Assume the contrary. Then from $\{a, b\}$-parallelity we get $a - b = c - d$ which is equivalent to $a - c = b - d$. Using $\{a, b\}$-perpendicularity it follows that $b - d = a - c = d - b$, hence $d = b$. This implies that C is not a 4-conflict, a contradiction. Being $\{a, b\}$-perpendicular or $\{a, b\}$-parallel covers every situation mainly because of Lemma 2 used in considering all configurations of the marks of C. The possible configurations are as follows.

$$a < b < c < d \quad (1) \qquad a < c < d < b \quad (3) \qquad c < a < d < b \quad (5)$$
$$a < c < b < d \quad (2) \qquad c < a < b < d \quad (4) \qquad c < d < a < b \quad (6)$$

In each configuration, by Lemma 2, the two leftmost marks measure the same distance as the rightmost two. It is thus immediate that in Configurations (1) and (6) C is $\{a, b\}$-parallel and in Configurations (3) and (4) C is $\{a, b\}$-perpendicular. Since $a - c = b - d$ is equivalent to $a - b = c - d$, conflict C is $\{a, b\}$-parallel in Configurations (2) and (5). □

We can furthermore observe that large-enough sets of $\{a, b\}$-perpendicular or $\{a, b\}$-parallel conflicts have only trivial intersections and that they induce additional conflicts.

Lemma 6 *Let $H_R = (R, E)$ be a conflict hypergraph and $a, b \in R$. If $C_1, C_2 \in E$ are distinct $\{a, b\}$-perpendicular 4-conflicts then we have $C_1 \cap C_2 = \{a, b\}$ and $C_1 \cup C_2 \setminus \{a, b\} \in E$. Moreover, if $C_1, C_2, C_3 \in E$ are distinct $\{a, b\}$-parallel 4-conflicts then we have $C_1 \cap C_2 \cap C_3 = \{a, b\}$ and $C_1 \cup C_2 \setminus \{a, b\} \in E$.*

Proof Without loss of generality assume that $a < b$. Consider first $C_1, C_2 \in E$ such that both are $\{a, b\}$-perpendicular. For the sake of contradiction, assume that $C_1 = \{a, b, \gamma, \delta\}$ and $C_2 = \{a, b, \gamma, \epsilon\}$. We consider all possible configurations of γ relative to δ and ϵ. First, if $\gamma < \delta, \epsilon$, then it follows that $\delta = \epsilon$, because $a - \gamma = \delta - b = \epsilon - b$ by definition. If $\delta < \gamma < \epsilon$, then $a - \delta = \gamma - b$ and $a - \gamma = \epsilon - b$ which also leads to $\delta = \epsilon$ (the case $\epsilon < \gamma < \delta$ is analogous). Finally if $\epsilon, \delta < \gamma$ then $a - \epsilon = \gamma - b = a - \delta$ and again $\delta = \epsilon$. Thus no two 4-conflicts that are $\{a, b\}$-perpendicular intersect in a mark other than a and b.

The existence of the additional conflict can be seen as follows. Let conflict $\{a, b, \gamma_1, \delta_1\} \in E$ and conflict $\{a, b, \gamma_2, \delta_2\} \in E$ be $\{a, b\}$-perpendicular and, without loss of generality, assume that $\gamma_1 < \delta_1$ and $\gamma_2 < \delta_2$. By definition $a - \gamma_1 = \delta_1 - b$ and $a - \gamma_2 = \delta_2 - b$. Subtracting the first equation from the second one, we get $\gamma_1 - \gamma_2 = \delta_2 - \delta_1$ implying that, if $|\{\gamma_1, \delta_1, \gamma_2, \delta_2\}| = 4$, then $\{\gamma_1, \delta_1, \gamma_2, \delta_2\}$ is a conflict in H_R. We already know that $\{a, b, \gamma_1, \delta_1\} \cap \{a, b, \gamma_2, \delta_2\} = \{a, b\}$. This implies that indeed $|\{\gamma_1, \delta_1, \gamma_2, \delta_2\}| = 4$ and thus yields the first part of Lemma 6.

Now consider distinct $C_1, C_2, C_3 \in E$, each of them $\{a, b\}$-parallel. Let $C' = \{a, b, \gamma\}$ and for the sake of contradiction assume that $C_1 = C' \cup \{\delta\}, C_2 = C' \cup \{\epsilon\}, C_3 = C' \cup \{\zeta\}$. Note that without loss of generality we may assume $\delta < \gamma < \epsilon$. Then, applying the definition of $\{a, b\}$-parallel we have $a - b = \delta - \gamma = \gamma - \epsilon$ and either $a - b = \zeta - \gamma$ or $a - b = \gamma - \zeta$. Thus either $\delta = \zeta$ or $\epsilon = \zeta$. This implies that at most two $\{a, b\}$-parallel 4-conflicts intersect in a fixed mark other than a and b.

Let us prove that if $C_1 = \{a, b, \gamma_1, \delta_1\}, C_2 = \{a, b, \gamma_2, \delta_2\} \in E$ such that both are $\{a, b\}$-parallel 4-conflicts, then also $D := \{\gamma_1, \delta_1, \gamma_2, \delta_2\} \in E$. Assume without loss of generality that $\gamma_1 < \delta_1$ and $\gamma_2 < \delta_2$. Then, by definition of $\{a, b\}$-parallelity,

$$\delta_1 - \gamma_1 = a - b = \delta_2 - \gamma_2. \tag{7}$$

Note that neither $\gamma_1 = \gamma_2$ nor $\delta_1 = \delta_2$ as, in this case, Equation 7 would lead to $D = \{\gamma_1, \gamma_2, \delta_1, \delta_2\} = \{\gamma_1, \delta_1\}$ implying that $C_1 = C_2$. Hence, if $|D| = 3$, then either $\gamma_1 = \delta_2$ or $\delta_1 = \gamma_2$. Plugging one of these equalities into Equation 7 we obtain either $\delta_1 = 2\delta_2 - \gamma_2$ or $2\delta_1 - \gamma_1 = \delta_2$. This means that either δ_2 is half-way between γ_2 and δ_1 or δ_1 is half-way between γ_1 and δ_2. Thus, in this case $D \in E$. Finally, if $|D| = 4$ then Equation 7 directly implies that $D \in E$, as required. □

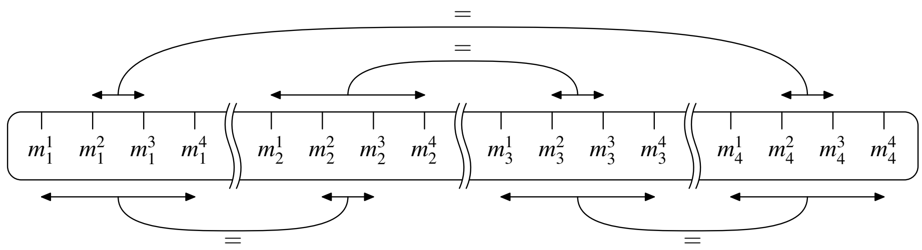


Figure 3: Schematic view of a ruler produced by Construction 1 (not to scale). All conflicts of the ruler are shown except the conflicts of the form $\{m_i^1, m_i^2, m_i^3, m_i^4\}$ where $1 \leq i \leq 4$.

