How to Put Through Your Agenda in Collective Binary Decisions

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We consider the following decision making scenario: a society of voters has to find an agreement on a set of proposals, and every single proposal is to be accepted or rejected. Each voter supports a certain subset of the proposals–the *favorite ballot* of this voter–and opposes the remaining ones. He accepts a ballot if he supports more than half of the proposals in this ballot. The task is to decide whether there exists a ballot approving a specified number of selected proposals (agenda) such that all voters (or a strict majority of them) accept this ballot.

We show that, on the negative side, both problems are NP-complete, and on the positive side, they are fixed-parameter tractable with respect to the total number of proposals or with respect to the total number of voters. We look into further natural parameters and study their influence on the computational complexity of both problems, thereby providing both tractability and intractability results. Furthermore, we provide tight combinatorial bounds on the worst-case size of an accepted ballot in terms of the number of voters.

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1. INTRODUCTION

Consider the following decision making scenario which may occur in contexts like coalition formation, the design of party platforms, the change of statutes of an association,

An extended abstract of this work appeared in the Proceedings of the 3rd International Conference on Algorithmic Decision Theory (ADT '13) [Alon et al. 2013a]. In this long version, we provide numerous details that have been omitted in the extended abstract and have the following new contributions: We extend the agenda model allowing the leader to specify a lower bound on the number of agenda proposals that should be contained in the solution ballot (instead of forcing all agenda proposals being part of the solution ballot). We adapt all algorithms to also work with this extended model. We show that requiring majoritywise acceptance instead of unanimous acceptance changes the computational complexity for the parameterization by the maximum size of the favorite ballots from FPT to W[1]-complete.
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or the agreement on contract issues: A leader has an agenda, that is, a set of proposals she wants to get realized. However, a set of proposals has to be approved or disapproved as a whole by a set of voters. Each voter has his favorite proposals he wants to support. A set of proposals is acceptable to a voter if he supports more than half of these proposals. Now, the leader might have a hidden agenda, that is, she is searching for a set of proposals containing at least some of her favorite proposals such that a majority of voters accepts this set. Can the leader efficiently find such a successful set of proposals satisfying her agenda? What changes when this set of proposals has to be acceptable to *all* voters and not just to a majority?

1.1. Mathematical model

Let $\mathcal{V}$ be a society of n voters and $\mathcal{P}$ be a set of m proposals. Each voter may support any number of proposals in $\mathcal{P}$ and rejects all the others. Subsets of $\mathcal{P}$ are called *ballots*. The favorite ballot $B_i \subseteq \mathcal{P}$ of a voter i ($1 \leq i \leq n$) consists of all proposals he supports.

The voters evaluate a ballot $Q \subseteq \mathcal{P}$ according to the size of the intersection of Q and their favorite ballots. More precisely, voter i *accepts* Q if and only if a strict majority of proposals from Q is also contained in his favorite ballot, that is,

$$|B_i \cap Q| \ > \ |Q|/2.$$

We say that in this case voter i is *happy* with Q.

The central question is whether there exists a ballot Q that (a) satisfies a hidden agenda and that (b) is acceptable to the society. The agenda is encoded by the agenda set $Q_+ \subseteq \mathcal{P}$ and by a lower bound q_+ on the number of agenda proposals that are to be contained in Q. The society's acceptance in (b) might be a *unanimous* acceptance or a *majority* acceptance. This leads to the following two problems which only differ in the respective questions asked.

UNANIMOUSLY ACCEPTED BALLOT (UNAAB)
Input: A set $\mathcal{P}$ of m proposals; a society $\mathcal{V}$ of n voters with favorite ballots $B_1, \dots, B_n \subseteq \mathcal{P}$; an agenda (Q_+, q_+), $Q_+ \subseteq \mathcal{P}$ and $q_+ \in \mathbb{N}$.
Question: Is there a ballot $Q \subseteq \mathcal{P}$ with $|Q_+ \cap Q| \geq q_+$ which *every* single voter i accepts (that is, $|B_i \cap Q| > |Q|/2$)?

MAJORITYWISE ACCEPTED BALLOT (MAJAB)
Input: A set $\mathcal{P}$ of m proposals; a society $\mathcal{V}$ of n voters with favorite ballots $B_1, \dots, B_n \subseteq \mathcal{P}$; an agenda (Q_+, q_+), $Q_+ \subseteq \mathcal{P}$ and $q_+ \in \mathbb{N}$.
Question: Is there a ballot $Q \subseteq \mathcal{P}$ with $|Q_+ \cap Q| \geq q_+$ which a *strict majority* of the voters accepts (that is, $|B_i \cap Q| > |Q|/2$)?

Sometimes we only use the term agenda (without "hidden"), although we always assume the agenda to be unknown for the voters. One important special case of UNAAB or MAJAB is when there is no agenda, that is, $Q_+ = \emptyset$ or $q_+ = 0$. In this case, the only question is whether there is a ballot acceptable to the society. For the rest of the paper, we assume without loss of generality that $q_+ = 0$ if and only if $Q_+ = \emptyset$.

The following example demonstrates that the solution sizes to our problems are *not* monotone, that is, a solution ballot of size h does not imply a solution of a size smaller or larger than h. This is in notable contrast to many natural decision problems, such as all problems we reduce from in this paper.

Example 1. Consider the society $\mathcal{V} = \{1, 2, 3, 4\}$ of voters and the set $\mathcal{P} = \{p_1, p_2, p_3, p_4, p_5\}$ of proposals. The favorite ballots are given as $B_1 = \{p_1, p_2, p_4\}$, $B_2 = \{p_1, p_2, p_5\}$, $B_3 = \{p_1, p_3, p_5\}$, and $B_4 = \{p_2, p_3, p_4\}$.

	p_1	p_2	p_3	p_4	p_5
B_1	+	+	-	+	-
B_2	+	+	-	-	+
B_3	+	-	+	-	+
B_4	-	+	+	+	-

	p_1	p_2	p_3	p_4	p_5
B_1	+	+	-	+	-
B_2	+	+	-	-	+
B_3	+	-	+	-	+
B_4	-	+	+	+	-

Fig. 1. Illustration of Example 1. There is one column for each proposal from $\mathcal{P}$ and one row for each favorite ballot of a voter. There is a "+" in row i and column j if and only if $p_j \in B_i$; otherwise there is a "$-$". We use this way of illustrating subset families several times in this paper. Here, we highlight the only two unanimously accepted ballots from Example 1 (left: $\{p_1, p_2, p_3\}$, right: $\{p_1, p_2, p_3, p_4, p_5\}$).

Suppose that there is no agenda. Then the only unanimously accepted ballots are $\{p_1, p_2, p_3\}$ of size three and $\{p_1, p_2, p_3, p_4, p_5\}$ of size five (see also Figure 1). This shows that the set of the sizes of all solution ballots may contain gaps.

