How to Put Through Your Agenda in Collective Binary Decisions*

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Abstract. We consider the following decision scenario: a society of voters has to find an agreement on a set of proposals, and every single proposal is to be accepted or rejected. Each voter supports a certain subset of the proposals–the *favorite ballot* of this voter–and opposes the remaining ones. He accepts a ballot if he supports more than half of the proposals in this ballot. The task is to decide whether there exists a ballot approving a set of selected proposals (agenda) such that all voters (or a strict majority of them) accept this ballot.
On the negative side both problems are NP-complete, and on the positive side they are fixed-parameter tractable with respect to the total number of proposals or with respect to the total number of voters. We look into further natural parameters and study their influence on the computational complexity of both problems, thereby providing both tractability and intractability results. Furthermore, we provide tight combinatorial bounds on the worst-case size of an accepted ballot in terms of the number of voters.

1 Introduction

Consider the following decision scenario which may occur in contexts like coalition formation, the design of party platforms, the change of statutes of an association, or the agreement on contract issues: A leader has an agenda, that is, a set of proposals she wants to get realized. However, a set of proposals has to be approved or disapproved as a whole by a set of voters. Each voter has his favorite proposals he wants to support. A set of proposals is acceptable to a voter if he supports more than half of these proposals. Now, the leader is searching for a set of proposals containing her personal agenda such that a majority of

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voters accepts this set. Can the leader efficiently find such a successful set of proposals realizing her agenda? What about when this set of proposals has to be acceptable to *all* voters and not just to a majority?

Mathematical model. Let $\mathcal{V}$ be a society of n voters and $\mathcal{P}$ be a set of m proposals. Each voter may support any number of proposals in $\mathcal{P}$ and rejects all the others. Subsets of $\mathcal{P}$ are called *ballots*. The favorite ballot $B_i \subseteq \mathcal{P}$ of a voter i ($1 \leq i \leq n$) consists of all proposals he supports.

The voters evaluate a ballot $Q \subseteq \mathcal{P}$ according to the size of the intersection of Q and their favorite ballots. More precisely, voter i *accepts* Q if and only if a strict majority of proposals from Q is also contained in his favorite ballot, that is,

$$|B_i \cap Q| \;>\; |Q|/2.$$

We say that in this case voter i is *happy* with Q.

The central question is whether there exists a ballot Q that (a) contains a *given* agenda and that (b) is acceptable to the society. The agenda in (a) is a set $Q_+ \subseteq \mathcal{P}$ of proposals that have to be contained in Q, that is, $Q_+ \subseteq Q$. The society's acceptance in (b) might be a *unanimous* acceptance or a *majority* acceptance. This leads to the following two problems which only differ in the respective questions asked.

Unanimously Accepted Ballot (UnaAB)
Input: A set $\mathcal{P}$ of m proposals; a society $\mathcal{V}$ of n voters with favorite ballots $B_1, \ldots, B_n \subseteq \mathcal{P}$; an agenda $Q_+ \subseteq \mathcal{P}$.
Question: Is there a ballot $Q_+ \subseteq Q \subseteq \mathcal{P}$ which *every* single voter i accepts (that is, $|B_i \cap Q| > |Q|/2$)?

Majoritywise Accepted Ballot (MajAB)
Input: A set $\mathcal{P}$ of m proposals; a society $\mathcal{V}$ of n voters with favorite ballots $B_1, \ldots, B_n \subseteq \mathcal{P}$; an agenda $Q_+ \subseteq \mathcal{P}$.
Question: Is there a ballot $Q_+ \subseteq Q \subseteq \mathcal{P}$ which a *strict majority* of the voters accepts (that is, $|B_i \cap Q| > |Q|/2$)?

One important special case of UnaAB or MajAB is when the agenda is empty, that is, $Q_+ = \emptyset$. In that case, the only question is whether there is a ballot acceptable to the society.

Interestingly, the following example demonstrates that the solutions sizes to our problems are *not* monotone, that is, a solution ballot of size h does not imply a solution of a size smaller or larger than h. This is in notable contrast to many natural decision problems, such as all problems we reduce from in this paper.

Example 1. Consider the society $\mathcal{V} = \{1, 2, 3, 4\}$ of voters and the set $\mathcal{P} = \{p_1, p_2, p_3, p_4, p_5\}$ of proposals. The favorite ballots are given as $B_1 = \{p_2, p_3, p_4\}$, $B_2 = \{p_1, p_3, p_5\}$, $B_3 = \{p_1, p_2, p_4\}$, and $B_4 = \{p_1, p_2, p_3\}$.

Suppose that the hidden agenda is empty. Then the only unanimously accepted ballots are $\{p_1, p_2, p_3\}$ of size three and $\{p_1, p_2, p_3, p_4, p_5\}$ of size five. This shows that the set of the sizes of all solution ballots may contain gaps.

With regard to majority acceptance, if the hidden agenda is $\{p_5\}$, then ballots $\{p_1, p_2, p_5\}$, $\{p_2, p_3, p_5\}$, and $\{p_1, p_2, p_3, p_4, p_5\}$ are the only ballots that are acceptable to a strict majority of voters. Again, the set of the sizes of all solution ballots contains gaps. If the hidden agenda is empty, then ballot $\{p_1, p_2, p_3, p_4\}$ is also acceptable to a strict majority of voters.