Let us briefly remark that it is indeed possible for two $\{a, b\}$-parallel 4-conflicts C_1 and C_2 to have $|C_1 \cap C_2| = 3$. This is the case for $\{0, 1, 2, 3\}$ and $\{0, 1, 3, 4\}$ for example, which both are $\{0, 1\}$-parallel.

From Lemmas 5 and 6 we immediately get that the graph shown in Figure 2b cannot be an induced subgraph in a conflict hypergraph: If there are three conflicts intersecting in two marks a, b, at least two of them are $\{a, b\}$-perpendicular or $\{a, b\}$-parallel. Hence, there is at least one additional conflict.

We can furthermore derive from Lemmas 5 and 6 that conflict hypergraphs do not contain a certain kind of induced "hypercycle". Let R be a ruler and $H_R = (R, E)$ its conflict hypergraph. A *2-hypercycle* is a sequence $S = \{m_0^1, m_0^2\}, \ldots, \{m_\ell^1, m_\ell^2\}$ of unordered pairs of distinct marks $m_i^1, m_i^2 \in R$ such that for every i, j, $0 \leq i < j \leq \ell$, we have $\{m_i^1, m_i^2\} \cap \{m_j^1, m_j^2\} = \emptyset$ and $\{m_i^1, m_i^2, m_{i+1}^1, m_{i+1}^2\} \in E$. (Indices taken modulo $\ell + 1$.) Here, $\ell + 1$ is the *length* of S. A *chord* for the hypercycle S is a conflict $\{m_i^1, m_i^2, m_j^1, m_j^2\} \in E$ such that i and j are not consecutive, that is $|i - j| > 1$. The following Proposition 2 shows that conflict hypergraphs do not contain sufficiently large induced 2-hypercycles of odd length.

Proposition 2 *Let $H_R = (R, E)$ be a conflict hypergraph. Every 2-hypercycle in H_R of odd length at least five has a chord.*

Proof Let $S = \{m_0^1, m_0^2\}, \ldots, \{m_\ell^1, m_\ell^2\}$ be a 2-hypercycle of odd length in H_R. Consider i, $0 \leq i \leq \ell$, and the conflict $C_i = \{m_i^1, m_i^2, m_{i+1}^1, m_{i+1}^2\} \in E$. Lemma 5 gives that C_i is either $\{m_i^1, m_i^2\}$-perpendicular or $\{m_i^1, m_i^2\}$-parallel. Note that, if any conflict $C = \{a, b, c, d\}$ is $\{a, b\}$-perpendicular then it is also $\{c, d\}$-perpendicular and analogously if C is $\{a, b\}$-parallel then it is also $\{c, d\}$-parallel. Let us thus call C_i perpendicular if it is $\{m_i^1, m_i^2\}$-perpendicular and parallel if it is $\{m_i^1, m_i^2\}$-parallel. Since S has odd length, there are thus two consecutive conflicts both being perpendicular or parallel, say conflicts $\{m_i^1, m_i^2, m_{i+1}^1, m_{i+1}^2\}$ and $\{m_{i+1}^1, m_{i+1}^2, m_{i+2}^1, m_{i+2}^2\}$. Lemma 6 now implies that $\{m_i^1, m_i^2, m_{i+2}^1, m_{i+2}^2\} \in E$ and thus, since the length is at least five, this conflict is a chord in S. $\square$

In contrast to the absence of induced 2-hypercycles of odd length, we note that there are rulers of arbitrarily large size whose conflict hypergraphs are 2-hypercycles of even length.

Construction 1 Suppose that we want to construct a ruler with $4n$ marks, $n \geq 3$, with a corresponding conflict hypergraph that is a 2-hypercycle of length $2n$. We iteratively define groups $M_1, \ldots, M_n$ of marks where each group M_i consists of the four marks $m_i^1, m_i^2, m_i^3, m_i^4$. Carefully choosing these marks will ensure that the only conflicts in the constructed ruler are of the forms $\{m_i^1, m_i^2, m_i^3, m_i^4\}$ and $\{m_i^1, m_i^4, m_{i+1}^2, m_{i+1}^3\}$, except for $i = n - 1, n$. A scheme of a ruler that is to be constructed, and the corresponding conflicts, is shown in Figure 3 for $n = 4$.

We first define the distances of marks within a group without placing the marks explicitly on the ruler. We do this by defining the marks m_i^1, m_i^3, and m_i^4 relative to the mark m_i^2 in each group M_i. Using appropriate definitions we introduce the desired conflicts and we ensure that no other conflicts arise from the distances measured *within* one group. The marks m_i^2 for all groups M_i, $i > 1$, are placed in the last step, and hence the placement of all marks is made explicit. The marks m_i^2 are defined in such a way that the group M_i is far away from all previous groups M_j, $j < i$. We show that this ensures that there are no unwanted conflicts arising from distances measured *between* two groups.

