On Bounded-Degree Vertex Deletion Parameterized by Treewidth☆

Nadja Betzler[a,1], Robert Bredereck[a,1], Rolf Niedermeier[a], Johannes Uhlmann[a,2]

[a] *Institut für Softwaretechnik und Theoretische Informatik, TU Berlin, Germany*
{robert.bredereck, rolf.niedermeier}@tu-berlin.de
{nadja.betzler, johannes.uhlmann}@campus.tu-berlin.de

Abstract

Given an undirected graph G and an integer $d \geq 0$, the NP-hard Bounded-Degree Vertex Deletion problem asks to delete as few vertices as possible from G such that the resulting graph has maximum vertex degree d. Our main result is to prove that Bounded-Degree Vertex Deletion is W[1]-hard with respect to the parameter treewidth. As a side result, we obtain that the NP-hard Vector Dominating Set problem is W[1]-hard with respect to the parameter treewidth. On the positive side, we show that Bounded-Degree Vertex Deletion becomes fixed-parameter tractable when parameterized by the combined parameter treewidth and number of vertices to delete, and when parametrized by the feedback edge set number.

Keywords: parameterized complexity, structural parameterization, tree-likeness, Vector Dominating Set, k-dependent set, co-k-plexes

1. Introduction

This work is mainly concerned with the following graph modification problem, here stated in its decision version.

> Bounded-Degree Vertex Deletion (BDD)
> *Given:* An undirected graph $G = (V, E)$, and integers $d \geq 0$ and $k \geq 0$.
> *Question:* Is there a subset $V' \subseteq V$ with $|V'| \leq k$ whose removal from G yields a graph in which each vertex has degree at most d?

BDD finds applications in computational biology [16] and its "dual problem" to find maximum s-plexes has applications in social network analysis [28, 1, 22].

☆This research was started while all authors were with Friedrich-Schiller-Universität Jena.
[1]Supported by the DFG, research project PAWS, NI 369/10.
[2]Supported by the DFG, research project PABI, NI 369/7.

There is a substantial amount of theoretical work related to its algorithmic complexity [2, 10, 8, 12, 21, 25, 28]. A famous special case of BDD is Vertex Cover, where $d = 0$. Whereas we look at the problem from the viewpoint of minimizing the number of deleted vertices, Dessmark et al. [12] rather studied restrictions to special graph classes of the dual problem to maximize the number of vertices remaining in the graph and referred to these vertices as *k-dependent sets*. Balasundaram et al. [2] referred to this degree-bounded generalization of independent sets as *co-k-plexes* and developed constant-factor approximation algorithms for the problem. Finally, related problems for directed and undirected graphs, which model problems in voting theory and social network analysis. have been studien in companion work [4, 3]. In this paper, we study how the treewidth of the underlying graph influences the parameterized computational complexity of BDD.

Let n denote the number of vertices in the input graph. Dessmark et al. [12, Theorem 5.1] stated (using different terminology) that BDD can be solved in $O(n^{\text{tw}+1})$ time, where tw denotes the treewidth of the underlying graph. There are several fixed-parameter tractability results for BDD for *constant* d and parameter solution size k [10, 16, 22, 25]. For unbounded values of d, BDD becomes W[2]-hard for parameter k [16], excluding hope for fixed-parameter tractability, that is, for an algorithm solving BDD in $f(k) \cdot n^{O(1)}$ time for a computable function f only depending on k [14, 19, 23]. On the contrary, other results [25, 16, 22] show that BDD is fixed-parameter tractable with respect to the combined parameter (k, d). We remark that, using Courcelle's theorem [11], it is not hard to see that BDD is fixed-parameter tractable for the combined parameter (tw, d).[3]

Our central contribution is to show that BDD is W[1]-hard when parameterized by the treewidth tw, thus destroying hope for fixed-parameter tractability with respect to the parameter tw. While BDD is hard for the single parameters k and tw, we show that BDD becomes fixed-parameter tractable when parameterized by the combined parameter (tw, k) and when parameterized by the "feedback edge set number", that is, the number of edges to delete from a graph in order to make it a forest.

Related domination problems. Our results rely on relations between BDD and two NP-hard variants of the classical Dominating Set problem, namely, Vector Dominating Set and Capacitated Dominating Set. A *dominating set* of an undirected graph $G = (V, E)$ is a subset $V' \subseteq V$ such that every vertex v from V is in V' or has a neighbor w in V', that is, w *dominates* v. In the considered variants, one additionally has as input a nonnegative integer x_i for every $v_i \in V$. For a vector dominating set V' it is required that every $v_i \notin V'$ needs to have at least x_i neighbors in V'. In contrast, in a capacitated dominating set, every vertex $v_i \in V'$ can dominate at most x_i of its neighbors. While

[3] Indeed, this works in analogy to proving the fixed-parameter tractability of the closely related Minimum Degree Deletion problem for the parameter treewidth [9].

the corresponding Capacitated Dominating Set problem has been shown to be W[1]-hard with respect to the parameter treewidth [13], to our knowledge the parameterized complexity of Vector Dominating Set for the parameter treewidth has been open so far.