Related work. While the two problems we introduce and study seem to be new, the investigation of situations where a society has to decide upon binary (that is, yes-or-no) issues is common within the theory and practice of decision making. For instance, Laffond and Lainé [26] recently investigated the conditions under which issue-wise majority voting allows for reaching several types of compromise. An alternative to issue-wise evaluation is to compare issue sets (which correspond to ballots in our setting) using the *symmetric difference from a voter's favorite issue set* [8,25,26]. A small symmetric difference is good, and a large symmetric difference is bad. This way of comparing issue sets is very close to the way we study in our paper: A voter accepts a ballot Q if and only if the symmetric difference from his favorite ballot B to Q is smaller than the symmetric difference from B to the empty ballot. Typically, the studies in this context focus on proving desirable properties or on showing how to deal with certain paradoxes. Computational complexity studies are established for related binary decision making problems like judgment aggregation [4,14], lobbying [7,9,15,16], or control of multiple referenda [10]. In the context of judgment aggregation, Alon et al. [2] investigated the computational complexity of control by bundling issues which is also related to "vote on bundled proposals" as considered in this paper.

The scenario considered in our work is also (weakly) related to the concepts of collective domination [13] and proportional representation [28]—in both cases one has to select certain alternatives (proposals in our context) that provide a "good representation" of the voter's will. Herein, extending the Condorcet winner principle to Condorcet winner sets plays a central role. In our work, we also deal with "collectively winning ballots", namely more than half of the proposals in such a ballot are supported by a voter.

Finally, we mention in passing that central computational complexity results of our work are cast within the framework of parameterized complexity analysis, which due to its refined view on algorithmic (in)tractability fits particularly well with voting and related problems [5].

Our contributions. We analyze the combinatorial and algorithmic behavior of Unanimously Accepted Ballot and Majoritywise Accepted Ballot. In particular, we investigate the role of the following natural parameters:

- the number m of proposals,
- the number n of voters,
- the size h of the solution ballot Q, that is, $h = |Q|$,
- the maximum size b_{max} of favorite ballots, that is, $b_{\text{max}} = \max_{i \in \mathcal{V}} |B_i|$, and
- the difference b_{gap} between $\lceil (m+1)/2 \rceil$ and the minimum size of favorite ballots, that is, $b_{\text{gap}} = \lceil (m+1)/2 \rceil - \min_{i \in \mathcal{V}} |B_i|$.

Table 1. Parameterized complexity results on two central problems. An entry "ILP-FPT" means fixed-parameter tractability based on a formulation as an integer linear program. Note that all our "intractability" results also hold for the case of $Q_+ = \emptyset$.

Parameters	UnaAB	MajAB
Number m of proposals	FPT, no polynomial kernel (Thm. 2)	
Number n of voters	ILP-FPT, no polynomial kernel (Thm. 3)	
Parameter h	W[2]-complete (Thm. 4)	W[2]-hard (Thm. 4)
Parameter b_{max}	FPT, no polynomial kernel (Thm. 5)	in W[1] (Thm. 5)
Parameter b_{gap}	NP-complete already for $b_{\text{gap}} = 1$ (Thm. 6)	

The parameter b_{gap} measures how far a given instance is from being trivial in terms of the number of proposals: If each voter's favorite ballot contains at least $\lceil (m+1)/2 \rceil$ proposals, then choosing $Q = \mathcal{P}$ makes every voter happy, so the instance is a trivial yes-instance. While the parameters n and m are naturally related to the "dimensions" of the input, the parameters h, b_{max}, and b_{gap} measure certain degrees of contradiction or inhomogeneity in an instance.

Section 2 is devoted to computational complexity results. The main picture is summarized in Table 1. Not too much of a surprise, Unanimously Accepted Ballot and Majoritywise Accepted Ballot turn out to be NP-complete. More surprisingly, this remains so even when the input ballots are almost trivial, that is, $b_{\text{gap}} = 1$. Namely, if $|B_i| \geq \lceil (m+1)/2 \rceil$ for all voters i, then all voters accept the ballot $\mathcal{P}$. But if every voter i only satisfies the slightly weaker condition $|B_i| \geq \lfloor m/2 \rfloor$, then both problems already become NP-complete. Next, formulating the problems as integer linear programs (ILPs) where the number of variables only depends (exponentially) on n implies fixed-parameter tractability with respect to the parameter n. Using simple brute-force search, one easily obtains that both problems are fixed-parameter tractable with respect to the parameter m. As to efficient and effective preprocessing by polynomial-time data reduction, however, we show that neither for parameter n nor for parameter m polynomial-size problem kernels exist unless an unlikely collapse in complexity theory occurs. As to the parameter h, we prove parameterized intractability—more precisely, W[2]-completeness for Unanimously Accepted Ballot and W[2]-hardness for Majoritywise Accepted Ballot. While the two problems behave in almost the same way with respect to the parameters n, m, and h, the situation may change for the parameter b_{max}: While Unanimously Accepted Ballot is shown fixed-parameter tractable, for Majoritywise Accepted Ballot we only could show containment in W[1] and leave hardness as an open question.