First, define $m_1^2 := 0$ and $m_1^3 := 1$. Next, set $i := 2$ and choose any distance $\delta_i \in \mathbb{N} \setminus \{0\}$ such that neither δ_i, nor $\delta_i + m_{i-1}^3 - m_{i-1}^2$, nor $2\delta_i + m_{i-1}^3 - m_{i-1}^2$ is measured by any pair of marks already defined that both belong to any group M_j, $j < i$. Note that this is always possible, since we may choose δ_i arbitrarily large. Next, define

$$
\begin{aligned}
m_{i-1}^1 &:= m_{i-1}^2 - \delta_i, & m_i^3 &:= m_i^2 + m_{i-1}^3 - m_{i-1}^2 + 2\delta_i, \text{ and}\\
m_{i-1}^4 &:= m_{i-1}^3 + \delta_i.
\end{aligned}
$$

In this way, we introduce the following new distances measured by marks in one group:

$$
\begin{aligned}
m_{i-1}^2 - m_{i-1}^1 &= m_{i-1}^4 - m_{i-1}^3 = \delta_i,\\
m_{i-1}^3 - m_{i-1}^1 &= m_{i-1}^4 - m_{i-1}^2 = \delta_i + m_{i-1}^3 - m_{i-1}^2, \text{ and}\\
m_{i-1}^4 - m_{i-1}^1 &= m_i^3 - m_i^2 = 2\delta_i + m_{i-1}^3 - m_{i-1}^2.
\end{aligned}
$$

Hence the conflicts $\{m_{i-1}^1, m_{i-1}^2, m_{i-1}^3, m_{i-1}^4\}$ and $\{m_{i-1}^1, m_{i-1}^4, m_i^2, m_i^3\}$ are introduced. Furthermore, the distances avoided in choosing δ_i are exactly the new distances measured by pairs of marks *within* one of the groups. That is, no further conflicts involving only two groups are introduced. Moreover, as $m_{i-1}^3 - m_{i-1}^2$ is odd, it is also maintained that $m_i^3 - m_i^2$ is odd. This is used in defining the last groups of marks. Increment i and repeat the above steps until $i = n - 2$. Next, let $\delta_n \in \mathbb{N}$ such that none of the following distances is measured by any pair of marks already defined that both belong to any group M_j, $j < n$:

$$
\begin{aligned}
&\delta_n, && (m_{n-1}^3 - m_{n-1}^2 + 2\delta_n - 1)/2,\\
&m_{n-1}^3 - m_{n-1}^2 + \delta_n, && (m_{n-1}^3 - m_{n-1}^2 + 2\delta_n - 1)/2 + 1, \text{ and}\\
&m_{n-1}^3 - m_{n-1}^2 + 2\delta_n.
\end{aligned}
$$

Note that both distances on the right are integers as $m_{n-1}^3 - m_{n-1}^2$ is odd. Define the final marks by letting

$$
\begin{aligned}
m_{n-1}^1 &:= m_{n-1}^2 - \delta_n, & m_n^1 &:= m_n^2 - (m_{n-1}^3 - m_{n-1}^2 + 2\delta_n - 1)/2,\\
m_{n-1}^4 &:= m_{n-1}^3 + \delta_n, & m_n^4 &:= m_n^2 + (m_{n-1}^3 - m_{n-1}^2 + 2\delta_n - 1)/2 + 1, \text{ and}\\
m_n^3 &:= m_n^2 + 1.
\end{aligned}
$$

Thus the following distances are measured by the new marks within one of the last two groups:

$$
\begin{aligned}
m^2_{n-1} - m^1_{n-1} &= m^4_{n-1} - m^3_{n-1} = \delta_n, \\
m^3_{n-1} - m^1_{n-1} &= m^4_{n-1} - m^2_{n-1} = \delta_n + m^3_{n-1} - m^2_{n-1}, \\
m^4_{n-1} - m^1_{n-1} &= m^4_n - m^1_n = 2\delta_n + m^3_{n-1} - m^2_{n-1}, \\
m^2_n - m^1_n &= m^4_n - m^3_n = (m^3_{n-1} - m^2_{n-1} + 2\delta_n - 1)/2, \\
m^3_n - m^1_n &= m^4_n - m^2_n = (m^3_{n-1} - m^2_{n-1} + 2\delta_n - 1)/2 + 1, \text{ and} \\
m^3_n - m^2_n &= m^3_1 - m^2_1 = 1.
\end{aligned}
$$

Hence, in this way we introduce the conflicts $\{m^1_{n-1}, m^2_{n-1}, m^3_{n-1}, m^4_{n-1}\}$, $\{m^1_{n-1}, m^4_{n-1}, m^1_n, m^4_n\}$, and $\{m^1_n, m^2_n, m^3_n, m^4_n\}$ as well as the conflict $\{m^2_1, m^3_1, m^2_n, m^3_n\}$ which closes the 2-hypercycle. Each of these conflicts is between two groups of marks. Similarly as above, the choice of δ_n ensures that no further conflicts involving two groups are introduced.

Finally, we define the marks m^2_i and hence make the positions of all marks explicit. Let $i := 2$ and define m^2_i as any sufficiently large integer such that $m^1_i - m^4_{i-1} \geq \max\{\text{maxdist}, m^4_{i-1} - m^1_1\} + 1$, where maxdist is the maximum distance measured by any pair of marks within one of the groups M_i. Note that, in this way, the ruler induced by the marks M_j, $j \leq i$, has at least double the length of the ruler induced by the marks M_k, $k \leq i - 1$, plus the length of the ruler induced by M_i. Hence, the new distances introduced *between* two groups are larger than any distance between two groups previously present. Also, as any two groups are at least maxdist apart, distances within M_i do not conflict with any other distance between two groups. Hence, placing m_i in this way does not introduce any new conflicts. Increment i and repeat the above steps until all marks are placed, finishing the construction.

From the above observations we now obtain the following theorem.

Theorem 2 *Every characterization of conflict hypergraphs through forbidden subgraphs or forbidden induced subgraphs contains infinitely many of them.*

Proof Assume there was such a characterization that is finite. It includes a subgraph H (induced subgraph, respectively) that is contained in infinitely many 2-hypercycles of odd length and not contained in infinitely many 2-hypercycles of even length. In particular, there is one 2-hypercycle C containing H and having more vertices than H and another 2-hypercycle C' that does not contain H and has more vertices than C. However, since C has more vertices than H it follows that H consists of a collection of degree-zero vertices and "2-hyperpaths" (that is, 2-hypercycles with a hyperedge removed). Thus, any 2-hypercycle larger than C also contains H and so does C' in particular. This is absurd and hence there is no finite characterization of conflict hypergraphs through forbidden (induced) subgraphs. □

3.3 A Note on Conflict Hypergraph Recognition

It is interesting to determine the computational complexity of deciding whether a given hypergraph is a conflict hypergraph for some ruler or not. This can give a hint on how complex a succinct description of a characterization through forbidden (induced) subgraphs is. For example, bipartite graphs are characterized through the set of odd-length cycles as forbidden subgraphs. However, the description of this set is very simple in some sense, and it is indeed decidable in linear time whether a given graph is bipartite. For conflict hypergraphs the recognition problem is defined as follows.

Conflict Hypergraph Recognition
Input: A 3,4-hypergraph G.
Question: Is there a ruler R such that its conflict hypergraph H_R is isomorphic to G?