Our main result, the W[1]-hardness of BDD with respect to treewidth, follows by a parameterized reduction from Capacitated Dominating Set. Moreover, the positive result that BDD is fixed-parameter tractable with respect to the combined parameter (tw, k) relies on a simple reduction from BDD to Vector Dominating Set, for which fixed-parameter tractability for a corresponding combined parameter has been shown by Raman et al. [26]. Combining this parameterized reduction with the W[1]-hardness result for BDD implies the W[1]-hardness for Vector Dominating Set with respect to treewidth.

Preliminaries. We work with undirected and simple graphs. For a graph $G = (V, E)$ and a vertex set X we write $G - X$ as an abbreviation for the induced subgraph $G[V \setminus X]$. Unless stated otherwise, let $n := |V|$ and $m := |E|$.

A famous parameter measuring the tree-likeness of an undirected graph is the *treewidth* [5, 7]. Many NP-complete graph problems become easy when the input instance is a tree. The notion of *treewidth*, introduced by Robertson and Seymour [27], tries to capture the "tree-likeness" of a graph in the sense that "tree-like" graphs have small treewidth. Many in general NP-hard graph problems can then be solved in polynomial or even linear time when the underlying graph has a treewidth bounded by a constant [7, 19, 23].

A *tree decomposition* of a graph $G = (V, E)$ is a pair $\langle \{X_i \mid i \in I\}, T\rangle$, where each X_i is a subset of V, called *bag*, and $T = (I, F)$ is a tree with node set I and edge set F. The following must hold:

1. $\bigcup_{i \in I} X_i = V$;
2. for every edge $\{u, v\} \in E$, there is an $i \in I$ such that $\{u, v\} \subseteq X_i$;
3. for all $i, j, l \in I$, if j lies on the path between i and l in T, then $X_i \cap X_l \subseteq X_j$.

The *width* of $\langle \{X_i \mid i \in I\}, T\rangle$ is $\max\{|X_i| \mid i \in I\} - 1$. The *treewidth* of G is the minimum width over all tree decompositions of G. Trees have treewidth one.

Parameterized complexity is a two-dimensional framework for studying the computational complexity of problems [14, 19, 23]. One dimension is the input size n (as in classical complexity theory), and the other one is the *parameter* k (usually a positive integer). A problem is called *fixed-parameter tractable* if it can be solved in $f(k) \cdot n^{O(1)}$ time, where f is a computable function only depending on k. Notably, a problem can usually be parameterized in several natural ways, particularly leading to a multivariate complexity analysis where combined parameters are studied [15, 24].

Downey and Fellows [14] developed a formal framework for showing *fixed-parameter intractability* by means of *parameterized reductions*. A parameterized reduction from a parameterized problem P to another parameterized problem P' is a function that, given an instance (x, k), computes in $f(k) \cdot n^{O(1)}$ time an instance (x', k') (with k' only depending on k) such that (x, k) is a yes-instance of problem P if and only if (x', k') is a yes-instance of problem P'. The basic complexity class for fixed-parameter intractability is called $W[1]$, followed by the next level $W[2]$. There is good reason to believe that $W[1]$-hard problems are not fixed-parameter tractable [14, 19, 23]. In this sense, $W[1]$-hardness is the parameterized complexity analog of NP-hardness.

2. Parameter treewidth

The main result of this section is to show that BDD is W[1]-hard when parameterized by treewidth (Section 2.1). When treewidth together with solution size forms a combined parameter, then BDD becomes fixed-parameter tractable (Section 2.2).

2.1. Single parameter treewidth

We show W[1]-hardness of Bounded-Degree Vertex Deletion with respect to the parameter treewidth. To this end, we present a parameterized reduction from Capacitated Dominating Set, which is defined next.

Let $G_u = (V, E)$ be an undirected graph and let $\text{cap} : V \to \mathbb{N}$ be a *capacity function* such that $0 \le \text{cap}(v) \le \deg(v)$, where $\deg(v)$ is the degree of vertex $v \in V$. We call $G = (V, E, \text{cap})$ *capacitated graph*. For $V' \subseteq V$, a *subset map* $M \subseteq (V \setminus V') \times V'$ maps a vertex $x \in V \setminus V'$ to a vertex $y \in V'$ if $(x, y) \in M$. We denote by $\text{sat}(M, v) := |\{(x, v) \in M\}|$ the *saturation* of a vertex v under the subset map M, that is, the number of vertices mapped by M to v. We call $S \subseteq V$ a *capacitated dominating set* if there exists a subset map M for S mapping every vertex from $(V \setminus S)$ to one of its neighbors from S such that $\text{sat}(M, s) \le \text{cap}(s)$ for all $s \in S$. Herein, M is called *feasible subset map* for S.

> Capacitated Dominating Set
> *Given:* A capacitated graph $G = (V, E, \text{cap})$ and a positive integer k.
> *Question:* Is there a capacitated dominating set for G containing at most k vertices?

Capacitated Dominating Set is W[1]-hard with respect to the combined parameter (tw, k) [13]. In the following, we describe a parameterized reduction to Bounded Degree Deletion such that the treewidth of the BDD graph only depends on the treewidth of the Capacitated Dominating Set graph.