In Section 3, we provide an in-depth combinatorial analysis concerning the dependence of the size of a solution ballot on the parameter n. In particular, we show the upper bound $(n+1)^{(n+1)/2}$ and the lower bound $n^{n/2-o(n)}$ for Unanimously Accepted Ballot with $Q_+ = \emptyset$, thus achieving asymptotically almost matching bounds. Analogous results hold for MAJORITYWISE ACCEPTED BALLOT. In Section 4, we conclude with some open questions for future research.

Due to the lack of space, we only sketch the ideas of the proofs for some of our results.

Parameterized complexity preliminaries. The concept of parameterized complexity was pioneered by Downey and Fellows [12] (see also [18,27] for more recent textbooks). A parameterized problem is a language $L \subseteq \Sigma^* \times \Sigma^*$, where Σ is an alphabet. The second component is called the *parameter* of the problem. Typically, the parameter or the "combined" ones are non-negative integers. A parameterized problem L is *fixed-parameter tractable* if there is an algorithm that decides in $f(k) \cdot |x|^{O(1)}$ time whether $(x, k) \in L$, where f is an arbitrary computable function depending only on k. Correspondingly, FPT denotes the class of all fixed-parameter tractable parameterized problems. A core tool in the development of fixed-parameter algorithms is polynomial-time preprocessing by *data reduction* [6,22]. Here, the goal is to transform a given problem instance (x, k) in polynomial time into an equivalent instance (x', k') with parameter $k' \leq k$ such that the size of (x', k') is upper-bounded by some function g only depending on k. If this is the case, we call instance (x', k') a (problem) *kernel* of size $g(k)$. If g is a polynomial, then we say that this problem has a *polynomial-size problem kernel*, in short, *polynomial kernel*.

Fixed-parameter intractability under some plausible complexity-theoretic assumptions can be shown by means of *parameterized reductions*. A parameterized reduction from a parameterized problem P to another parameterized problem P' is a function that, given an instance (x, k), computes in $f(k) \cdot |x|^{O(1)}$ time an instance (x', k') (with k' only depending on k) such that (x, k) is a yes-instance for P if and only if (x', k') is a yes-instance for P'. The two basic complexity classes for fixed-parameter intractability are W[1] and W[2]. A parameterized problem L is W[1]- or W[2]-hard if there is a parameterized reduction from a W[1]- or W[2]-hard problem to L. For instance, both INDEPENDENT SET and HITTING SET are known to be NP-complete [20]. However, when parameterized by the solution size, INDEPENDENT SET is W[1]-complete while HITTING SET is W[2]-complete [12].

2 Computational Complexity

The following observation is used many times in our proofs.

Observation 1 *Let i and j be two voters that are both happy with some $Q \subseteq \mathcal{P}$.*

(i) Then $B_i \cap B_j \neq \emptyset$.
(ii) If $B_i \cap B_j = \{p\}$, then $p \in Q$ and furthermore $|B_i \cap Q| = |B_j \cap Q|$.

The next observation basically says that UNAAB can be many-one reduced in polynomial time to MAJAB with the same agenda. This implies that the "majority problem" is computationally at least as hard as the "unanimous problem".

Observation 2 *Let I_{una} be a* UNAAB *instance with n voters, and let I_{maj} be a* MAJAB *instance with $2n-1$ voters such that*

- *I_{una} and I_{maj} both have the same proposal set $\mathcal{P}$ and the same agenda Q_+,*
- *the voters from I_{una} and the first n voters from I_{maj} have the same favorite ballots $B_1, \dots, B_n$, and*
- *the remaining $n-1$ voters from I_{maj} support no proposals.*

Then, $Q \subseteq \mathcal{P}$ is a solution for I_{una} if and only if Q is a solution for I_{maj}.

We will use the NP-complete HITTING SET (HS) problem [20] to show many of our intractability results. Given a finite set U, subsets $S_1, \dots, S_r$ of U, and a nonnegative integer k, HS asks whether there is a *hitting set* of size k, that is, whether there is a size-k set $U' \subseteq U$ such that $S_i \cap U' \neq \emptyset$, $i \in \{1, \dots, r\}$. The following reduction from HS to UNAAB is used several times in our intractability proofs. Note that, due to Observation 2, it implies a reduction to MAJAB.

Reduction 1 Let $(U, S_1, \dots, S_r, k)$ be an instance of HS. Construct an instance of UNAAB as follows. The proposal set $\mathcal{P}$ consists of all the elements of U, of k new dummy proposals, and of a special proposal α. There are $r+2$ voters. For $1 \leq i \leq r$, the favorite ballot B_i consists of the elements from S_i together with all dummy proposals. Furthermore, $B_{r+1} = U \cup \{\alpha\}$ and B_{r+2} consists of α together with all dummy proposals. Finally, set $Q_+ = \emptyset$.

Lemma 1 *Reduction 1 is a parameterized reduction where the parameters h, n, and m are linearly bounded in the parameters k, r, and $|U|$, respectively. More precisely, $h = 2k+1$, $n = r+2$, and $m = |U| + k + 1 \leq 2|U| + 1$.*

2.1 NP-completeness

We show that UNAAB and MAJAB are NP-complete even if $Q_+ = \emptyset$. This implies that there is no hope for fixed-parameter tractability parameterized by $|Q_+|$.