We give an upper bound on the complexity of Conflict Hypergraph Recognition by showing that this problem lies in NP. It would be interesting to see whether it is also in P, which likely entails further insight into forbidden subgraph characterizations for conflict hypergraphs. To show that Conflict Hypergraph Recognition is in NP we give a non-deterministic algorithm that solves an integer linear program as a subproblem. In the Integer Linear Program problem, one is given an integer matrix A and a compatible integer vector b and it is asked whether there is an integer vector x such that $Ax \leq b$. This problem is known to be NP-complete [19].

Theorem 3 Conflict Hypergraph Recognition *lies in NP.*

Proof A given hypergraph $G = (V, E)$ is a conflict hypergraph for some ruler R if and only if there is a total ordering $v_1, \ldots, v_n$ of the vertices and a mapping $\phi : V \to R$ such that

(i) $\phi(v_i) < \phi(v_{i+1})$ for all $1 \leq i < n$,
(ii) for all 4-edges $\{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4} \mid i_1 < i_2 < i_3 < i_4\} \in E$ it holds that $\phi(v_{i_2}) - \phi(v_{i_1}) = \phi(v_{i_4}) - \phi(v_{i_3})$ (see Lemma 2), and analogously for all 3-edges, and
(iii) for all "4-non-edges" $\{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4} \mid i_1 < i_2 < i_3 < i_4\} \notin E$ it holds that $\phi(v_{i_2}) - \phi(v_{i_1}) \neq \phi(v_{i_4}) - \phi(v_{i_3})$, and analogously for all 3-non-edges.

Thus, a non-deterministic algorithm may proceed as follows. It first non-deterministically guesses an ordering of the vertices of the input graph. It creates an integer linear program by first adding constraints corresponding to conditions (i) and (ii). (It is trivial to replace equality constraints by two less-than-or-equal constraints. The constraints of form $a < b$ can be replaced by $a + 1 \leq b$ since we only deal with integer coefficients and variables.) Then the algorithm iterates over all 4-non-edges and, for each of them, guesses whether the difference of the first two marks should be smaller or larger than the difference of the second two marks. It does so analogously for all 3-non-edges. The algorithm adds the corresponding constraints to the integer linear program and solves it.

If there is a solution to an integer linear program obtained in this way, then this solution implies a corresponding ruler R such that H_R is isomorphic to G. In the other direction, by the equivalence noted above, if G is a conflict hypergraph, then there must be a solution to one of the possibilities for the obtained linear programs. □

4 Effective Data Reduction in Polynomial Time

In this section, we present data reduction rules for the Golomb Subruler (GSR) problem parameterized by the number k of deleted marks. We use the hypergraph characterization and structural observations from Section 3 to derive data reduction rules such that after $O((n + m) \log n)$ processing time, an equivalent instance with at most $9k^3 + 6k^2 + k$ marks and at most $3(k^3 + k^2)$ conflicts remains.

Using the conflict hypergraphs, one can regard GSR as a special case of the Hitting Set problem. A natural approach is thus to apply reduction rules for Hitting Set to the hypergraphs. We refer to the work of van Bevern [5] and references therein for a comprehensive overview on provably efficient data reduction for Hitting Set. For example, if constant d is the maximum number of vertices in an edge, then there are data reduction rules for Hitting Set that yield a problem kernel with $O(k^{d-1})$ vertices [1]. One has to bear in mind, however, that crucial reduction rules from the literature destroy the conflict hypergraph property by removing edges and/or by inserting edges of size two. Furthermore, the constants in the upper bound on the number of remaining vertices either depend exponentially on d, leading to large constants hidden in the O-notation, or one has to spend $O(2^d k^{d-1} \log(k) \cdot m)$ time computing a certain set of "weakly related edges" to get a bound of $(2d-1)k^{d-1} + k$ on the number of vertices (see the analysis of Abu-Khzam's [1] reduction rules by van Bevern [5]). Moreover, a bound of $O(k^{d-\epsilon})$ on the number of edges is not obtainable for Hitting Set within polynomial preprocessing time unless coNP ⊆ NP/poly and the polynomial hierarchy collapses [10].

We give here reduction rules for GSR and, using the special structure of conflict hypergraphs, we show that they simultaneously yield a small constant in the upper bound on the number of remaining marks, have efficient running times, and break the $O(k^4)$-barrier for the upper bound on the number of remaining conflicts. Our reduction rules operate on the conflict hypergraphs and only delete marks and all conflicts they are contained in. They thus preserve the conflict hypergraph property.

For our preprocessing procedure, we employ two modified high-degree reduction rules. When exhaustively applied, one of these rules suffices to upper-bound the number of 3-conflicts in the conflict hypergraph, the other rule handles the number of 4-conflicts. With the help of these two bounds, we are then able to bound the number of marks in a reduced instance. In the following description of the reduction rules, we assume that the conflict hypergraph of the input ruler has been computed and is continuously updated alongside the ruler. First we need the following simple rule.

Reduction Rule 1 (Isolated marks) *If there is a mark that is not present in any conflict, then remove it.*

It is clear that such marks never have to be deleted in order to make the input ruler conflict-free.

The next two "high-degree" rules are similar in spirit to rules of Abu-Khzam [1], but differ in decisive details. The following reduction rule is based on the small hand forbidden subgraph (Proposition 1).

Reduction Rule 2 (High degree for 3-conflicts) *If there is a mark v that is contained in more than $3k$ 3-conflicts, then remove v from the ruler, remove any conflict containing v and decrement k by one.*

Lemma 7 *Rule 2 is correct. If Rule 2 cannot be applied to a ruler R, and R can be made Golomb with at most k mark deletions, then H_R has at most $3k^2$ 3-conflicts.*

Proof Assume that there are more than $3k$ 3-conflicts incident to one mark v in H_R. By Proposition 1, there are at most three 3-conflicts that intersect in two marks. Hence, the deletion of any mark other than v can destroy at most three conflicts incident to v. Thus, if v is not deleted, then more than k marks are necessary to hit all conflicts incident to v.

Now let (R, k) be a yes-instance that does not fulfill the conditions of Rule 2. Every 3-conflict in H_R has to be hit but one mark can hit at most $3k$ 3-conflicts and, thus, H_R has at most $3k^2$ 3-conflicts. □

To lift the high-degree concept to 4-conflicts, we need the following auxiliary lemma. It can be seen as an analogous replacement for Proposition 1.