With regard to majority acceptance, if the hidden agenda is $(\{p_4, p_5\}, 2)$, then the ballots $\{p_1, p_4, p_5\}$, $\{p_2, p_4, p_5\}$, and $\{p_1, p_2, p_3, p_4, p_5\}$ are the only ballots that are acceptable to a strict majority of voters (the first ballot is accepted by the voters 1–3, the second ballot is accepted by the voters 1, 2, and 4, and the third ballot is accepted by all voters). Again, the set of the sizes of all solution ballots contains gaps.

1.2. Related work

Our model can be seen as special case of an approval voting system with threshold functions as introduced by Fishburn and Pekeč [2004]. In this model, the profile also consists of a set $\mathcal{P}$ of m proposals and a society $\mathcal{V}$ of n voters with favorite ballots $B_1, \ldots, B_n \subseteq \mathcal{P}$. Additionally, one specifies the set ζ of *admissible committees* which is a family of subsets of $\mathcal{P}$ without the empty set. The winners are defined by the *choice function* which maps each profile to a non-empty subfamily of ζ. The choice function C_t for some threshold function $t : \zeta \to \mathbb{R}^+$ (with $\mathbb{R}^+$ being the set of positive rational numbers) is defined by

$$Q \in C_t \leftrightarrow Q \in \zeta \land \forall Q' \in \zeta : |\{i : B_i \cap Q \geq t(Q)\}| \geq |\{i : B_i \cap Q' \geq t(Q')\}|.$$

Now, setting the admissible committees ζ to all subsets of $\mathcal{P}$ without the empty set and defining the strict majority threshold function SMT by $\mathrm{SMT}(Q) := \frac{|Q|+1}{2}$, the choice function C_{SMT} can be read as follows: A ballot Q *represents* a voter if and only if a strict majority of the proposals from Q are supported by the voter. The choice function C_{SMT} selects the ballots that represent the most voters [Kilgour and Marshall 2012]. In our model, we ask whether there is a ballot that represents all (resp. a strict majority of the) voters and which contains q_+ proposals from Q_+ Except for t being the constant 1-function where NP-hardness for computing the choice function is known [Fishburn and Pekeč 2004], no computational complexity analysis has been undertaken for C_{SMT} [Fishburn and Pekeč 2004; Kilgour 2010; Kilgour and Marshall 2012].

The scenario considered in our work is also related to the concepts of collective domination [Elkind et al. 2011], approval-based multiwinner rules [Aziz et al. 2014; Kilgour 2010; Kilgour and Marshall 2012], and proportional representation in multiwinner elections [Betzler et al. 2013; Chamberlin and Courant 1983; Lu and Boutilier 2011; Monroe 1995; Potthoff and Brams 1998; Procaccia et al. 2008; Skowron et al. 2013a] as well as in resource allocation [Skowron et al. 2013b; Skowron et al. 2014]—in all cases one has to select certain alternatives (proposals in our context) that provide a "good representation" of the voters' will. In our work, we deal with "collectively winning ballots", namely more than half of the proposals in such a ballot are supported by a voter. The literature contains many different concepts for "good representations" of the voters' will in the literature which are designed for different application scenarios. The approval-based multiwinner rule Satisfaction Approval Voting (SAV), introduced by Kilgour [2010], has a similar flavor as our model since it also aims to maximize the voters' satisfaction with the selected committee. However, in SAV the committee size is fixed, the satisfaction is measured by "the number of selected favorites divided by the total number of the voter's favorites" (instead of "the number of selected favorites divided by the total number of selected alternatives" as in our model) and the optimization criterion is utilitarian (instead of egalitarian as in our model). In contrast to our model, winner determination for SAV is polynomial-time solvable. Similarly to our unanimous model variant, where *each* voter must be satisfied, proportional representation aims to assign to *each* voter a good representative, but the votes are usually based on linear preferences or utility functions.

Further related models have been considered in the theory and practice of decision making. For instance, Laffond and Lainé [2012] recently investigated the conditions under which issue-wise majority voting allows for reaching several types of compromise. An alternative to issue-wise evaluation is to compare issue sets (which correspond to ballots in our setting) using the *symmetric difference from a voter's favorite issue set* [Çuhadaroğlu and Lainé 2012; Laffond and Lainé 2009; Laffond and Lainé 2012]. A small symmetric difference is good, and a large symmetric difference is bad. This way of comparing issue sets is very close to our model: A voter accepts a ballot Q if and only if the symmetric difference from his favorite ballot B to Q is smaller than the symmetric difference from B to the empty ballot. Typically, the studies in this context use issue sets to analyze properties or paradoxes of an issue-wise voting process and do not provide computational complexity results.

Computational complexity studies are established for related decision making problems like judgment aggregation [Baumeister et al. 2011; Endriss et al. 2012], approval-based multiwinner determination [Aziz et al. 2014], lobbying [Bredereck et al. 2014b; Christian et al. 2007; Binkele-Raible et al. 2014; Erdélyi et al. 2007], proportional representation in multiwinner elections [Potthoff and Brams 1998; Procaccia et al. 2008; Lu and Boutilier 2011; Betzler et al. 2013; Skowron et al. 2013a] as well as in resource allocation [Skowron et al. 2013b; Skowron et al. 2014], or control of multiple referenda [Conitzer et al. 2009]. In the context of judgment aggregation, Alon et al. [2013b] investigated the computational complexity of control by bundling issues which is also related to "vote on bundled proposals" as considered in our paper. Baumeister et al. [2013] investigated the computational complexity of control by bundling judges.

Finally, we mention in passing that central computational complexity results of our work are cast within the framework of parameterized complexity analysis, which due to its refined view on algorithmic (in)tractability fits particularly well with voting and related problems [Betzler et al. 2012; Bredereck et al. 2014a].

1.3. Our contributions

We analyze the combinatorial and algorithmic behavior of UNANIMOUSLY ACCEPTED BALLOT and MAJORITYWISE ACCEPTED BALLOT. In particular, we investigate the role of the following natural parameters:

— the size $|Q_+|$ of the agenda set,
— the number m of proposals,
— the number n of voters,

Table I. Parameterized complexity results on two central problems. The entry "ILP-FPT" means fixed-parameter tractability based on a formulation as an integer linear program. Note that all our "intractability" results except for the W[1]-completeness result for MAJAB parameterized by $b_{\max}$ also hold for the case without hidden agenda.