Theorem 1. *Both* UNANIMOUSLY ACCEPTED BALLOT *and* MAJORITYWISE ACCEPTED BALLOT *are NP-complete even if $Q_+ = \emptyset$.*

Proof (Sketch). Containment in NP is easy to see; the hardness result is achieved due to Observations 1 and 2 and Lemma 1. □

2.2 Few proposals or few voters

Complementing our intractability result from Theorem 1, we show that instances with few proposals or few voters are tractable. More precisely, we show that the considered problems are polynomial-time solvable for a fixed number of proposals or a fixed number of voters and the degree of the polynomial is a constant. However, we also show that under plausible complexity-theoretic assumptions these problems do not admit polynomial-time preprocessing algorithms that reduce the size of an instance to be polynomially bounded by the the number m of proposals or the number n of voters. In other words, UnaAB and MajAB are unlikely to allow for polynomial kernels with respect to the parameters n or m, respectively.

Theorem 2. *Parameterized by the number m of proposals,* Unanimously Accepted Ballot *and* Majoritywise Accepted Ballot *are fixed-parameter tractable. Unless* NP $\subseteq$ coNP/poly*, both problems do not admit a polynomial kernel even if $Q_+ = \emptyset$.*

Proof (Sketch). For the fixed-parameter tractability result, one guesses a ballot Q with $Q_+ \subseteq Q \subseteq \mathcal{P}$ and checks whether this is a solution for UnaAB (resp. MajAB). This takes $O(2^m \cdot n^c)$ time with c being a constant. As for the non-existence of a polynomial kernel for UnaAB, this is due to the non-existence of a polynomial kernel of HS parameterized by $|U|+k+1$ [11] and due to Lemma 1. Together with Observation 2, the non-existence of polynomial kernels transfers to MajAB even if $Q_+ = \emptyset$. ☐

Theorem 3. *Parameterized by the number n of voters,* Unanimously Accepted Ballot *and* Majoritywise Accepted Ballot *are fixed-parameter tractable. Unless* NP $\subseteq$ coNP/poly*, both problems do not admit a polynomial kernel even if $Q_+ = \emptyset$.*

Proof. We first describe how to formulate MajAB as an integer linear program (ILP) and show how to modify the ILP to also work for UnaAB. Let N_V be the number of proposals that are accepted by the voter set V, that is, $N_V := |\{j \mid (\forall i \in V : j \in B_i) \wedge (\forall i' \notin V : j \notin B_{i'})\}|$. As the proposals counted by N_V only depend on V, we refer to V as a *proposal type*. Let x_V be the number of proposals of type V in the ballot Q. Further, let N_V^+ be the number of proposals in Q_+ that are accepted by the voter set $V \subseteq \mathcal{V}$, that is, $N_V^+ := |Q_+ \cap \{j \mid (\forall i \in V : j \in B_i) \wedge (\forall i' \notin V : j \notin B_{i'})\}|$. For each voter i we introduce a binary variable z_i that may only have value 1 if voter i is happy with Q (and may have value 0 in any case). Then Q must satisfy the following constraints (1)–(3).

$$\sum_{i=1}^{n} z_i \geq \frac{n+1}{2} \tag{1}$$

$$\sum_{\substack{V\subseteq\mathcal{V}:\\ i\in V}} x_V - \sum_{\substack{V\subseteq\mathcal{V}:\\ i\notin V}} x_V \geq m(z_i - 1) + 1 \qquad \forall\, i \in \{1,\dots,n\} \tag{2}$$

$$N_V \geq x_V \geq N_V^+ \qquad \forall\, V \subseteq \mathcal{V} \tag{3}$$

Constraint (1) requires that a strict majority of voters is happy with Q. Constraint set (2) ensures that voter i is happy if variable z_i is set to 1. Constraint set (3) requires ballot Q to contain all proposals in Q_+ and restricts the number of proposals of each type in Q to those actually present.

Our ILP contains at most 2^n variables x_V and n variables z_i. The total number of constraints is at most $2^n + n + 1$. Since an ILP with ρ variables and L input bits can be solved in $O(\rho^{2.5\rho+o(\rho)}L)$ time [24,19], MajAB is fixed-parameter tractable with respect to the number n of voters.

If we delete constraint (1) and the variables z_i, and replace the right-hand side of constraint (2) with 1, then we gain an ILP for UnaAB with at most 2^n variables and $2^n + n$ constraints. Thus, UnaAB is also fixed-parameter tractable with respect to parameter n.

Unless NP $\subseteq$ coNP/poly, even if $Q_+ = \emptyset$, both problems do not have a polynomial kernel with respect to the parameter n: Reduction 1 is a polynomial-time reduction from the NP-complete problem Hitting Set; the number n of voters in the reduced instance is linearly bounded by the number r of sets in the instance one reduces from; and $Q_+ = \emptyset$. A polynomial kernel of UnaAB with $Q_+ = \emptyset$ parameterized by n would yield a polynomial kernel for Hitting Set parameterized by r. However, this is not possible unless NP $\subseteq$ coNP/poly (e.g. [23, Lemma 14]). Thus, even if $Q_+ = \emptyset$, UnaAB does not admit a polynomial kernel. Neither does MajAB admit a polynomial kernel even if $Q_+ = \emptyset$ due to Observation 2. □

2.3 Small Ballots

We perform a parameterized complexity analysis concerning parameters based on the ballot sizes. We start with the size h of the solution ballot. For technical reasons, we need to assume that h is given as part of the input when dealing with the parameterized problems.