Lemma 8 *Let (R, k) be a yes-instance of* GSR *and let $a < b$ be two marks in R. The conflict hypergraph H_R has at most $3k$ 4-conflicts that contain $\{a, b\}$.*

Proof We mainly use Lemmas 5 and 6 and the case distinction for 4-conflicts that is presented there. We first prove that if there are more than k 4-conflicts that are $\{a, b\}$-perpendicular (see Lemma 5), then R cannot be made Golomb by at most k mark deletions. Then we proceed to show that if there more than $2k$ 4-conflicts that are $\{a, b\}$-parallel, then R cannot be made Golomb by at most k mark deletions. Thus, if there are more than $3k$ conflicts that intersect in two marks a, b, then, by Lemma 5, either more than k conflicts are $\{a, b\}$-perpendicular or more than $2k$ conflicts are $\{a, b\}$-parallel; and, thus, R cannot be made Golomb with k mark deletions.

Assume that there are more than k $\{a, b\}$-perpendicular 4-conflicts in H_R. By Lemma 6 none of these conflicts intersect in marks other than a, b. Hence, since any two such 4-conflicts imply a third conflict by Lemma 6, there are more than $k + 1$ pairwise disjoint unordered pairs of marks such that the union of any two such pairs is a conflict in H_R. Thus, to destroy all these conflicts, one has to remove more than k marks. It follows that in a yes-instance of GSR, there are at most k 4-conflicts that are $\{a, b\}$-perpendicular.

Now assume that there are more than $2k$ 4-conflicts in H_R, each of them $\{a, b\}$-parallel. Since any two of these 4-conflicts imply a third conflict by Lemma 6, there are more than $2k + 1$ unordered pairs of marks such that the union of any two pairs forms a conflict in H_R. Furthermore, at most two such pairs of marks share a mark and, thus, to destroy all the conflicts induced by these pairs, more than k marks have to be removed. It follows that in a yes-instance of GSR there are at most $2k$ 4-conflicts that are $\{a, b\}$-parallel. □

Using Lemma 8 we now obtain the following data reduction rule.

Reduction Rule 3 (High degree for 4-conflicts) *If there is a mark v that is contained in more than $3k^2$ 4-conflicts, then remove v from the ruler, remove any conflicts containing v, and decrement k by one.*

Lemma 9 *Rule 3 is correct. If Rule 3 cannot be applied to a ruler R and R can be made Golomb with at most k mark deletions, then H_R has at most $3k^3$ 4-conflicts.*

The proof of Lemma 9 is analogous to the proof of Lemma 7: Observe that for the correctness, we can substitute Lemma 8 for Proposition 1 to obtain that if v is not deleted, then (R, k) cannot be a yes-instance.

Concluding, we obtain the following theorem.

Theorem 4 *For input instances of* Golomb Subruler *there is a polynomial-time procedure that yields equivalent input instances with at most $9k^3 + 6k^2 + k$ marks. The conflict hypergraph of the ruler of a resulting instance has at most $3k^3$ 4-conflicts and $3k^2$ 3-conflicts. The procedure can be carried out in $O((n + m) \log n)$ time if the conflict hypergraph is given.*

Proof The general procedure for a given ruler R and conflict hypergraph $H_R = (R, E)$ is as follows: Apply Rule 2, apply Rule 3, and iterate until neither applies anymore. Then apply Rule 1 until it does not apply anymore. To describe the procedure in more detail let us fix some notation. The 3-*degree* of a mark v, denoted by $\deg_3(v)$, is the number of 3-conflicts in H that contain v. The 4-degree and $\deg_4(v)$ is defined analogously. Note that $\deg(v) = \deg_3(v) + \deg_4(v)$.

In order to apply the high-degree rules, we use two maps Δ_3, Δ_4 that keep track of vertices that have a given 3-degree and 4-degree, respectively. That is, Δ_3 maps each integer $\delta \in \{\deg_3(v) : v \in R\}$ to a list containing the marks $v \in R$ such that $\delta = \deg_3(v)$. The second map Δ_4 is defined analogously. Using an appropriate implementation for the maps, for example red-black trees, we can support insertion, deletion and update of a key/value pair as well as finding the maximum key in logarithmic time in the number of key/value pairs present in the map. Thus, to initialize the maps we can iterate over all marks in R and update the maps according to their degrees in $O(n \log n + m)$ time.

Now to apply Rule 2 and Rule 3, we find the maximum keys in Δ_3, Δ_4 ($O(\log n)$ time) to check whether the preconditions of the rules are satisfied. If so, then we find a mark v in the list mapped to a maximum key, and find its set of neighbors $\bigcup_{e \in E\colon v \in e} e \setminus v$ in $O(\deg(v))$ time. Then, we delete v and the conflicts it is contained in from R and H_R in $O(\deg(v))$ time and update the key/value pairs of its neighbors in the maps Δ_3, Δ_4 in $O(\deg(v) \log n)$ time. Applying the reduction rules for one mark v is thus possible in $O(\deg(v) \log n)$ time. Since both Rule 2 and Rule 3 can be applied at most once for every mark and since $\sum_{v \in R} \deg(v) \leq 4m$, the running time of the whole data reduction procedure is bounded by $O((n + m) \log n)$.

The upper bound on the number of 3- and 4-conflicts follows from Lemma 7 and Lemma 9. In a yes-instance there is a set S of at most k marks such that every conflict in the conflict hypergraph has a non-empty intersection with S. This means that in each of the 4-conflicts there are at most three marks not in S, summing up to $9k^3$. Analogously, in each 3-conflict there are at most two marks not in S, summing up to $6k^2$. Thus, summing the marks that are in 4-conflicts but not in S, the marks in 3-conflicts but not in S and the marks of S, we get that there are at most $9k^3 + 6k^2 + k$ marks in a yes-instance. □

Note that we can only achieve a running time of $O((n + m) \log n)$ if the conflict hypergraph of the given ruler is also given. In the worst case, its computation would imply an additional running time of $O(n^3)$ (Theorem 1).

Remarks There is a much simpler strategy to apply Rule 2 and Rule 3 that yields a running time of $O(k(n + m))$: For every application, build an array that contains $\deg_3$ and $\deg_4$ for every vertex by iterating over every edge. Then, choose a vertex of high degree and carry out the rules. This procedure takes $O(n + m)$ time and iterates at most k times. Thus, depending on whether $k \leq \log n$ or not, one can choose the strategy appropriate in practice.