Parameters	UNAAB	MAJAB
Size $\|Q_+\|$ of the agenda	NP-complete already for $\|Q_+\| = 0$ (Thm. 2.1)	
Number m of proposals	FPT, no polynomial kernel (Thm. 2.2)	
Number n of voters	ILP-FPT, no polynomial kernel (Thm. 2.3)	
Parameter h	W[2]-complete (Thm. 2.4)	W[2]-hard (Thm. 2.4)
Parameter $b_{\max}$	FPT, no polynomial kernel (Thm. 2.5)	W[1]-complete (Thm. 2.5)
Parameter b_{gap}	NP-complete already for $b_{\text{gap}} = 1$ (Thm. 2.9)	

— the size h of the solution ballot Q, that is, $h = |Q|$,
— the maximum size $b_{\max}$ of favorite ballots, that is, $b_{\max} = \max_{i \in \mathcal{V}} |B_i|$, and
— the difference b_{gap} between $\lceil (m+1)/2 \rceil$ and the minimum size of favorite ballots, that is, $b_{\text{gap}} = \lceil (m+1)/2 \rceil - \min_{i \in \mathcal{V}} |B_i|$.

The parameter b_{gap} measures how far a given instance is from being trivial in terms of the number of proposals: If each voter's favorite ballot contains at least $\lceil (m+1)/2 \rceil$ proposals, then choosing $Q = \mathcal{P}$ makes every voter happy, so the instance is a trivial yes-instance. While the parameters n and m are naturally related to the "dimensions" of the input, the parameters h, $b_{\max}$, and b_{gap} measure certain degrees of contradiction or inhomogeneity in an instance. The parameter $|Q_+|$ measures the size of the agenda set and is an upper bound for q_+ (in non-trivial instances).

Section 2 is devoted to computational complexity results. The main picture is summarized in Table I. Not too much of a surprise, UNANIMOUSLY ACCEPTED BALLOT and MAJORITYWISE ACCEPTED BALLOT turn out to be NP-complete. More surprisingly, this remains so even when the input ballots are almost trivial, that is, $b_{\text{gap}} = 1$. Namely, if $|B_i| \geq \lfloor m/2 \rfloor + 1$ for all voters i, then all voters accept the ballot $\mathcal{P}$. But if every voter i only satisfies the slightly weaker condition $|B_i| \geq \lfloor m/2 \rfloor$, then both problems already become NP-complete. Next, formulating the problems as integer linear programs (ILPs) where the number of variables only depends (exponentially) on n implies fixed-parameter tractability with respect to the parameter n. Using simple brute-force search, one easily obtains that both problems are fixed-parameter tractable with respect to the parameter m. As to efficient and effective preprocessing by polynomial-time data reduction, however, we show that neither for parameter n nor for parameter m polynomial-size problem kernels exist unless an unlikely collapse in complexity theory occurs. As to the parameter h, we prove parameterized intractability—more precisely, W[2]-completeness for UNANIMOUSLY ACCEPTED BALLOT and W[2]-hardness for MAJORITYWISE ACCEPTED BALLOT. While the two problems behave in almost the same way with respect to the parameters n, m, and h, the situation changes for the parameter $b_{\max}$: While UNANIMOUSLY ACCEPTED BALLOT is shown fixed-parameter tractable, MAJORITYWISE ACCEPTED BALLOT is proven to be W[1]-complete. However, the corresponding W[1]-hardness proof requires a non-empty agenda set. W[1]-hardness for MAJORITYWISE ACCEPTED BALLOT without hidden agenda remains as an open question. All other intractability results hold even without hidden agenda.

In Section 3, we provide an in-depth combinatorial analysis concerning the dependence of the size of a minimal solution ballot on the parameter n. In particular, we show the upper bound $(n+1)^{(n+1)/2}$ and the lower bound $n^{n/2-o(n)}$ for UNANIMOUSLY ACCEPTED BALLOT without hidden agenda, thus achieving asymptotically almost matching bounds. Analogous results hold for MAJORITYWISE ACCEPTED BALLOT without hidden agenda. In Section 4, we conclude with some open questions for future research.

1.4. Parameterized complexity preliminaries

The concept of parameterized complexity was pioneered by Downey and Fellows [Downey and Fellows 2013] (see also further textbooks [Flum and Grohe 2006; Niedermeier 2006]). A parameterized problem is a language $L \subseteq \Sigma^* \times \Sigma^*$, where Σ is an alphabet. The second component is called the *parameter* of the problem. Typically, the parameter or the "combined" ones are non-negative integers. A parameterized problem L is *fixed-parameter tractable* if there is an algorithm that decides in $f(k) \cdot |x|^{O(1)}$ time whether $(x, k) \in L$, where f is an arbitrary computable function depending only on k. Correspondingly, FPT denotes the class of all fixed-parameter tractable parameterized problems. A core tool in the development of fixed-parameter algorithms is polynomial-time preprocessing by *data reduction* [Guo and Niedermeier 2007; Kratsch 2014]. Here, the goal is to transform a given problem instance (x, k) in polynomial time into an equivalent instance (x', k') with parameter $k' \leq k$ such that the size of (x', k') is upper-bounded by some function g only depending on k. If this is the case, we call instance (x', k') a (problem) *kernel* of size $g(k)$. If g is a polynomial, then we say that this problem has a *polynomial-size problem kernel*, in short, *polynomial kernel*.

Fixed-parameter intractability under some plausible complexity-theoretic assumptions can be shown by means of *parameterized reductions*. A parameterized reduction from a parameterized problem P to another parameterized problem P' is a function that, given an instance (x, k), computes in $f(k) \cdot |x|^{O(1)}$ time an instance (x', k') (with k' only depending on k) such that (x, k) is a yes-instance for P if and only if (x', k') is a yes-instance for P'. The two basic complexity classes for fixed-parameter intractability are W[1] and W[2]. A parameterized problem L is W[1]- or W[2]-hard if there is a parameterized reduction from a W[1]- or W[2]-hard problem to L. For instance, both INDEPENDENT SET and HITTING SET are known to be NP-complete [Garey and Johnson 1979]. However, when parameterized by the solution size, INDEPENDENT SET is W[1]-complete while HITTING SET is W[2]-complete [Downey and Fellows 2013]. There is good complexity-theoretic reason to believe that W[1]-hard and W[2]-hard problems are not fixed-parameter tractable [Downey and Fellows 2013; Flum and Grohe 2006; Niedermeier 2006].

2. COMPUTATIONAL COMPLEXITY RESULTS

The following observation is used many times in our proofs.

OBSERVATION 1. *Let i and j be two voters that are both happy with some $Q \subseteq \mathcal{P}$.*

(i) Then $B_i \cap B_j \neq \emptyset$.
(ii) If $B_i \cap B_j = \{p\}$, then $p \in Q$ and furthermore $|B_i \cap Q| = |B_j \cap Q|$.

PROOF. (i) Assume $B_i \cap B_j = \emptyset$. Since i is happy with Q it holds that $|B_i \cap Q| > |Q|/2$ and thus $|Q \setminus B_i| < |Q|/2$. As $B_j \cap Q \subseteq Q \setminus B_i$, voter j cannot be happy; a contradiction.