Theorem 4. *Parameterized by the size h of the solution ballot,* Unanimously Accepted Ballot *is W[2]-complete and* Majoritywise Accepted Ballot *is W[2]-hard. Both results hold even if $Q_+ = \emptyset$.*

Proof (Sketch). Reduction 1 is a parameterized reduction from the W[2]-hard Hitting Set parameterized by the size k of the hitting set to UnaAB parameterized by the size h of the solution ballot with $Q_+ = \emptyset$ (see Lemma 1). Because of Observation 2, this implies W[2]-hardness for MajAB parameterized by h even if $Q_+ = \emptyset$. To show that UnaAB is in W[2], we reduce from UnaAB parameterized by h to the W[2]-complete Independent Dominating Set parameterized by the solution size k [12]. □

The membership of MajAB parameterized by the size h of the solution ballot for the class W[2] remains open. Note that the W[2]-hardness reduction in the proof of Theorem 4 does not rely on (an upper bound for) h being given as part of the input. That is, the problem is computationally hard also for the cases where the size of ballot Q is not explicitly required to be bounded by h.

Except for the parameter h where we only know that MajAB is W[2]-hard while UnaAB is even W[2]-complete, all results shown so far are the same for unanimous acceptance and majority acceptance. The following theorem shows that this may change when considering the parameter $b_{\max}$ where UnaAB remains tractable but for MajAB we only know W[1]-membership.

Theorem 5. *Parameterized by the maximum size $b_{\max}$ of the favorite ballots,* Unanimously Accepted Ballot *can be solved in $O(b_{\max}^{2b_{\max}} \cdot nm)$ time implying fixed-parameter tractability; however, it admits no polynomial kernel unless NP $\subseteq$ coNP/poly even if $Q_+ = \emptyset$.* Majoritywise Accepted Ballot *parameterized by $b_{\max}$ is in W[1].*

Proof (Sketch). To show that UnaAB is solvable in $O(b_{\max}^{2b_{\max}} \cdot nm)$ time, we first observe that any solution Q must satisfy $|Q| \leq 2b_{\max}$. Based on this, we can design a depth-bounded search tree algorithm solving UnaAB where the number of branching possibilities in each step is at most $b_{\max}$ and the depth of the algorithm is at most $2b_{\max}$.

The non-existence of a polynomial kernel for UnaAB with respect to parameter m shown in Theorem 2 also holds for parameter $b_{\max}$, as $b_{\max} \leq m$.

Finally, to show the W[1] containment, we use a theorem from [18, Theorem 6.22.] which states that a parameterized problem L with parameter k is in W[1] if and only if there is a *tail-nondeterministic k-restricted nondeterministic random access machine (NRAM)* program deciding L. The description of a tail-nondeterministic $b_{\max}$-restricted NRAM program $\mathbb{P}$ for MajAB is omitted due to lack of space. □

Next, we discuss the relation between the parameters "maximum size $b_{\max}$ of the favorite ballots" and "the size $h_{\max}$ of the maximum symmetric difference between any two favorite ballots". As the following proposition shows, for the cases with $Q_+ = \emptyset$, the two parameters $h_{\max}$ and $b_{\max}$ are "equivalent" in terms of parameterized complexity theory: The fact that a parameter x is linearly bounded by a parameter y implies that the parameterization by x and the parameterization by y are in the same level of the W-hierarchy and yield the same parameterized hardness results.

Proposition 1. *For any instance of* Unanimously Accepted Ballot *or* Majoritywise Accepted Ballot *it holds that $h_{\max} \leq 2b_{\max}$, where $h_{\max}$ denotes the size of the maximum symmetric difference between two favorite ballots and $b_{\max}$ denotes the maximum size of the given favorite ballots. Instances of* Unanimously Accepted Ballot *or* Majoritywise Accepted Ballot *are yes-instances if $h_{\max} < b_{\max}/2$ and $Q_+ = \emptyset$.*

We conclude this section with the following theorem which uses the fact that an instance of UnaAB or MajAB is a trivial yes-instance if the minimum size of the favorite ballots is at least $\lceil (m+1)/2 \rceil$ where m denotes the total number of proposals in $\mathcal{P}$. However, both problems become NP-complete when this minimum size is one less than the guarantee $\lceil (m+1)/2 \rceil$, even if $Q_+ = \emptyset$. This implies that there is no hope for fixed-parameter tractability with respect to the "below guarantee parameter" b_{gap} which is the difference between $\lceil (m+1)/2 \rceil$ and the minimum size of the favorite ballots.