Let (G, k) be an instance of Capacitated Dominating Set with a capacitated graph $G = (V, E, \text{cap})$. Let $c^\star$ be the sum over all capacities. Let $V := \{v_1, \ldots, v_n\}$, that is, $|V| = n$. We construct an undirected graph G' that can be transformed into a graph with maximum degree n by deleting at most $n + c^\star$ vertices if and only if (G, k) is a yes-instance of Capacitated Dominating Set.

The graph G' is displayed in Figure 1. The vertex set V' of G' consists of the disjoint union of the vertex sets provided in Table 1. Furthermore, let $H := \bigcup_{1 \le i \le n} H_i$, $A := \bigcup_{1 \le i \le n} A_i$, and $B := \bigcup_{1 \le i \le n} B_i$.

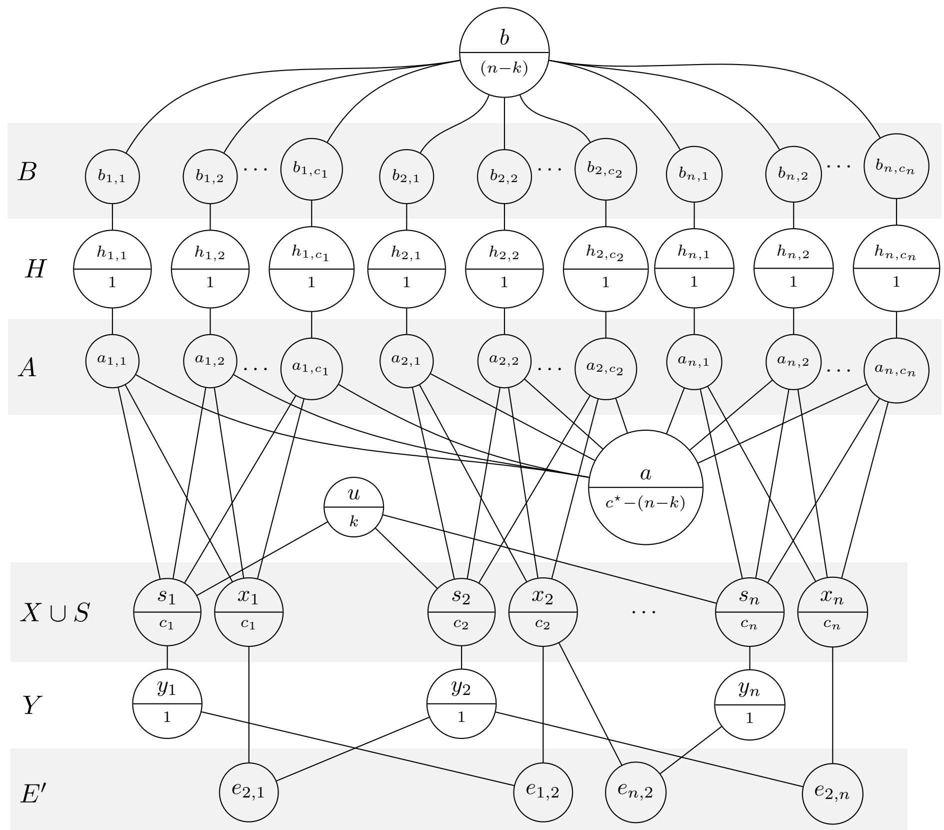

Figure 1: The graph G' obtained from the parameterized reduction from Capacitated Dominating Set to Bounded-Degree Vertex Deletion. The desired degree bound in the BDD instance is n, the number of vertices in the Capacitated Dominating Set instance. For vertices of degree greater than n the minimum number of neighbors that need to be deleted to reach degree at most n is displayed in the lower parts of the splitted circles. Here, $c_i := \text{cap}(v_i)$. The structure of the original graph G is reflected in the lowest three layers.

Table 1: Vertices of the Bounded-Degree Vertex Deletion graph G'.

$S := \{s_i \mid i = 1, \dots, n\}$,	dominating set selection
$X := \{x_i \mid i = 1, \dots, n\}$,	
$Y := \{y_i \mid i = 1, \dots, n\}$,	match check
$E' := \{e_{i,j}, e_{j,i} \mid \{v_i, v_j\} \in E\}$,	subset map selection
$\{a, b, u\}$,	
$H_i := \{h_{i,1}, \dots, h_{i,\text{cap}(v_i)}\}$ for $i \in \{1, \dots, n\}$,	
$A_i := \{a_{i,1}, \dots, a_{i,\text{cap}(v_i)}\}$ for $i \in \{1, \dots, n\}$,	
$B_i := \{b_{i,1}, \dots, b_{i,\text{cap}(v_i)}\}$ for $i \in \{1, \dots, n\}$,	
further degree-one vertices.	