(ii) First, assume $B_i \cap B_j = \{p\}$ and $p \notin Q$. Then, two voters with the favorite ballots $B_i \setminus \{p\}$ and $B_j \setminus \{p\}$ would be happy with Q, but $(B_i \setminus \{p\}) \cap (B_j \setminus \{p\}) = \emptyset$; a contradiction to (i). Second, assume $B_i \cap B_j = \{p\}$, $p \in Q$, and without loss of generality $|B_i \cap Q| > |B_j \cap Q|$. Since voter i is happy with Q it holds that $|B_i \cap Q| \geq |Q|/2 + 1$. Then, Q contains at least $|Q|/2$ proposals which are not p and thus not in B_j; a contradiction to the fact that voter j is happy. □

The next observation basically says that UNAAB can be many-one reduced in polynomial time to MAJAB with the same agenda. This implies that the "majority problem" is computationally at least as hard as the "unanimous problem".

$\mathcal{U} = \{1, \mathbf{2}, 3, 4, \mathbf{5}\}$

$\mathcal{S} = \{\ S_1 = \{\mathbf{2}, 4\}$
$S_2 = \{3, \mathbf{5}\}$
$S_3 = \{1, \mathbf{2}, 3\}$
$S_4 = \{4, \mathbf{5}\}\ \}$

$k = 2$
$r = 4$

	1	2	3	4	5	d_1	d_2	α
B_1	-	+	-	+	-	+	+	-
B_2	-	-	+	-	+	+	+	-
B_3	+	+	+	-	-	+	+	-
B_4	-	-	-	+	+	+	+	-
B_5	+	+	+	+	+	-	-	+
B_6	-	-	-	-	-	+	+	+

Fig. 2. Illustration of Reduction 1. Left: A HITTING SET instance with solution size $k = 2$ and number of subsets $r = 4$. The solution $\{2, 5\}$ is highlighted in boldface. Right: Constructed UNAAB instance with dummy proposals d_1 and d_2, special proposal α, and the dummy voters 5 and 6. The solution $\{2, 5, d_1, d_2, \alpha\}$ corresponding to the HITTING SET solution is highlighted in gray.

OBSERVATION 2. *Let I_{una} be a* UNAAB *instance with n voters, and let I_{maj} be a* MAJAB *instance with $2n - 1$ voters such that*

— *I_{una} and I_{maj} both have the same proposal set $\mathcal{P}$ and the same agenda (Q_+, q_+),*
— *the voters from I_{una} and the first n voters from I_{maj} have the same favorite ballots $B_1, \ldots, B_n$, and*
— *the remaining $n - 1$ voters from I_{maj} support no proposals.*

Then, $Q \subseteq \mathcal{P}$ is a solution for I_{una} if and only if Q is a solution for I_{maj}.

PROOF. For the "only if" part, assume there is a ballot Q with $|Q_+ \cap Q| \geq q_+$ and Q is accepted by all n voters from I_{una}. Then, the first n voters from I_{maj} also accept Q and I_{maj} has altogether $2n - 1$ voters. For the "if" part, observe that only the first n voters from I_{maj} can be happy with any ballot Q, because the remaining voters do not support any proposal. Hence, every Q that is accepted by a majority of voters from I_{maj} is accepted by all voters from I_{una}. □

We will use the NP-complete HITTING SET problem [Garey and Johnson 1979] to prove many of our intractability results. Given a finite set U, subsets $S_1, \ldots, S_r$ of U, and a nonnegative integer k, HITTING SET asks whether there is a *hitting set* of size k, that is, whether there is a size-k set $U' \subseteq U$ such that $S_i \cap U' \neq \emptyset$, $i \in \{1, \ldots, r\}$. The following reduction from HITTING SET to UNAAB is used several times in our intractability proofs. Note that, due to Observation 2, it implies a reduction to MAJAB.

Reduction 1. Let $(U, S_1, \ldots, S_r, k)$ be an instance of HITTING SET. Construct an instance of UNAAB as follows. The proposal set $\mathcal{P}$ consists of all the elements of U, of k new dummy proposals, and of a special proposal α. There are $r + 2$ voters. For $1 \leq i \leq r$, the favorite ballot B_i consists of the elements from S_i together with all dummy proposals. Furthermore, $B_{r+1} := U \cup \{\alpha\}$ and B_{r+2} consists of α together with all dummy proposals. Finally, set $Q_+ := \emptyset$ and $q_+ := 0$.

Reduction 1 is illustrated by an example in Figure 2.

LEMMA 1. *Reduction 1 is a parameterized reduction where the parameters h, n, and m are linearly bounded in the parameters k, r, and $|U|$, respectively. More precisely, $h = 2k + 1$, $n = r + 2$, and $m = |U| + k + 1 \leq 2|U| + 1$.*

PROOF. The instance constructed in Reduction 1 has $m = |U| + k + 1$ proposals and $n = r + 2$ voters. We will see in the following that the size h of the solution ballot Q is $2k+1$. The reduction runs in $O(|U| \cdot r)$ time. Since $Q_+ = \emptyset$, any ballot that makes every voter happy is a solution. Thus, it remains to show that $(U, S_1, \ldots, S_r, k)$ has a hitting set of size k if and only if there is a ballot $Q \subseteq \mathcal{P}$ with $|Q| \leq 2k + 1$ that makes every voter happy.

For the "only if" part, let H be a hitting set of size k, and let Q contain the k dummy proposals, the special proposal α, and the proposals in H. Then $|Q| = 2k + 1$, and it is easily seen that $|B_i \cap Q| \geq k + 1$ holds for all voters i.

For the "if" part, by applying Observation 1(ii) to B_{r+1} and B_{r+2}, ballot Q contains the special proposal α and furthermore Q contains the same number x of proposals from U as from the dummy proposals. Then $x \leq k$, as there are only k dummy proposals. For $1 \leq i \leq r$, the intersection $Q \cap B_i$ must contain at least one proposal from S_i and hence $Q \cap S_i \neq \emptyset$. Hence the $x \leq k$ elements in $Q \cap U$ form a hitting set. □

2.1. NP-completeness

We show that UNAAB and MAJAB are NP-complete even without hidden agenda. This implies that there is no hope for fixed-parameter tractability when parameterizing by $|Q_+|$ or q_+.

THEOREM 2.1. *Both* UNANIMOUSLY ACCEPTED BALLOT *and* MAJORITYWISE ACCEPTED BALLOT *are NP-complete even without hidden agenda.*

PROOF. Both problems are in NP: Checking whether a strict majority of voters or all voters are happy with a given proposal set Q such that $|Q_+ \cap Q| \geq q_+$ takes polynomial time. For the hardness proof, due to Observation 2, it is sufficient to show that UNAAB is NP-hard even without hidden agenda. This is achieved due to Lemma 1 and the fact that Reduction 1 is also a polynomial-time many-one reduction. □

2.2. Few proposals or few voters

Complementing our intractability result from Theorem 2.1, we show that instances with few proposals or few voters are tractable. More precisely, we show fixed-parameter tractability, that is, the considered problems are polynomial-time solvable for a fixed number of proposals or a fixed number of voters and the degree of the polynomial is a constant. However, we also show that under plausible complexity-theoretic assumptions these problems do not admit polynomial-time preprocessing algorithms that reduce the size of an instance to be polynomially bounded by the the number m of proposals or the number n of voters. In other words, UNAAB and MAJAB are unlikely to allow for polynomial kernels with respect to the parameters n or m, respectively.