Theorem 6. *An instance of* Unanimously Accepted Ballot *(resp.* Majoritywise Accepted Ballot*) is a yes-instance if each voter i satisfies $|B_i| >$*

$m/2$. Unanimously Accepted Ballot *(resp.* Majoritywise Accepted Ballot*) is* NP*-complete even if* $Q_+ = \emptyset$ *and each voter* i *satisfies* $|B_i| > m/2 - 1$.

Proof. As for the first statement, choosing $Q = \mathcal{P}$ makes every voter happy. To show the second statement, we many-one reduce from the NP-complete Vertex Cover (VC) problem. Given an undirected graph $G = (U, E)$ and an integer $k \leq |U|$, VC asks whether there is a *vertex cover* of at most k vertices, that is, whether there is a set $U' \subseteq U$ with $|U'| \leq k$ and $e \cap U' \neq \emptyset, \forall e \in E$.

Let $I = ((U, E), k)$ with vertex set $U = \{u_1, \ldots, u_r\}$ and edge set $E = \{e_1, \ldots, e_s\}$ be a VC instance. We first reduce from it to an instance I' for UnaAB and then extend this reduced instance I' to an instance I'' for MajAB.

Both instances I' and I'' have the same proposal set $\mathcal{P}$. It consists of one special proposal α, of all vertices in U, of k dummy proposals β_j ($1 \leq j \leq k$), and of $r-k$ additional dummy proposals $\gamma_{j'}$ ($1 \leq j' \leq r-k$). Thus, $|\mathcal{P}| = 2r+1$.

Instance I' contains four types of voters: one voter v_0, one voter $\overline{v}_0$, s *edge voter*s, and $r-k$ *vertex hater*s. Voter v_0 favors proposal α and all the r dummy proposals. Voter $\overline{v}_0$ also favors proposal α, and all the vertices in U. For $1 \leq i \leq s$, the ith edge voter's favorite ballot A_i consists of the two vertices in e_i, of all the k dummy proposals β_j, and of $r-k-2$ arbitrarily chosen dummy proposals from $\{\gamma_1, \ldots, \gamma_{r-k}\}$. For $1 \leq i' \leq r-k$, the favorite ballot $B_{i'}$ of vertex hater i' consists of α and of all dummy proposals but $\gamma_{i'}$. In total, the number of voters in I' is $s+r-k+2$, with each voter supporting at least $r = \lfloor |\mathcal{P}|/2 \rfloor$ proposals. Set $Q_+ = \emptyset$. Obviously, this reduction runs in polynomial time.

To show the reduction's correctness, we have to show that I has a vertex cover of size at most k if and only if there is a ballot $Q \subseteq \mathcal{P}$ that all the voters in I are happy with.

For the "only if" part, suppose that $U' \subseteq U$ with $|U'| \leq k$ is a vertex cover. We show that every voter is happy with $Q = \{\alpha\} \cup \{\beta_j \mid 1 \leq j \leq |U'|\} \cup U'$. First, the size of Q is $2|U'|+1$. To make a voter happy, at least $|U'|+1$ of his favorite proposals must be also in Q. Obviously, voters v_0, $\overline{v}_0$ and all vertex haters are happy with Q. For each $i \in \{1, \ldots, s\}$, $Q \cap A_i$ contains all dummy proposals β_j with $1 \leq j \leq |U'|$ and at least one vertex proposal $v_{j'}$ with $v_{j'} \in e_i \cap U'$ since U' is a vertex cover. This sums up to at least $|U'|+1$ proposals. Hence, every edge voter is also happy with Q.

For the "if" part, by applying Observation 1(ii) to the ballots of voters v_0 and $\overline{v}_0$, ballot Q must contain α, and furthermore, Q contains an equal number x of vertex proposals and dummy proposals. For each $i' \in \{1, \ldots, r-k\}$, ballot Q cannot contain dummy proposal $\gamma_{i'}$ since otherwise $|B_{i'} \cap Q| = x < \lfloor |Q|/2 \rfloor + 1$. Thus, vertex hater i' would not be happy. Therefore, the x dummy proposals must come from $\{\beta_1, \ldots, \beta_k\}$ and $x \leq k$. To make the ith edge voter happy, ballot Q must satisfy the condition $|Q \cap A_i| \geq x+1$. But since no edge voter favors proposal α, ballot Q must contain at least one proposal $u_j \in A_i$. By definition of A_i, the corresponding vertex u_j is incident to edge e_i. This implies that the x vertices in Q form a vertex cover for (U, E).

Next, we extend instance I' to instance I'' for MAJAB by adding $r-k$ *vertex lover*s who have the same favorite ballot U, and s *edge-inverse voter*s such that for $1 \le i \le s$, edge-inverse voter i's favorite ballot $C_i = (U \cup \{\gamma_1, \ldots, \gamma_{r-k}\}) \setminus A_i$. Thus, C_i and A_i are disjoint. In total, I'' has $2(s+r-k)+2$ voters. Since each of the newly added voters favors exactly r proposals, the constraint that each voter's proposal set has at least $r = \lfloor |\mathcal{P}|/2 \rfloor$ holds. This extension also runs in polynomial time.

Now we show the correctness of the extended reduction, that is, I has a vertex cover of size at most k if and only if there is a ballot $Q \subseteq \mathcal{P}$ which more than half of the voters in I'' are happy with.