Table 2: Edges and vertex degrees in the constructed graph G'. Unless stated otherwise, i,j $\in \{1, \dots, n\}$.

edges between
$e_{i,j}$ and x_j
$e_{i,j}$ and y_i
s_i and y_i
s_i and $u_1, \dots, u_k$
x_i and $a_{i,j}$ for $j = 1, \dots, \mathrm{cap}(i)$
s_i and $a_{i,j}$ for $j = 1, \dots, \mathrm{cap}(i)$
$h_{i,j}$ and $a_{i,j}$
$h_{i,j}$ and $b_{i,j}$
a and every vertex from A
b and every vertex from B

vertex	degree in G'
a	$n + c^\star - (n - k)$
b	$n + (n - k)$
Y, H	$n + 1$
u	$n + k$
$s_i \in S, x_i \in X$	$n + \mathrm{cap}(v_i)$
all remaining	$\leq n$

The basic idea of the construction is as follows. There are two types of *selection gadgets*. The first one selects k vertices corresponding to a capacitated dominating set in G (realized by the set S, that is, $s_i \in S$ is in the solution if v_i is in a capacitated dominating set in G). The second type of selection gadget selects $n - k$ elements of a subset map of the capacitated dominating set (realized by E'). Moreover, there is a *match gadget* checking whether the selected subset map elements belong to a capacitated dominating set built by the selected vertices (realized by Y). Finally, a third type of gadget (realized by A and X) ensures that there are at most $\mathrm{cap}(v_i)$ mapping edges selected for every selected vertex v_i. In addition, there are several sets of auxiliary vertices that, for example, ensure that the number of selected vertices is correct. In particular, the sets $H, B, \{a, b\}$ are needed to ensure that exactly $c^\star - (n - k)$ vertices from A are in any size-$(c^\star + n)$ solution. Note that this could be also achieved by introducing $c^\star + n$ new vertices adjacent to every vertex from A, but this would increase the treewidth of the graph.

The edges are defined in Table 2 (left). Moreover, we introduce additional degree-one vertices in order to end up with the degrees given by Table 2 (right). Note that if there is a solution containing a degree-one vertex, then there is also a solution of the same size without this vertex. Moreover, adding degree-one vertices does not affect the treewidth.

Lemma 1. *(G, k) is a yes-instance of* Capacitated Dominating Set *if and only if $(G', c^\star + n, n)$ is a yes-instance of* Bounded-Degree Vertex Deletion.

Proof. We start with the proof of the equivalence from left to right. Let (G, k) be a yes-instance of Capacitated Dominating Set with the capacitated graph $G := (V, E, \mathrm{cap})$. Furthermore, let D be a size-k capacitated dominating set for G, and let M be a feasible subset map for D. We construct a solution set L

for the BDD instance $(G', c^\star + n, n)$. For every $(v_i, v_j) \in M$, we add $e_{i,j}$ to L. Since M maps every vertex from $V \setminus D$ to one of its neighbors in D, at this point one has $|L| = n - k$. Furthermore, for every $v_j \in D$, we add s_j as well as $b_{j,1}, \dots, b_{j,\mathrm{sat}(M,v_j)}$ and $a_{j,\mathrm{sat}(M,v_j)+1}, \dots, a_{j,\mathrm{cap}(v_j)}$ to L. Finally, for every $v_j \in V \setminus D$, we add $a_{j,1}, \dots, a_{j,\mathrm{cap}(v_j)}$ to L. That is, we further add $\mathrm{cap}(v_i)$ vertices from $A \cup B$ for every $i \in \{1, \dots, n\}$ and k vertices from S. Thus, finally the cardinality of L is $c^\star + n$.

It remains to show that every vertex in $G' - L$ has degree at most n. We start with a and b. Since the total saturation of the vertices in D is $n - k$ and $b_{j,1}, \dots, b_{j,\mathrm{sat}(M,v_j)} \in L$ for $v_j \in D$, L contains $n - k$ neighbors of b. Similarly, one can verify that there are $c^\star - (n - k)$ neighbors of a in L.

Vertex u is adjacent to all vertices in S and $|L \cap S| = k$. Hence, k neighbors of u are deleted and the final degree of u is n. The vertices in Y have degree n in $G' - L$ because for each $y_i \in Y$ exactly one neighbor from $E \cup S$ is in L: Either $s_i \in L$ or $s_i \notin L$. In the latter case $e_{i,j}$ must be in L for j with $(v_i, v_j) \in M$. Since for each $h_i \in H$ either $a_i \in L$ or $b_i \in L$, every vertex from H has degree n in $G' - L$. Every vertex $s_i \in S$ is either in L (in the case of $v_i \in D$) or all $\mathrm{cap}(v_i)$ neighbors from A_i are in L (in the case of $v_i \in (V \setminus D)$). Hence, every vertex in S has degree n in $G' - L$. Finally, consider a vertex $x_j \in X$. We distinguish two cases:

- $v_j \in D$: Then, $\mathrm{sat}(M, v_j)$ many vertices from E are in L because D is a capacitated dominating set and M maps $\mathrm{sat}(M, v_j)$ many vertices to v_j. Furthermore, $\mathrm{cap}(v_j) - \mathrm{sat}(M, v_j)$ many vertices from A_j are in L. Thus, x_j has degree n in $G' - L$.

- $v_j \in (V \setminus D)$: Then, $\mathrm{cap}(v_j)$ many vertices from A_j are in L. Thus, x_j has degree n in $G' - L$.