THEOREM 2.2. *Parameterized by the number m of proposals,* UNANIMOUSLY ACCEPTED BALLOT *and* MAJORITYWISE ACCEPTED BALLOT *are fixed-parameter tractable. Unless NP $\subseteq$ coNP/poly, both problems do not admit a polynomial kernel even without hidden agenda.*

PROOF. A straightforward brute-force algorithm running in $O(2^m \cdot nm)$ time simply tries all ballots $Q \subseteq \mathcal{P}$ and checks for each ballot whether it is a solution for UNAAB, that is, whether $|Q_+ \cap Q| \geq q_+$ and $|Q \cap B_i| > |Q|/2$ for each voter i. Hence, UNAAB is in FPT. Analogously, MAJAB is also in FPT.

Unless NP ⊆ coNP/poly, both problems do not have a polynomial kernel with respect to the parameter m even without hidden agenda: Reduction 1 is a polynomial-time reduction from the NP-complete HITTING SET problem; the number m of proposals in the reduced instance equals $|U| + k + 1$; and $Q_+ = \emptyset$. A polynomial kernel of UNAAB parameterized by m would yield a polynomial kernel of HITTING SET parameterized by $|U| + k$. However, Dom et al. [2014] showed that the latter admits no polynomial kernels unless NP ⊆ coNP/poly.

Thus, even without hidden agenda, UNAAB does not admit a polynomial kernel with respect to the parameter m. Due to Observation 2, the non-existence of polynomial kernels transfers to MAJAB even without hidden agenda. □

THEOREM 2.3. *Parameterized by the number n of voters,* UNANIMOUSLY ACCEPTED BALLOT *and* MAJORITYWISE ACCEPTED BALLOT *are fixed-parameter tractable. Unless NP ⊆ coNP/poly, both problems do not admit a polynomial kernel even without hidden agenda.*

PROOF. We first describe how to formulate MAJAB as an integer linear program (ILP) and show how to modify the ILP to also work for UNAAB. Let N_V be the number of proposals that are accepted by the voter set V and rejected by the rest, that is, $N_V := |\{j \mid (\forall i \in V : j \in B_i) \wedge (\forall i' \notin V : j \notin B_{i'})\}|$. As the proposals counted by N_V only depend on V, we refer to V as a *proposal type*. Let x_V be the number of proposals of type V in the ballot Q. Further, let N_V^+ be the number of proposals in Q_+ that are accepted by the voter set $V \subseteq \mathcal{V}$ and rejected by the rest, that is, $N_V^+ := |Q_+ \cap \{j \mid (\forall i \in V : j \in B_i) \wedge (\forall i' \notin V : j \notin B_{i'})\}|$. Since ballot Q must contain at least q_+ proposals in Q_+, we introduce one variable x_V^+ for each proposal type V. Variable x_V^+ denotes the number of proposals of type V that are in ballot Q as well as in ballot Q_+. For each voter i we introduce a binary variable z_i that may only have value 1 if voter i is happy with Q. Then Q must satisfy the following constraints (1)–(5).

$$\sum_{i=1}^{n} z_i \geq \frac{n+1}{2} \tag{1}$$

$$\sum_{V \subseteq \mathcal{V}} x_V^+ \geq q_+ \tag{2}$$

$$\sum_{\substack{V \subseteq \mathcal{V}: \\ i \in V}} x_V - \sum_{\substack{V \subseteq \mathcal{V}: \\ i \notin V}} x_V \geq 1 - (m+1)(z_i - 1) \qquad \forall i \in \{1, \ldots, n\} \tag{3}$$

$$N_V^+ \geq x_V^+ \geq 0 \qquad \forall V \subseteq \mathcal{V} \tag{4}$$

$$N_V \geq x_V \geq x_V^+ \qquad \forall V \subseteq \mathcal{V} \tag{5}$$

Constraint (1) requires that a strict majority of voters is happy with Q, while Constraint (2) requires that the number of proposals in $Q \cap Q_+$ is at least q_+. Constraint set (3) ensures that if variable z_i is set to 1, then voter i is happy: if $z_i = 1$, then the right-hand side equals 1 and the number of proposals which voter i accepts (that is, $\sum_{V \subseteq \mathcal{V}: i \in V} x_V$) is larger than the number of proposals which voter i rejects (that is, $\sum_{V \subseteq \mathcal{V}: i \notin V} x_V$). If $z_i = 0$, then the right-hand side equals $-m$ and the constraint is fulfilled even if voter i rejects the all proposals. Constraint set (4) (respectively Constraint set (5)) restricts the number of proposals of each type in $Q \cap Q_+$ (respectively in Q) to those actually present. Moreover, Constraint set (5) expresses the relation of variables x_V^+ and x_V. It requires that the number of proposals of each type in Q is at least the number of proposals of this type in both Q and Q_+.

Our ILP contains at most 2^n variables x_V, 2^n variables x_V^+, and n variables z_i. The total number of constraints is at most $2 \cdot 2^n + n + 2$. Since the "integer feasibility problem" (that is, an ILP without objective function) with ρ variables and L input bits can be solved in $O(\rho^{2.5\rho+o(\rho)}L)$ time [Lenstra 1983; Kannan 1987; Frank and Tardos 1987], MAJAB is fixed-parameter tractable with respect to the number n of voters.

If we delete Constraint (1) and the variables z_i, and replace the right-hand sides of Constraint set (3) with 1, then we gain an ILP for UNAAB with at most $2 \cdot 2^n$ variables and $2 \cdot 2^n + n + 1$ constraints. Thus, UNAAB is also fixed-parameter tractable with respect to parameter n.

Unless NP $\subseteq$ coNP/poly, even without hidden agenda, both problems do not have a polynomial kernel with respect to the parameter n: Reduction 1 is a polynomial-time reduction from the NP-complete problem HITTING SET; the number n of voters in the reduced instance is linearly bounded by the number r of sets in the instance one reduces from; and $Q_+ = \emptyset$. A polynomial kernel of UNAAB with $Q_+ = \emptyset$ parameterized by n would yield a polynomial kernel for HITTING SET parameterized by r. However, this is not possible unless NP $\subseteq$ coNP/poly (cf. [Hermelin et al. 2013, Theorem 5]). Thus, even without hidden agenda, UNAAB does not admit a polynomial kernel. Neither does MAJAB admit a polynomial kernel even without hidden agenda due to Observation 2. □

2.3. Small Ballots

In this subsection, we perform a parameterized complexity analysis concerning parameters based on the ballot sizes. We start with the size h of the solution ballot. For technical reasons, we need to assume that h is given as part of the input when dealing with the parameterized problems and denotes an upper bound for the size of the solution ballot Q.