For the "only if" part, the ballot Q as constructed in the "only if" part above makes all voters in I' happy. This sums up to $s+r-k+2$. Since I'' contains all the voters from I' and has $2(s+r-k)+2$ voters, this also means that more than half of the voters in I'' is happy with Q.

For the "if" part, for $1 \le i \le s$, the ith edge voter and the ith edge-inverse voter do not share a common favorite proposal. Furthermore, no vertex hater's favorite ballot intersects any vertex lover's favorite ballot. Hence, by applying Observation 1(i), any ballot can make at most s voters from the edge voters and the edge-inverse voters happy, and can make at most $r-k$ voters from the vertex haters and the newly constructed vertex lovers happy. But I'' has $2(s+r-k)+2$ voters. This means that in order to be a solution ballot for I'', Q must make both v_0 and $\overline{v}_0$ happy. By applying Observation 1(ii), Q must then contain α, and, furthermore, Q contains the same number x of vertex proposals and dummy proposals. The ballot Q cannot make any vertex lover happy since his favorite ballot and Q have an intersection of size x which is smaller than $\lfloor |Q|/2 \rfloor + 1$. Thus, Q needs to make all vertex haters happy. Then, Q cannot contain any dummy proposal $\gamma_{i'}$ since otherwise the vertex hater i' is not happy due to $|B_{i'} \cap Q| = x < \lfloor |Q|/2 \rfloor + 1$. Hence, Q contains x dummy proposals from $\{\beta_1, \ldots, \beta_k\}$ with $x \le k$. Then, no edge-inverse voter is happy with Q since at most x proposals from his favorite ballot are in Q. This means that all edge voters must be happy with Q. To make the ith edge voter happy, Q must intersect with A_i in at least one vertex $u_j \in A_i$. By definition of A_i, the corresponding vertex u_j is incident to edge e_i. Thus, the x vertices in Q form a vertex cover for (U, E). $\square$

3 Combinatorial Bounds on Minimal Accepted Ballots

We say that a unanimously (resp. majoritywise) accepted ballot is *minimal* if no proper subset of it is also unanimously (resp. majoritywise) accepted. In this section, we investigate the largest possible size of a minimal unanimously accepted ballot for the situation with n voters and $Q_+ = \emptyset$. We derive (almost tight) upper and lower bounds on this quantity. From this bound, a similar result can be derived for majoritywise accepted ballots.

It is not hard to see that both upper and lower bounds come down to studying the case where the set $\mathcal{P}$ of all proposals already is a minimal accepted ballot: Such instances cannot have smaller solutions (giving a lower bound), and upper bounds directly carry over to $Q \subseteq \mathcal{P}$ by considering a restricted instance with $\mathcal{P}' := Q$. To make the question more amenable to combinatorial tools we translate it into a problem on a sequence of vectors with $\{-1, 1\}$-entries: Given n voters and m proposals we create m vectors $x_1, \ldots, x_m \in \{-1, 1\}^n$; the ith entry in vector x_j is 1 if the jth proposal is contained in the favorite ballot of voter i, else it is -1. In this formulation, a unanimously accepted ballot Q corresponds to a subset of the vectors whose vector sum is positive in each coordinate: Considering some voter i, for each proposal in $B_i \cap Q$ we incur 1, for each proposal in $Q \setminus B_i$ we incur -1. If $|B_i \cap Q| > |Q|/2$ then this gives a positive sum in coordinate i; the converse is true as well.

Let us normalize the question a little more. First of all, no minimal ballot can be of even size: Otherwise all coordinate sums would be even and hence each sum is at least 2; then however we may discard an arbitrary vector and still retain sums of at least 1 each. Secondly, it is clear that replacing $+1$ entries by -1 entries does not introduce additional subsequences with positive coordinate sums. Thus, we may restrict ourselves to the case where the coordinate sums over the minimal sequence of m vectors are all equal to 1 (all sums are odd and such a replacement lowers a sum by exactly 2).

Now, a collection of vectors is called a *minimal majority sequence of dimension n* (an n-mms for short) if all its coordinate-wise sums are 1 and no proper subsequence of the vectors has a positive sum in each coordinate. Note that an n-mms cannot contain a nonempty subsequence S whose sum is at most 0 in each coordinate, since otherwise the sum of the vectors that are in this n-mms but not in S must be positive in each coordinate—a contradiction to the minimality of an n-mms. Thus, the definition of an n-mms is equivalent to that all its coordinate-wise sums are 1 and no nonempty subsequence has sum of at most 0 in each coordinate. The *length* of the sequence is the number m of its elements. Let $f(n)$ denote the maximum possible length of an n-mms. In this section, we show that $f(n) \approx n^{n/2+o(n)}$.