Since all remaining vertices have degree at most n in G', it follows that $(G', c^\star + n, n)$ is a yes-instance of BDD.

Now, we show the other direction of the equivalence. Let $(G' := (V', E'), c^\star + n, n)$ be a yes-instance of BDD. Furthermore, let $L \subseteq V'$ be a size-$(c^\star + n)$ solution, that is, every vertex in $G' - L$ has degree at most n. We start with proving two claims to show which types of vertices must belong to L.

Claim 1. *For every solution L, $|L \cap A| = c^\star - (n - k)$ and $|L \cap B| = (n - k)$.*

Proof of Claim 1. We first show that any size-$(n + c^\star)$ solution L contains at most $c^\star$ vertices from $A \cup B \cup H \cup \{a, b\}$. Assume towards a contradiction that $|L \cap (A \cup B \cup H \cup \{a, b\})| > c^\star$ and, hence, $|L \setminus (A \cup B \cup H \cup \{a, b\})| \leq n - 1$. Note that there are n vertices in Y and for each vertex $y_i \in Y$ either y_i or a neighbor of y_i must be deleted. Since the neighborhoods for any two vertices in Y are disjoint and no vertex in Y is adjacent to any vertex from $(A \cup B \cup H \cup \{a, b\})$, it follows that $|L \setminus (A \cup B \cup H \cup \{a, b\})| \geq n$: a contradiction.

Now, since $|H| = c^\star$, every vertex from H has degree $n + 1$, and since any two vertices from H do not have a common neighbor, at least $c^\star$ vertices from

$A \cup B \cup H$ must belong to L. This implies that a and b cannot be in L and one hence must delete exactly $n-k$ vertices from B and $c^\star - (n-k)$ vertices from A to ensure that a and b end up with degree n. This finishes the proof of Claim 1.

Claim 2. *For every solution L, $|L \cap S| = k$ and $|L \cap E'| = n-k$.*

Proof of Claim 2. Since $|Y| = n$, every vertex from Y has degree $n+1$, and since no pair of vertices from Y has a common neighbor, at least n vertices from $E' \cup Y \cup S$ must belong to L. Due to Claim 1 exactly $c^\star$ solution vertices belong to $A \cup B$, implying that all n remaining solution vertices must belong to $E' \cup Y \cup S$. Moreover, at least $\sum_{x_i \in X} \operatorname{cap}(x_i) = c^\star$ neighbors of X must belong to L. Since $|L \cap A| = c^\star - (n-k)$, the total amount of deleted vertices from A that have a neighbor in X is $c^\star - (n-k)$. Thus, at least $n-k$ vertices from E' must be deleted. Moreover, there must be at least k vertices from S in the solution to ensure that u has degree n. This finishes the proof of Claim 2.

Now, due to Claims 1 and 2, we know that L consists of $c^\star - (n-k)$ vertices from A, k vertices from S, and $n-k$ vertices from E'. It remains to show that the selected k vertices from S correspond to a capacitated dominating set in G and the selected vertices from E' to a corresponding feasible subset map. Consider $D' := \{v_j \mid s_j \in L\}$ and $M' := \{(v_i, v_j) \mid e_{i,j} \in L\}$. We show the following claim.

Claim 3. *The vertex set D' is a capacitated dominating set of size at most k and M' is a feasible subset map for D' in G.*

Proof of Claim 3. We first show by contradiction that every vertex $x_i \in X$ with $x_i \notin L$ has $\deg(x_i) = n$ in $G' - L$ (we refer to this as **Observation 1**). Assume that there is an x_i in X with degree less than n. Then, L must contain at least $c^\star + 1$ neighbors of X. However, since each x_i has only neighbors in $A \cup E'$, it follows from Claims 1 and 2 that there are at most $c^\star$ neighbors of X in the solution, a contradiction.

Now, we show that for every "non-solution vertex", there is a selected edge adjacent to a "dominator vertex", that is,

$$\forall v_i \in (V \setminus D') : \exists v_j \in D' : (v_i, v_j) \in M'.$$

For every $y_i \in Y$, there must be a neighbor in L. Due to Claims 1 and 2 such a neighbor can either be s_i or $e_{i,j}$ for a j. Since $|S \cap L| = k$, for the $n-k$ vertices with $s_i \notin L$ there must be a j such that $e_{i,j} \in L$. It remains to show that if $e_{i,j} \in L$, then s_j must be in L as well. If $e_{i,j} \in L$, then there must be a vertex from A_j that is not in L since, otherwise, x_j would have degree less than n in $G' - L$ (contradicting Observation 1). Thus, there are more than n neighbors of s_j in $G' - L$ and s_j must be in L.

We have shown that D' is a size-k dominating set and M' is a corresponding subset map. It remains to show that M' is feasible, that is, $\operatorname{sat}(M', v) \leq \operatorname{cap}(v)$ for every $v \in D'$. To this end, assume that $\operatorname{sat}(M', v_i) > \operatorname{cap}(v_i)$. Then, x_i has more than $\text{cap}(v_i)$ neighbors in L and, therefore, x_i has degree less than n in $G' - L$, a contradiction to Observation 1.