We note that in the conference version of this article [32], we erroneously claimed our data reduction rules to yield a problem kernel. This implied that the encoding of the marks in the resulting instances is polynomially bounded by the parameter k, which is not necessarily the case. However, since the length of the rulers currently considered is in the hundreds [12], the encoding of the marks fits into register space and operations on them work in constant time. Thus, it is more important to upper bound the combinatorial explosion in the running times of search algorithms for Golomb Subruler which depends on the number of marks present rather than their encoding length.

Additional Reduction Rules The following two rules are not needed to prove the bound on the number of remaining marks. However, since they can be applied efficiently, it should not be detrimental to apply them in a preprocessing algorithm.

Reduction Rule 4 (Isolated conflicts) *If there is a conflict that does not intersect any other conflict, then remove it and all marks it comprises from the ruler and decrement k by one.*

Reduction Rule 5 (Leaf conflicts) *If there is a conflict e that intersects the union of all other conflicts in exactly one mark v, then remove e and all marks it comprises from the ruler, remove all conflicts incident to v, and decrement k by one.*

The first rule is trivial. For the second rule, at least one mark in e has to be deleted. However, any mark in e can destroy only the conflict e except v. Thus we may assume every Golomb

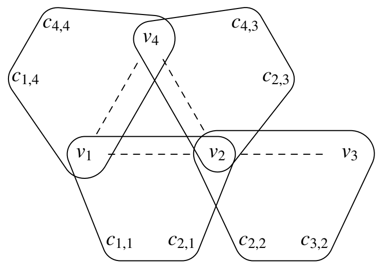


Figure 4: A graph (vertices v_1 through v_4 and dashed edges) and the corresponding hypergraph produced by Construction 2 (vertices $v_i, c_{i_j,j}$ and solid hyperedges).

deletion set to contain v. The running time of an exhaustive application of Rule 5 when applying it in a breadth-first search fashion can easily be seen to be $O(n + m)$. We omit the straightforward details.

5 A Simplified Hardness Construction

Meyer and Papakonstantinou [24] showed that Golomb Subruler (GSR) is NP-hard via a reduction from an NP-hard SAT variant. However, the construction of the ruler corresponding to the SAT formula is involved and hard to comprehend. Using our hypergraph characterization of rulers (Section 3), we provide a reduction from the NP-complete Independent Set problem yielding a much simpler construction. Along these lines, we additionally observe that GSR is hard even when there are no three marks that measure the same distance twice and even when each mark in the input instance is bounded by a polynomial of the input size. This implies that there is no pseudo-polynomial algorithm for GSR unless P = NP which, to our knowledge, was not known before. We also note that the corresponding reduction implies a W[1]-hardness result, that is, presumable fixed-parameter intractability, for a modified version of GSR, where the size of the sought ruler depends on the number of conflicts.

In Independent Set a graph $G = (V, E)$ and an integer $\ell \geq 1$ are given and it is asked whether there is a vertex set $I \subseteq V$ in G such that no edge of G is contained in I and $|I| \geq \ell$. In Hypergraph Independent Set also hypergraphs instead of graphs are allowed as input. For readability we opt to use the word "edges" for vertex sets of cardinality two and "hyperedges" for vertex sets of higher order in this section.

The basic idea of our reduction from Independent Set is to output instances of Hypergraph Independent Set that constitute conflict hypergraphs for some rulers. Since the marks of a Golomb ruler R form an independent set in all conflict hypergraphs of superrulers of R, in this way one achieves a reduction from Independent Set to GSR.

Construction 2 Let a graph G and an integer ℓ constitute an instance of Independent Set. Let $v_1, \ldots, v_n$ be the vertices of G and let $e_1, \ldots, e_m$ be the edges in G. Construct the hypergraph H from G as follows: Add all vertices of G to H. For every edge $e_j = \{v_{i_j}, v_{k_j}\} \in E(G)$, introduce two new vertices $c_{i_j,j}, c_{k_j,j}$ into H and add the hyperedge $e_j \cup \{c_{i_j,j}, c_{k_j,j}\}$ to H. The hypergraph H and the integer $\ell + 2m$ constitute an instance of Hypergraph Independent Set. See also Figure 4.

Lemma 10 *Construction 2 is a polynomial-time many-one reduction from* Independent Set *to* Hypergraph Independent Set.

Proof Clearly, Construction 2 is polynomial-time computable. Let v_k, e_j, and $c_{i_j,j}$ be as in the construction. To prove the correctness let I be an independent set in G with $|I| \geq \ell$. Adding

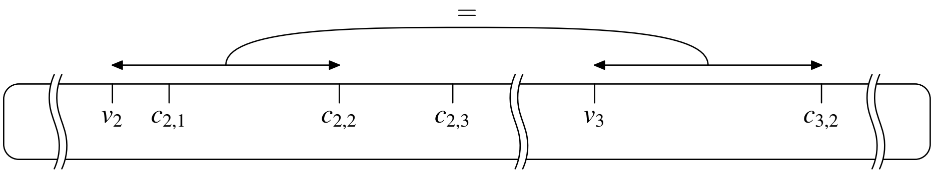


Figure 5: Parts of a ruler produced by Construction 3 from the hypergraph shown in Figure 4.

the vertices $c_{i_j,j}, c_{k_j,j}$ for every $1 \leq j \leq m$ yields a vertex set I'. The set I' has size $\ell + 2m$ and is an independent set for H because the intersection $I' \cap e$ for any $e \in E(H)$ contains at most one vertex $v \in V(G)$.

Now assume that there is an independent set I' of size at least $\ell+2m$ in H. The set $I' \cap V(G)$ might not be an independent set because there might be edges $e_j = \{v_{i_j}, v_{k_j}\} \in E(G)$ with $e_j \subseteq I'$. For every such edge e_j however either $c_{k_j,j}$ or $c_{i_j,j}$ is not in I', since I' is an independent set in H. Without loss of generality, let $c_{k_j,j} \notin I'$. We can replace v_{k_j} with $c_{k_j,j}$ in I'. Doing this for every edge e_j as above, we obtain a set I''. The set I'' is independent in H, has size at least $\ell + 2m$, and $I := I'' \cap V(G)$ is independent in G. Since I'' contains at most $2m$ vertices $c_{i_j,j}, c_{k_j,j}$, at least ℓ vertices are in I. □

In order to prove NP-hardness for GSR, we now give a method to construct a ruler R from a hypergraph H produced by Construction 2 such that the conflict hypergraph H_R of R is isomorphic to H. In order to carry out Construction 3 below we need to construct Golomb rulers with a certain number $\mu + 2$ of marks such that each mark lies between 0 and $O(\mu^3)$. It has previously been noted by Dimitromanolakis [11] and Drakakis [16] that this is possible in polynomial time.