Theorem 7. *The maximum possible length $f(n)$ of a minimal majority sequence of dimension n satisfies*

$$n^{n/2-o(n)} \leq f(n) \leq (n+1)^{(n+1)/2}.$$

Proof (Sketch). One way to obtain an upper bound on $f(n)$ is to apply a known result of Sevastyanov [29]. It asserts that any sequence of vectors whose sum is the zero vector, where the vectors lie in an arbitrary n-dimensional normed space R and each of them has norm at most 1, can be permuted so that all initial sums of the permuted sequence are of norm at most n. Given an n-mms $v_1, \ldots, v_m \in \{-1, 1\}^n$, append to it the vector $-\mathbf{1}$ where $\mathbf{1}$ is the all-1-vector of length n to get a zero-sum sequence of $m+1$ vectors in R^n, where the ℓ_∞ norm of each vector is 1. By the above mentioned result there is a permutation $u_1, u_2, \ldots, u_{m+1}$ of these vectors so that the ℓ_∞-norm of each initial sum $\sum_{i=1}^{j} u_i$ is at most n. If $m+1 > (2n+1)^n$ then, by the pigeonhole principle, some two distinct initial sums are equal, and their difference gives a proper subsequence of the original mms with sum either the zero vector (if this difference does not include the vector $-\mathbf{1}$), or $\mathbf{1}$ (if it does). In both cases, this contradicts the assumption that the original sequence is an mms. This shows that $f(n) \leq (2n+1)^n$. See [1] for a similar argument.

The proof of the stronger upper bound stated in Theorem 7 is similar to that of a result of Huckeman, Jurkat, and Shapley (cf. [21]) and is based on some simple facts from convex geometry. The details and the proof for the lower bound are omitted. □

The proof combines the main result of Alon and Vu [3] with arguments from Linear Algebra, Geometry, and Discrepancy Theory. Instead of turning to the proof, let us give a corollary for the effect on our two central problems.

Corollary 1. *Consider a* Unanimously Accepted Ballot *instance with n voters. If there exists a unanimously accepted ballot, then there also exists one of size at most $(n+1)^{(n+1)/2}$. This bound is essentially tight, as there exist choices of accepted ballots such that any unanimously accepted ballot has size at least $n^{n/2-o(n)}$. For* Majoritywise Accepted Ballot*, the corresponding upper and lower bounds are respectively $(t+1)^{(t+1)/2}$ and $t^{t/2-o(t)}$, where $t = \lceil (n+1)/2 \rceil$ denotes the majority threshold.*

Proof. As the correspondence between favorite ballots and vector sequences has been thoroughly discussed above for the unanimous case, we now concentrate on the majority case.

To see the lower bound for the majority case, we start from a lower bound example for the unanimous case with t old voters and a minimum accepted ballot size of $t^{t/2-o(t)}$, and we add $n-t < n/2$ new voters with empty favorite ballots to it. Note that the resulting instance has a total of n voters and that its majority threshold indeed is t. Then any majoritywise accepted ballot must be unanimously accepted by the t old voters, so that the minimum majoritywise accepted ballot has size at least $t^{t/2-o(t)}$.

For the upper bound, consider any majoritywise accepted ballot Q for n voters and consider any minimal majority of t voters that (amongst themselves) unanimously accept this ballot. Then any other unanimously accepted ballot for these voters is also majoritywise accepted by all n voters, so that we get the desired upper bound of $(t+1)^{(t+1)/2}$ on the size of Q. □

4 Open Questions and Conclusion

We have introduced new and naturally motivated problems in computational social choice, and we studied their computational complexity and started an analysis of their combinatorial properties. We conclude this paper with a few challenges for future research.

First, recall that in Proposition 1 we stated upper bounds on $h_{\max}$ (the size of the maximum symmetric difference between two favorite ballots) in terms of linear functions in $b_{\max}$ (the maximum ballot size of voters). Hence, parameterized hardness results with respect to $b_{\max}$ transfer to the parameterization by $h_{\max}$. In the case of empty agenda, that is, $Q_+ = \emptyset$, however, we have no good lower bounds for $h_{\max}$ in terms of $b_{\max}$. Thus, it remains to classify the parameterized computational complexity of both Unanimously Accepted Ballot and Majoritywise Accepted Ballot using parameter $h_{\max}$. Notably, in the cases of $Q_+ = \emptyset$ the parameters $h_{\max}$ and $b_{\max}$ are linearly related so that the same parameterized complexity results will hold for both parameterizations.

Second, with respect to parameter h (the size of the solution ballot Q), we established W[2]-hardness for Majoritywise Accepted Ballot even if $Q_+ = \emptyset$, but we left open the precise location of this problem in the parameterized complexity hierarchy. It might be W[2]-complete, but all we currently know is that it is contained in W[2] (Maj), a class presumably larger than W[2] [17].

Third, the combinatorial bounds from Section 3 do not hold for instances with nonempty agenda, since such bounds cannot be independent of $|Q_+|$. For cases with nonempty agenda there are similar bounds with an extra factor of $|Q_+|$. A detailed analysis could be part of investigations of weighted variants of our problems. In this regard, weights on the voters, weights on the proposals, or weights on the acceptance threshold of the voters seem to be well-motivated.

Fourth, can we avoid Integer Linear Programs for showing fixed-parameter tractability with respect to the parameter number n of votes and provide direct combinatorial algorithms beating the ILP-based running times? In this context, the exponential lower bound on the number of proposals in ballots accepted by society from Section 3 might be relevant.

Finally, it remains a puzzling open question whether Majoritywise Accepted Ballot parameterized by $b_{\max}$ is fixed-parameter tractable—we could only show containment in W[1].

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