Altogether, it follows that D' is a capacitated dominating set for G' and M' is a feasible subset map for D'. □

It remains to ensure that the treewidth of G' is bounded by a function of the treewidth of G.

Lemma 2. *Let (G, k) be an instance of* Capacitated Dominating Set *and let $(G', c^\star + n, d)$ be the corresponding instance of* Bounded-Degree Vertex Deletion. *Let* tw *be the treewidth of G. It holds that G' has treewidth at most* $\text{tw}^2 + 3 \cdot \text{tw}$.

Proof. Let $\mathcal{T}$ be a tree decomposition of G with maximum bag size $\text{tw} + 1$. Let $V := \{v_1, \dots, v_n\}$ denote the vertices of G and let E be the set of edges of G. We modify $\mathcal{T}$ such that it is a tree decomposition of G':

1. For every bag containing two vertices $v_i, v_j \in V$ with $\{v_i, v_j\} \in E$ add $e_{i,j}$ and $e_{j,i}$ to the bag. One adds at most tw^2 new vertices per bag.
2. For every bag containing $v_i \in V$ replace v_i by the three vertices x_i, s_i, and y_i. This contributes with $3 \cdot \text{tw}$ to the treewidth.
3. Add a, b, and u to every bag. The treewidth is further increased by 3.
4. Adding the vertices from H, A, and B can be done with bags of size seven: For each $h_{i,j} \in H$ create a new bag $B_{i,j}$ containing $h_{i,j}$, $a_{i,j}$, $b_{i,j}$, x_i, s_i, a, and b. Add $B_{i,j}$ as a new leaf to a bag containing x_i and s_i to the decomposition tree.

Clearly, the degree-one vertices can be added without further increasing the treewidth. Moreover, it is easy to check that the constructed tree decomposition is a correct tree decomposition for G'. □

Combining Lemma 1 and Lemma 2, we arrive at our main theorem.

Theorem 1. Bounded-Degree Vertex Deletion *is W[1]-hard with respect to the parameter "treewidth of the input graph".*

2.2. Combined parameter treewidth and solution size

Whereas BDD is W[2]-hard with respect to the parameter solution size [16] and W[1]-hard with respect to the parameter treewidth (Theorem 1), next we show that it becomes fixed-parameter tractable when combining both parameters. To this end, we employ a close connection to the Vector Dominating Set problem.

BDD is a special case of Vector Dominating Set which is defined as follows:

> Vector Dominating Set (VDS)
> *Given:* A graph $G = (V, E)$ with $V := \{v_1, \dots, v_n\}$, an integral threshold vector $l = \{l(v_1), \dots, l(v_n)\}$, and a positive integer k.
> *Question:* Is there a set $V' \subseteq V$ with $|V'| \leq k$ such that $|N(v) \cap V'| \geq l(v)$ for all $v \in V \setminus V'$?

Clearly, if $l(v) = \max\{0, \deg(v) - d\}$, then VDS and BDD coincide. The parameterized complexity of VDS has been investigated by Raman et al. [26] for different classes of threshold vectors and special graph classes. In particular, they showed that VDS can be solved in $k^{O(\rho \cdot k^2)} n^{O(1)}$ time on ρ-degenerated graphs. Herein, a graph is *ρ-degenerated* if every induced subgraph of G has a vertex of degree at most ρ. Note that $\rho \leq \text{tw}$ (see e.g. Bodlaender [5]).

As stated above, one can transform a BDD instance, consisting of a graph $G = (V, E)$, a degree bound d, and an integer k, to an equivalent VDS instance by setting $l(v) := \max\{0, \deg(v) - d\}$ for all $v \in V$. However, Raman et al. [26] required that $l(v) \geq 1$. This can be achieved by the following transformation. Given a VDS instance where $l(v) = 0$ for some $v \in V$, build an equivalent instance as follows. Let $V_0 := \{v \in V \mid l(v) = 0\}$. Add two new vertices x and y, the edges $\{x, y\}$ and $\{y, w\}$ for all $w \in V_0$, and set $l(v) := 1$ for all $v \in V_0 \cup \{x, y\}$. Observe that the original instance has a solution of size k if and only if the new instance has a solution of size $k + 1$. Moreover, by the above transformation the treewidth and degeneracy increases by at most two. Hence, we arrive at the following.

Corollary 1.

1. Vector Dominating Set *is W[1]-hard with respect to the parameter "treewidth of the input graph".*
2. Bounded-Degree Vertex Deletion *can be solved in* $k^{O(\text{tw} \cdot k^2)} n^{O(1)}$ *time, hence it is fixed-parameter tractable with respect to the combined parameter* (tw, k).

3. Parameter Feedback Edge Set

The previous section showed that there is no hope for fixed-parameter tractability for BDD with respect to the single parameter treewidth. We contrast this result by showing that, using the feedback edge set number, which upper-bounds treewidth, as parameter, one can achieve fixed-parameter tractability. Clearly, the feedback edge set number can also be considered as a measurement of tree-likeness. Another interesting parameter in this direction is the "feedback vertex set number", that is, the minimum number of vertices to delete to make a graph a forest. However, the parameterized complexity with respect to the feedback vertex set number remains open.