To show parameterized intractability for the parameter h, we use the following lemma that basically says that for UNAAB we may assume without loss of generality that ballot Q has size exactly h.

LEMMA 2. *Given an instance I of* UNANIMOUSLY ACCEPTED BALLOT*, we define a modified instance I' by duplicating each voter and by adding $\lceil h/2 \rceil$* dummy *proposals and $\lceil h/2 \rceil$* copy *proposals so that each original voter additionally supports all dummy proposals and each duplicate voter additionally supports all the copy proposals. Then, there is a unanimously accepted ballot of size at most h for I if and only if there is a unanimously accepted ballot of size exactly h for I'.*

PROOF. For the "only if" part, suppose that a ballot Q with $|Q| \leq h$ is accepted by all voters in I. We assume that $|Q|$ and h are odd since every voter that is happy with a ballot of even size is still happy if one removes an arbitrary proposal. Then, adding $(h - |Q|)/2$ dummy proposals and $(h - |Q|)/2$ copy proposals to Q results in a ballot of size h that is accepted by all voters in I'.

For the "if" part, suppose that a ballot Q' with $|Q'| = h$ is accepted by all voters in I'. Then delete all dummy and copy proposals from Q' to obtain a ballot Q that is accepted by all voters in I. By applying Observation 1(i) to any original voter and its duplicate one knows that Q is non-empty. Assume towards a contradiction that there is a voter in I that is not happy with Q. Since the corresponding voter in I' is happy with Q', ballot Q' contains more dummy proposals than copy proposals. Then, the corresponding duplicate voter is not happy with Q', a contradiction.

	p_1	$\boldsymbol{p_2}$	$\boldsymbol{p_3}$	$\boldsymbol{p_4}$
B_1	+	**+**	**-**	**-**
B_2	+	**-**	**+**	**-**
B_3	-	**+**	**+**	**+**

$h = 3$, $Q_+ = \{p_2, p_3, p_4\}$, $q_+ = 2$
$[h] = \{1, 2, 3\}$, $\mathcal{H} = \{H_1, H_2, H_3\}$
$H_1 = \{1, 2\}$, $H_2 = \{2, 3\}$, $H_3 = \{1, 3\}$

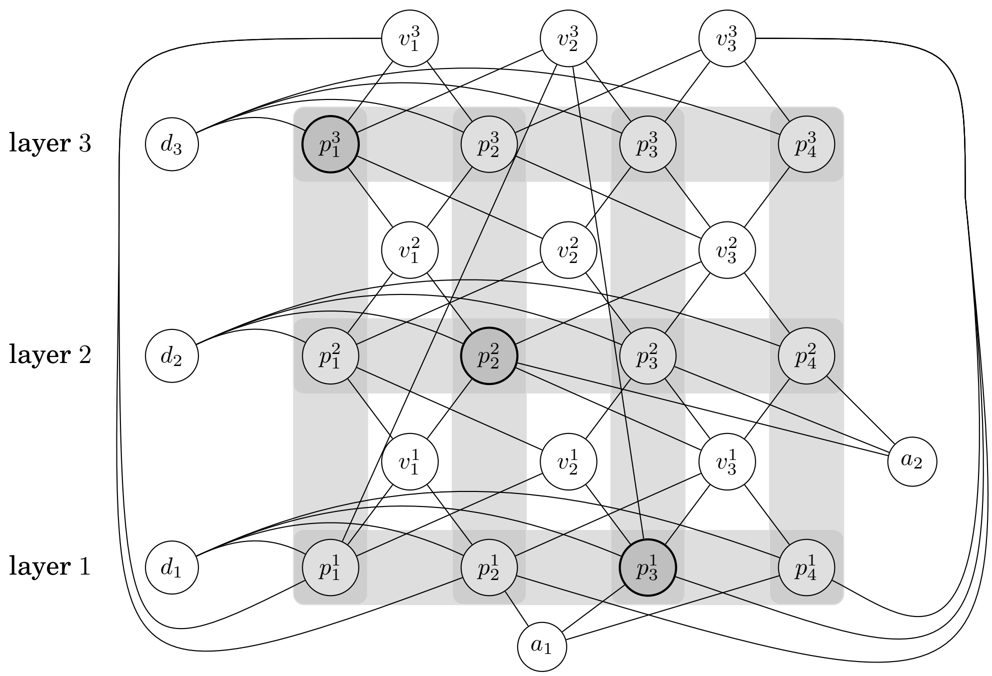


Fig. 3. Illustration of the reduction from UNAAB to INDEPENDENT DOMINATING SET (IDS). Top: UNAAB instance with four proposals and three voters and the corresponding auxiliary set family $\mathcal{H}$. All entries in columns corresponding to proposals from the agenda are written in boldface. All voters are happy with the ballot $\{p_1, p_2, p_3\}$. Bottom: The corresponding graph of the IDS instance. Vertices inside gray bars form cliques. The three vertices p_1^3, p_2^2, and p_3^1 form an independent dominating set.

Note that I and I' have the same agenda (Q_+, q_+) (we initially copied I to obtain I') and adding or removing proposals that are not in Q_+ from a ballot Q has no influence on whether $|Q_+ \cap Q| \geq q_+$. □

Due to Lemma 2 we can assume without loss of generality that given any upper bound h on the size of the solution ballot every unanimously accepted ballot has size exactly h. We use this in the W[2]-membership proof for UNAAB parameterized by h leading to the following theorem.

THEOREM 2.4. *Parameterized by the size h of the solution ballot,* UNANIMOUSLY ACCEPTED BALLOT *is W[2]-complete and* MAJORITYWISE ACCEPTED BALLOT *is W[2]-hard. Both results hold even without hidden agenda.*

PROOF. Due to Lemma 1, Reduction 1 is a parameterized reduction from the W[2]-hard HITTING SET parameterized by the size k of the hitting set to UNAAB parameterized by the size h of the solution ballot without hidden agenda. Because of Observation 2, this implies W[2]-hardness for MAJAB parameterized by the size h of the solution ballot even without hidden agenda.

To show that UNAAB is in W[2], we describe a parameterized reduction from UNAAB parameterized by the size h of the solution ballot to INDEPENDENT DOMINATING SET (IDS) parameterized by the solution size k. (The W[2]-membership of IDS follows by a straightforward reduction to WEIGHTED WEFT-t CIRCUIT SATISFIABILITY [Downey and Fellows 2013].) Given an undirected graph $G = (U, E)$ and an integer k, IDS asks whether there is an *independent dominating set* of k vertices, that is, whether there exists a $U' \subseteq U$ with $|U'| = k$ such that no pair of vertices from U' is adjacent and each vertex from $U \setminus U'$ is adjacent to at least one vertex from U'.