Construction 3 Let H be a hypergraph derived from a graph as in Construction 2 and let $\mu = \max\{n, m\}$. For notational convenience denote $c_{i,0} := v_i$, $1 \leq i \leq n$. Let $g_0 < \ldots < g_{\mu+1}$ be the marks of a Golomb ruler such that $g_0 = 0$ and $g_{\mu+1} \in O(\mu^3)$. Furthermore, define $\phi\colon V(H) \to \mathbb{N}$ by $\phi(c_{i,k}) := 2g_{\mu+1}g_i + g_k$. Construct a ruler R by letting $R := \{\phi(c_{i,k})\colon c_{i,k} \in V(H)\}$.

The intuition behind this construction is to have a "big" ruler whose marks represent the vertices v_i of the original graph. The marks of the big ruler are the $2g_{\mu+1}g_i$ summands in the definition of ϕ. Then, in between each two consecutive marks $m_1 < m_2$ of the big ruler, there is a smaller ruler that represents the incident edges of the vertex corresponding to m_1. The distance between a big mark and a small mark occurs exactly twice in the constructed ruler and, thus, yields a conflict according to an edge. We will show that these conflicts are in fact the only conflicts occurring in the constructed ruler. For an illustration of the construction see Figure 5.

Lemma 11 *Construction 3 is polynomial-time computable and H is isomorphic to H_R.*

Proof As noted above, to carry out Construction 3 we need to construct Golomb rulers that contain $\mu + 2$ marks and are of length at most $O(\mu^3)$. This can be done in polynomial time [11, 16]. Using this, the first part of the lemma follows. In order to prove the second part of the lemma, we show that the function ϕ as defined in Construction 3 is a hypergraph isomorphism between H and H_R. Thus, we prove that for each hyperedge in H, there is a corresponding conflict in H_R and vice-versa.

First consider a hyperedge in H and let this hyperedge contain the vertices $c_{i_1,0}, c_{i_1,j}, c_{i_2,0}$, and $c_{i_2,j}$. Then, the four marks $\phi(c_{i_1,0}), \phi(c_{i_1,j}), \phi(c_{i_2,0})$, and $\phi(c_{i_2,j})$ form a conflict in H_R, because

$$2g_{\mu+1}g_{i_1} - (2g_{\mu+1}g_{i_1} + g_j) = 2g_{\mu+1}g_{i_2} - (2g_{\mu+1}g_{i_2} + g_j).$$

Next, consider a conflict in H_R, that is, there are positive, not necessarily distinct integers $1 \leq i_1, i_2, i_3, i_4 \leq n$ and $0 \leq j_1, j_2, j_3, j_4 \leq m$ such that $\phi(c_{i_1,j_1}) - \phi(c_{i_2,j_2}) = \phi(c_{i_3,j_3}) - \phi(c_{i_4,j_4})$, that is

$$(2g_{\mu+1}g_{i_1} + g_{j_1}) - (2g_{\mu+1}g_{i_2} + g_{j_2}) = (2g_{\mu+1}g_{i_3} + g_{j_3}) - (2g_{\mu+1}g_{i_4} + g_{j_4}). \tag{8}$$

Note that by allowing the indices j_k to assume the value 0 we also catch conflicts that contain marks corresponding to vertices v_{i_k}.

Observe that by taking both sides of Equation (8) modulo $2g_{\mu+1}$, we get $g_{j_1} - g_{j_2} = g_{j_3} - g_{j_4}$ because $g_i < g_{\mu+1}$ for $0 \leq i \leq \mu$. This yields $J := |\{j_1, j_2, j_3, j_4\}| \leq 2$, because by construction $\{g_i : 0 \leq i \leq \mu + 1\}$ is a Golomb ruler and $(J = 3) \vee (J = 4)$ would imply that it contains a conflict. We claim that we can assume that

$$i_1 = i_2 \qquad \text{and} \qquad i_3 = i_4. \tag{9}$$

Provided that this is true, neither $J = 1$ nor $(j_1 = j_2) \wedge (j_3 = j_4)$ holds, since otherwise we have $(c_{i_1,j_1} = c_{i_2,j_2}) \wedge (c_{i_3,j_3} = c_{i_4,j_4})$ implying that Equation (8) does not represent a conflict in H_R. Furthermore, observe that the case $(j_1 = j_4) \wedge (j_2 = j_3)$ reduces to $J = 1$. Thus, it follows that $(j_1 = j_3) \wedge (j_2 = j_4)$. Now assume for the sake of contradiction that $j_1 \neq 0 \neq j_2$. This implies that there are four vertices $c_{i_1,j_1}, c_{i_1,j_2}, c_{i_2,j_1}, c_{i_2,j_2}$ in H. Then, however, by Construction 2, there are two edges between the vertices v_{i_1}, v_{i_2} in the graph that H has been constructed from; this is a contradiction. Thus, without loss of generality, let $j_1 = 0$. Then, each conflict consists of marks of the form

$$\phi(c_{i_1,0}) - \phi(c_{i_1,j_2}) = \phi(c_{i_3,0}) - \phi(c_{i_3,j_2}) \qquad \text{where} \qquad 1 \leq j_2 \leq m,$$

that is, each conflict represents a hyperedge in H.

We now have that ϕ is a hypergraph isomorphism if Condition (9) holds. For this, recall that we have already established $g_{j_1} - g_{j_2} = g_{j_3} - g_{j_4}$ and, hence, also $g_{i_1} - g_{i_2} = g_{i_3} - g_{i_4}$. Since we have chosen the g_i's to be marks of a Golomb ruler, we again get that $|\{i_1, i_2, i_3, i_4\}| \leq 2$. Now, if $i_1 \neq i_2$, then we get $(i_1 = i_3) \wedge (i_2 = i_4)$ by a similar argument as above. Thus, we obtain an equation that is equivalent to Equation (8) such that Condition (9) holds, by simply adding $\phi(c_{i_2,j_2}) - \phi(c_{i_3,j_3})$ to both sides of Equation (8) and renaming the vertices appropriately. □

Lemma 11 implies the following theorem:

Theorem 5 Golomb Subruler *is NP-complete even if all conflicts in the input instance are 4-conflicts and all integers in the input instance are upper-bounded by a polynomial in the input length.*

The Parameterized Complexity of GOLOMB SUBRULER It would be interesting to know whether GOLOMB SUBRULER is fixed-parameter tractable when parameterized with the size of the sought ruler. By our hypergraph characterization this problem is a special case of HYPERGRAPH INDEPENDENT SET parameterized with the size of the sought independent set. HYPERGRAPH INDEPENDENT SET has been proven W[1]-hard by Nicolas and Rivals [25]. However, the forbidden subgraphs we have found for conflict hypergraphs prevent their proof from being applicable to GOLOMB SUBRULER. We still conjecture that the restriction given by these subgraphs is not so crucial as to make GOLOMB SUBRULER fixed-parameter tractable with this parameterization. A hint towards this might be the fact that the NP-hardness proof for GOLOMB SUBRULER we have given in this section also yields a W[1]-hardness result for a related problem. Intuitively, the following problem asks for a Golomb subruler that keeps at least two marks for each conflict.