We now show that BDD is fixed-parameter tractable with respect to the parameter feedback edge set number s_e. Let (G, d, k) be an instance of BDD and let E_f be a feedback edge set of size s_e. Note that E_f can be computed in linear time using depth-first search. For every $\{x, y\} \in E_f$, branch into three cases. First, if x is in the solution, then delete x and decrease k by one. Second, if y is in the solution, then delete y and decrease k by one. Third, if neither x nor y are in the solution, then transform the graph as follows. Remove the edge $\{x, y\}$, add two new vertices a_x and a_y, and add the edges $\{a_x, x\}$ and $\{a_y, x\}$. Moreover, we mark the vertices x, y, a_x, a_y as *unremovable*, that is, they cannot be part of the solution in the considered branching case.

After exhaustive branching, every edge from E_f is either deleted or "cut" into two parts. Hence, it remains to solve the "annotated version" of BDD with unremovable vertices when restricted to forests. To this end, if there is an unremovable vertex with more than d unremovable neighbors, then return "no". Otherwise, for every single tree a minimum number of vertices that need to be deleted can be computed as follows. Root the tree arbitrarily and process the tree according to a bottom-up traversal. Let x be the first node with $\deg(x) \geq d + i$ with $i \geq 1$.

If $\deg(x) = d + 1$ and the parent p of x is "removable",
  then delete p.
Otherwise, if x is removable,
  then delete x,
  else distinguish two further cases.
    If p is removable,
      then delete p and $i - 1$ removable children of x,
      else, delete i removable children from x.

It is easy to verify that the given case distinction is correct and can be accomplished in $O(n^2)$ time with n being the number of vertices [9]. Hence, one arrives at the following.

Theorem 2. Bounded-Degree Vertex Deletion *can be solved in* $O(3^{s_e} \cdot n^2)$ *time with* s_e *being the size of a feedback edge set.*

4. Outlook and Open Questions

Having shown that there is presumably no hope for fixed-parameter tractability of Bounded-Degree Vertex Deletion parameterized by treewidth (assuming an unbounded value d for the maximum degree), the following are natural next steps for future research.

- Combine the treewidth parameter with other parameters. We have seen that BDD becomes fixed-parameter tractable when parameterized by treewidth and solution size, and when parameterized by treewidth and degree bound d (the latter due to its reliance on Courcelle's theorem, which only gives a classification result). Besides identifying further parameters that might be combined with the treewidth, one might try to improve the corresponding upper bounds of the existing results, perhaps even going for problem kernel results [6, 20].

- Study the parameterized complexity of BDD with respect to "weaker" structural parameters than treewidth is. For instance, we already identified that BDD becomes fixed-parameter tractable when parameterized by the feedback edge set number of the underlying graph. Clearly the treewidth of a graph is always upper-bounded by its feedback edge set number, which means that the feedback edge set number is the weaker parameter in the view of parameterized complexity analysis. An interesting parameter in the middle between feedback edge set number and treewidth regards the size of a feedback vertex set. We stress that although it directly follows from the construction by Dom et al. [13] that Capacitated Dominating Set is W[1]-hard with respect to the feedback vertex number, our W[1]-hardness reduction for BDD only holds for the parameter treewidth. Hence, the parameterized complexity of BDD with respect to the feedback vertex set number remains open. Since the feedback vertex set number can be much smaller than the feedback edge set number this question seems to be of particular interest.

Moreover, an other interesting parameter that is weaker than treewidth regards the vertex cover number. See Fellows et al. [17] and Fiala et al. [18] for several problems that parameterized by treewidth are W[1]-hard but become fixed-parameter tractable when parameterized by the vertex cover number.

As a general side remark, note that even when a problem turns out to be fixed-parameter tractable with respect to a single parameter, from a practical point of view it still may make sense to combine this parameter with other parameters in order to achieve more efficient algorithms. This is a general theme of multivariate algorithmics [15, 24]. Finally, a completely different issue would be to provide a missing thorough study concerning the polynomial-time approximability of BDD also when restricted to special cases such as bounded treewidth graphs.

[1] B. Balasundaram, S. Butenko, and I. V. Hicks. Clique relaxations in social network analysis: The maximum k-plex problem. *Operations Research*, 59(1):133–142, 2011.

[2] B. Balasundaram, S. S. Chandramouli, and S. Trukhanov. Approximation algorithms for finding and partitioning unit-disk graphs into co-k-plexes. *Optimization Letters*, 4(3):311–320, 2010.

[3] N. Betzler, R. Bredereck, R. Niedermeier, and J. Uhlmann. On making a distinguished vertex minimum degree by vertex deletion. In *Proceedings of the 37th International Conference on Current Trends in Theory and Practice of Computer Science (SOFSEM '11)*, volume 6543 of *LNCS*, pages 123–134. Springer, 2011.

[4] N. Betzler and J. Uhlmann. Parameterized complexity of candidate control in elections and related digraph problems. *Theoretical Computer Science*, 410(52):5425–5442, 2009.

[5] H. L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoretical Computer Science*, 209(1–2):1–45, 1998.