In the following, we use $[z]$ to denote the set $\{1, \ldots, z\}$ with $z \in \mathbb{N}$. Let $I := (\mathcal{P}, \mathcal{V}, (Q_+, q_+), h)$ be an instance of UNAAB with n voters and m proposals where h denotes an upper bound for the size of the solution ballot. Let $B_1, \ldots, B_n$ denote the favorite ballots of the voters. We construct an instance $I' = ((U, E), k)$ of IDS as follows. The vertex set U consists of proposal vertices, voter vertices, agenda vertices, and dummy vertices. There are h layers of *proposal vertices* each containing one vertex for each proposal. We say that the proposal vertex p_j^ℓ corresponds to proposal j in layer ℓ. We enumerate all subsets of $[h]$ of size $\lceil h/2 \rceil$. In the following let H_x denote the xth of these subsets and set $\mathcal{H} := \{H_1, \ldots, H_{h^*}\}$ with $h^* = \binom{h}{\lceil h/2 \rceil}$. For each voter i there is one *voter vertex* for each element in $\mathcal{H}$. We say that voter vertex v_i^s corresponds to voter i and subset H_s. There are q_+ *agenda vertices* $a_1, \ldots, a_{q_+}$ and h *dummy vertices* $d_1, \ldots, d_h$.

The edge set E is constructed as follows. Two proposal vertices are adjacent if they correspond to the same proposal or are in the same layer, that is, $\{p_j^\ell, p_{j'}^{\ell'}\} \in E$ if $j = j'$ or $\ell = \ell'$. Each dummy vertex d_ℓ is adjacent to all proposal vertices in layer ℓ, that is, $\{d_\ell, p_j^\ell\} \in E$ for each $j \in [m]$. Proposal vertex p_j^ℓ and voter vertex v_i^s are adjacent if voter i supports proposal j and $\ell \in H_s$. Proposal vertex p_j^ℓ and agenda vertex a_ℓ are adjacent if proposal j is from Q_+.

We set the size k of the independent dominating set to h. This completes the construction which is illustrated in Figure 3.

Now, we highlight two properties of the constructed instance which help to prove the correctness of the reduction. First, the proposal vertices from the same layer and the proposal vertices corresponding to the same proposal form a complete subgraph, respectively. Hence, an independent dominating set may contain at most one proposal vertex from each layer and no two proposal vertices corresponding to the same proposal. Second, taking a different proposal vertex from each layer into the dominating set is the only way to form an independent set of size k such that all dummy vertices and all other proposal vertices are dominated.

Due to Lemma 2 it remains to show that every voter in I is happy with some ballot Q with $|Q| = h$ and $|Q_+ \cap Q| \geq q_+$ if and only if the constructed graph has an independent dominating set of size $k = h$.

For the "only if" part, suppose that every voter is happy with the ballot $Q := \{j_1, \ldots, j_h\}$ and $|Q_+ \cap Q| \geq q_+$. Without loss of generality let $\{j_1, \ldots, j_{q_+}\} \subseteq Q_+$, that is, we fix an ordering of the proposals in Q such that the first q_+ proposals are from Q_+. We show that the vertex set $U' = \{p_{j_1}^1, \ldots, p_{j_h}^h\}$ is an independent dominating set for I. As discussed above, U' is an independent set and all dummy vertices as well as proposal vertices are either in U' or adjacent to a vertex in U'. So, suppose for the sake of contradiction that a voter vertex v_i^s is not dominated by U'. This means that $U' \cap \{p_j^\ell \mid j \in B_i \wedge \ell \in H_s\} = \emptyset$. Hence, at most $\lfloor h/2 \rfloor$ layers may contain proposal vertices p_j^ℓ with $j \in B_i$ that are in vertex set U'. Since U' does not contain two proposal vertices from the same layer, there are at most $\lfloor h/2 \rfloor$ proposal vertices p_j^ℓ in U' with $j \in B_i$. Thus, voter i is not happy with Q, a contradiction. Analogously, suppose that agenda vertex a_ℓ is not dominated by U'. However, we already know that $p_{j_\ell}^\ell \in Q_+$ and there is an edge between $p_{j_\ell}^\ell$ and a_ℓ, a contradiction.

For the "if" part, due to the observations above we know that the vertices forming an independent dominating set must be proposal vertices (from h different layers). Now, suppose that $U' := \{p_{j_1}^1, \ldots, p_{j_h}^h\}$ is an independent dominating set of size h. First, we show that every voter in I is happy with $Q := \{j_1, \ldots, j_h\}$. The ballot Q is of size h because U' is an independent set, and hence, there are no two proposals vertices in U' corresponding to the same proposal. Suppose for the sake of contradiction that voter i is not happy with Q. Then $|Q \cap B_i| \leq \lfloor h/2 \rfloor$ which means $|Q \setminus B_i| \geq \lceil h/2 \rceil$. Let $X = \{\ell \in [h] \mid p_j^\ell \in U' \wedge j \in Q \setminus B_i\}$ and let v_i^s be a voter vertex with $H_s \subseteq X$. This vertex exists since $|X| \geq \lceil h/2 \rceil$. Let p_j^ℓ be a proposal vertex in U' that dominates v_i^s. Then $\ell \in H_s \subseteq X$ and $j \in B_i$, a contradiction. Second, we show that $|Q \cap Q_+| \geq q_+$. Recall that U' contains h proposal vertices from h different layers corresponding to h different proposals. In particular, this implies that the q_+ agenda vertices are adjacent to q_+ proposal vertices from U' corresponding to pairwise different proposals. Hence, Q contains q_+ proposals from the agenda set Q_+. □

The membership of MAJAB parameterized by the size h of the solution ballot for the class W[2] remains open. Note that the W[2]-hardness reduction in the proof of Theorem 2.4 does not rely on (an upper bound for) h being given as part of the input. That is, the problem is computationally hard also for the cases where the size of ballot Q is not explicitly required to be bounded by h.

Except for the parameter h where we only know that MAJAB is W[2]-hard while UNAAB is even W[2]-complete, all results shown so far are the same for unanimous acceptance and majority acceptance. The following theorem shows that this changes when considering the parameter $b_{\max}$ where UNAAB remains fixed-parameter tractable but for MAJAB we show W[1]-completeness.

THEOREM 2.5. *Parameterized by the maximum size $b_{\max}$ of the favorite ballots,*

(*1*) UNANIMOUSLY ACCEPTED BALLOT *can be solved in $O(b_{\max}^{2b_{\max}} \cdot nm)$ time implying fixed-parameter tractability; however, even without hidden agenda it admits no polynomial kernel unless NP ⊆ coNP/poly, and*
(*2*) MAJORITYWISE ACCEPTED BALLOT *parameterized by $b_{\max}$ is W[1]-complete.*

In the remainder of this subsection we prove Theorem 2.5. The non-existence of a polynomial kernel for UNAAB with respect to parameter m shown in Theorem 2.2 also holds for parameter $b_{\max}$, as $b_{\max} \leq m$. Although there is no hope for polynomial kernels, we at least show fixed-parameter tractability for UNAAB parameterized by $b_{\max}$ by a depth-bounded search tree algorithm.