> Input: A ruler $R \subseteq \mathbb{N}$ and $\ell \in \mathbb{N}$.
> Question: Is there a Golomb ruler $R' \subseteq R$ such that $|R'|$ is at least ℓ plus two times the number of hyperedges in the conflict hypergraph H_R of the ruler R?

Corollary 1 *The above problem is W[1]-hard with respect to parameter ℓ.*

Proof The reduction used for Theorem 5 from INDEPENDENT SET to GOLOMB SUBRULER maps a graph G and size ℓ of the sought independent set to a ruler R and the sought Golomb subruler size $\ell + 2m$, where m is the number of hyperedges in H_R. INDEPENDENT SET parameterized with the size of the sought independent set is W[1]-hard [15]. The reduction identifies this parameter of INDEPENDENT SET and the parameter of the above problem, making it a parameterized reduction. □

6 Conclusion

In this work, we continued to study the algorithmic complexity of GOLOMB SUBRULER (GSR). In particular, we initiated research on its parameterized complexity and studied combinatorial properties of GSR instances. Indeed, GSR can be considered as a puzzling special case of the HITTING SET problem. Some preliminary experimental investigations indicated that our data reduction rules and simple search tree strategies may be beneficial in practical studies for Golomb ruler construction. However, it currently seems most promising to try to combine the data reduction with known approaches such as the Distributed.net project [12].

Golomb ruler construction leads to numerous challenges for algorithmic and complexity-theoretic research. For instance, there is the unsettled computational complexity of constructing shortest Golomb rulers of order n [11, 24]. This has been open for many years. Moreover there are numerous natural variants of Golomb ruler construction [24, 31]. In this paper, we focused on GSR introduced by Meyer and Papakonstantinou [24]. Even restricting attention to GSR, a number of interesting research challenges remain: Which graphs constitute a complete forbidden (induced) subgraph characterization of conflict hypergraphs (see Section 3)? Can a given hypergraph be recognized to be a conflict hypergraph in polynomial time, or is this task NP-hard? Are there other interesting (structural) parameterizations for GSR in the spirit of multivariate algorithmics [17, 21, 27]?

Acknowledgments We thank Falk Hüffner for assistance with Subsection 3.3 and we are grateful to anonymous referees for their valuable feedback leading to significant improvements of the presentation.

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Appendix A Further Forbidden Subgraphs

Proposition 3 (Forbidden subgraph "large hand") *The graph shown in Figure 2c is a forbidden subgraph in a conflict hypergraph.*

Proof In a 4-conflict there are two pairs of vertices that measure the same distance. We choose one unordered pair from $\{a, b, c\}$ and, thus, define the distance that caused the conflict. Then, for the fourth mark, there are only two possible positions left. Multiplying this with the number of possible unordered pairs, one gets six as an upper bound for such edges intersecting in three marks.

In order to prove an upper bound of three, we show that in the previous argument every edge is actually counted twice. Assume $a < b$ has been chosen as pair. Then a fourth mark d can assume only two values, given by

$$d = c - (b - a) \qquad \text{or} \qquad d = c + (b - a).$$

We can rewrite these conditions as

$$d = a + (c - b) \qquad \text{or} \qquad d = b + (c - a).$$

Now observe that the conditions correspond also to the case that the chosen pair is $b < c$ or $a < c$, respectively. That is, every possible location for d is counted twice. This means that there are at most three 4-conflicts that intersect in three marks. □

Proposition 4 (Forbidden subgraph "rotor") *The graph shown in Figure 2d is a forbidden subgraph in a conflict hypergraph.*

Proof First, by definition of the rotor graph, a, c, d are distinct. We fix a total ordering of the three marks in $\{a, c, d\}$ and then try to position b in that ordering. We find that all possible locations lead to equality of two of the marks a, c, d, a contradiction. Because of the symmetry of the graph we can look at one specific ordering without loss of generality. Hence, let $a < c < d$. Now assume that $b < c$. Because of the conflict $\{b, c, d\}$, mark c is half-way between b and d. The conflict $\{a, b, d\}$ implies that either $a = c$ (a contradiction) or $a < b$. But in the latter case, because a, b are in one conflict with c and in one with d, we have $c = d$ which again is a contradiction. The case $b > c$ is symmetric. □

Proposition 5 (Forbidden induced subgraph "scissors") *The graph shown in Figure 2e is a forbidden induced subgraph in a conflict hypergraph.*

Proof We show that, in the configuration shown in Figure 2e, an edge comprising d_1, d_2 and one mark $m \in \{a, b, c\}$ must also be present.

We again use the fact that 4-conflicts are due to two pairs of them having the same distances. Choose two pairs from $\{a, b, c\}$ corresponding to the two conflicts and hence defining a distance each conflict arises from. If the chosen pairs comprise the same marks, then the proposition holds: If a, b is the pair measuring the same distance in both conflicts, then

$$|a - b| = |c - d_1| \qquad \text{and} \qquad |a - b| = |c - d_2|,$$

and, hence, $\{c, d_1, d_2\}$ is a conflict. The cases that a, c, or b, c are chosen in both conflicts are similar. If the two chosen pairs are not equal, then the pairs must share one mark. Without loss of generality, let the pairs be $a < b$ and $b < c$. The equations

$$d = c \pm (b - a) \qquad \text{and} \qquad e = a \pm (c - b)$$

hold for appropriate choices of + or − instead of ±. Note that the sign before $(b - a)$ cannot be negative at the same time with the sign of $(c - b)$ being positive. Otherwise this would imply that $d = e$. In any other case, the two terms on the right-hand side of the equations differ only in the sign of exactly two variables. This means that there exists an $m \in \{a, b, c\}$ such that the following equation holds:

$$|e - m| = |m - d|.$$

Thus there is an additional conflict $\{d, e, m\}$. $\square$