[6] H. L. Bodlaender. Kernelization: New upper and lower bound techniques. In *Proceedings of the 4th International Workshop on Parameterized and*

Exact Computation (IWPEC '09), volume 5917 of *LNCS*, pages 17–37. Springer, 2009.

[7] H. L. Bodlaender and A. M. C. A. Koster. Combinatorial optimization on graphs of bounded treewidth. *The Computer Journal*, 51(3):255–269, 2008.

[8] H. L. Bodlaender and B. van Antwerpen-de Fluiter. Reduction algorithms for graphs of small treewidth. *Information and Computation*, 167(2):86–119, 2001.

[9] R. Bredereck. Graph and Election Problems Parameterized by Feedback Set Numbers. Diplomarbeit, Institut für Informatik, Friedrich-Schiller-Universität Jena, Germany, 2010. Available online at http://arxiv.org/abs/1108.2191.

[10] Z.-Z. Chen, M. R. Fellows, B. Fu, H. Jiang, Y. Liu, L. Wang, and B. Zhu. A linear kernel for co-path/cycle packing. In *Proceedings of the 6th International Conference on Algorithmic Aspects in Information and Management (AAIM '10)*, volume 6124 of *LNCS*, pages 90–102. Springer, 2010.

[11] B. Courcelle. The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. *Information and Computation*, 85(1):12–75, 1990.

[12] A. Dessmark, K. Jansen, and A. Lingas. The maximum k-dependent and f-dependent set problem. In *Proceedings of the 4th International Symposium on Algorithms and Computation (ISAAC '93)*, volume 762 of *LNCS*, pages 88–98. Springer, 1993.

[13] M. Dom, D. Lokshtanov, S. Saurabh, and Y. Villanger. Capacitated domination and covering: A parameterized perspective. In *Proceedings of the 3rd International Workshop on Parameterized and Exact Computation (IWPEC '08)*, volume 5018 of *LNCS*, pages 78–90. Springer, 2008.

[14] R. G. Downey and M. R. Fellows. *Parameterized Complexity*. Springer, 1999.

[15] M. R. Fellows. Towards fully multivariate algorithmics: Some new results and directions in parameter ecology. In *Proceedings of the 20th International Workshop on Combinatorial Algorithms (IWOCA 2009)*, volume 5874 of *LNCS*, pages 2–10. Springer, 2009.

[16] M. R. Fellows, J. Guo, H. Moser, and R. Niedermeier. A generalization of Nemhauser and Trotter's local optimization theorem. *Journal of Computer and System Sciences*, In Press, Corrected Proof, 2011. Available online at http://dx.doi.org/10.1016/j.jcss.2010.12.001.

[17] M. R. Fellows, D. Lokshtanov, N. Misra, F. A. Rosamond, and S. Saurabh. Graph layout problems parameterized by vertex cover. In *Proceedings of the 19th International Symposium on Algorithms and Computation (ISAAC '08)*, volume 5369 of *LNCS*, pages 294–305. Springer, 2008.

[18] J. Fiala, P. A. Golovach, and J. Kratochvíl. Parameterized complexity of coloring problems: Treewidth versus vertex cover. *Theoretical Computer Science*, 412(23):2513–2523, 2011.

[19] J. Flum and M. Grohe. *Parameterized Complexity Theory.* Springer, 2006.

[20] J. Guo and R. Niedermeier. Invitation to data reduction and problem kernelization. *ACM SIGACT News*, 38(1):31–45, 2007.

[21] C. Komusiewicz, F. Hüffner, H. Moser, and R. Niedermeier. Isolation concepts for efficiently enumerating dense subgraphs. *Theoretical Computer Science*, 410(38-40):3640–3654, 2009.

[22] H. Moser, R. Niedermeier, and M. Sorge. Exact combinatorial algorithms and experiments for finding maximum k-plexes. *Journal of Combinatorial Optimization*, In Press, 2011. Available online at http://dx.doi.org/10.1007/s10878-011-9391-5.

[23] R. Niedermeier. *Invitation to Fixed-Parameter Algorithms.* Oxford University Press, 2006.

[24] R. Niedermeier. Reflections on multivariate algorithmics and problem parameterization. In *Proceedings of the 27th International Symposium on Theoretical Aspects of Computer Science (STACS 2010)*, volume 5 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 17–32, Dagstuhl, Germany, 2010. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[25] N. Nishimura, P. Ragde, and D. M. Thilikos. Fast fixed-parameter tractable algorithms for nontrivial generalizations of Vertex Cover. *Discrete Applied Mathematics*, 152(1–3):229–245, 2005.

[26] V. Raman, S. Saurabh, and S. Srihari. Parameterized algorithms for generalized domination. In *Proceedings of the 2nd Annual International Conference on Combinatorial Optimization and Applications (COCOA '08)*, volume 5165 of *LNCS*, pages 116–126. Springer, 2008.

[27] N. Robertson and P. D. Seymour. Graph minors II: Algorithmic aspects of tree-width. *Journal of Algorithms*, 7:309–322, 1986.

[28] S. B. Seidman and B. L. Foster. A graph-theoretic generalization of the clique concept. *Journal of Mathematical Sociology*, 6:139–154, 1978.