LEMMA 2.6. UNANIMOUSLY ACCEPTED BALLOT *can be solved in $O(b_{\max}^{2b_{\max}} \cdot nm)$ time.*

PROOF. Let $I = (\mathcal{P}, \mathcal{V}, (Q_+, q_+))$ denote an instance of UNAAB with $|B_i| \leq b_{\max}, i \in \mathcal{V}$. Observe that any ballot Q that makes every voter happy contains at most $2b_{\max} - 1$ proposals, otherwise the intersection of Q and any favorite ballot has at most $b_{\max}$ proposals but $b_{\max} \leq |Q|/2$. Using this observation, we describe a depth-bounded search tree algorithm solving the optimization version of UNAAB, that is, it computes a solution ballot Q with the largest intersection $Q \cap Q_+$ such that every voter is happy with Q, or it returns 'no' if there is no such ballot. The algorithm works as follows. Start with branching over the upper bound size $h \in \{1, \ldots, 2b_{\max} - 1\}$ of the solution and initialize $Q \leftarrow \emptyset$ in each branch. Repeat the following until all voters are *satisfied* or $|Q| = h$: Mark every voter i with $|B_i \cap Q| > \lfloor h/2 \rfloor$ as satisfied and branch into adding one proposal from $B_j \setminus Q$ to Q for an arbitrary unsatisfied voter j. Finally, if all voters are satisfied, then the computed ballot Q makes every voter happy since $|B_i \cap Q| > \lfloor h/2 \rfloor \geq |Q|/2$ for each $i \in \mathcal{V}$; otherwise discard this branch, because this path of the search tree cannot lead to a solution as the size of ballot Q reaches the upper bound h but there is some voter j with $|B_j \cap Q| \leq \lfloor h/2 \rfloor$. Finally, if $|Q| < h$, then fill up Q with $h - |Q|$ arbitrary proposals from Q_+. It is easy to verify that the ballot Q having the largest intersection with Q_+ among all ballots in the leaves of the search tree is an optimal solution, that is, among all possible ballots that make all voters happy, ballot Q also has the largest intersection with Q_+.

The search tree has depth at most $2b_{\max}$, since the size of Q is increased in each branching step and $|Q| \leq 2b_{\max}$. The number of branching possibilities in each step is at most $b_{\max}$. Altogether, the algorithm takes $O(b_{\max}^{2b_{\max}} \cdot nm)$ time, because each branching step needs $O(nm)$ time. □

In contrast to UNAAB, MAJAB becomes W[1]-complete for the parameter $b_{\max}$. We first prove the W[1]-hardness by a reduction from MAJORITY VERTEX COVER.

LEMMA 2.7. MAJORITYWISE ACCEPTED BALLOT *parameterized by the maximum size $b_{\max}$ of the favorite ballots is W[1]-hard.*

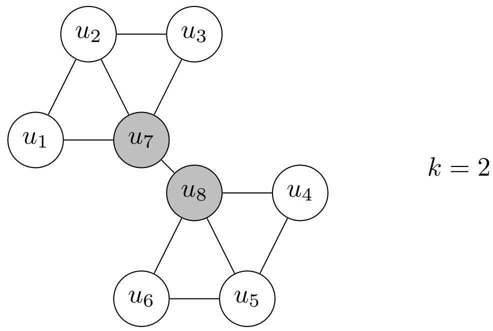
PROOF. We describe a parameterized reduction from the W[1]-hard MAJORITY VERTEX COVER (MVC) problem [Fellows et al. 2010] (cf. [Guo et al. 2007]). Given an undirected graph $G = (U, E)$ and an integer k, MVC asks whether there is a subset of k vertices which covers a majority of the edges of G, that is, is there a size-k subset $U' \subseteq U$ with $|\{e \in E \mid e \cap U' \neq \emptyset\}| > |E|/2$?

Let (G, k) be an instance of MVC with U denoting the set of vertices and $E = \{e_1, \ldots, e_s\}$ denoting the set of edges. Now, construct a MAJAB instance as follows. The proposal set $\mathcal{P}$ is defined as $U \cup D$ with D being a set of $k - 1$ dummy proposals. For each edge $e_j \in E$ there is one voter with the favorite ballot $B_j = e_j \cup D$. Furthermore, the agenda consists of $Q_+ = U$ and $q_+ = k$. This completes the construction which runs in polynomial time. It is illustrated by an example in Figure 4. Next, we show that (G, k) is a yes-instance for MVC if and only if the constructed instance is a yes-instance for MAJAB.

For the "only if" part, suppose that there is a subset $U' \subseteq U$ of k vertices covering more than $s/2$ edges. Then, $Q = U' \cup D$ is a solution for our MAJAB instance: Furthermore, $|Q| = 2k - 1$ and $|Q_+ \cap Q| = k$. Since $|B_j \cap Q| = |D| + |U' \cap e_j| \geq k$ for each covered edge e_j, more than $s/2$ voters are happy.

For the "if" part, we first assume that $s/2 \geq \binom{k+2}{2}$. Otherwise, add to the graph G a tree consisting of a root with $\binom{k+3}{2}$ children each of which has a single leaf. This results in an equivalent instance $((U', E'), k' = k + 1)$ with $|E'|/2 = s/2 + \binom{k'+2}{2} \geq \binom{k'+2}{2}$.

Suppose that there is a ballot $Q \subseteq \mathcal{P}$ which a majority of voters is happy with, and $|Q \cap Q_+| \geq k$. Note that adding all dummy proposals to Q also results in a feasible solution. Thus, we assume that Q contains all $k - 1$ dummy proposals. Then, ballot Q must contain at least one vertex proposal in each of the happy voters' favorite ballots. This implies that the edges corresponding to the happy voters are covered by the vertices corresponding to the voters in Q. As a majority of voters is happy, a majority of edges



	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	d_1
$\{u_1, u_2\}$	+	+	-	-	-	-	-	-	+
$\{u_2, u_3\}$	-	+	+	-	-	-	-	-	+
$\{u_1, u_7\}$	+	-	-	-	-	-	+	-	+
$\{u_2, u_7\}$	-	+	-	-	-	-	+	-	+
$\{u_3, u_7\}$	-	-	+	-	-	-	+	-	+
$\{u_7, u_8\}$	-	-	-	-	-	-	+	+	+
$\{u_4, u_5\}$	-	-	-	+	+	-	-	-	+
$\{u_5, u_6\}$	-	-	-	-	+	+	-	-	+
$\{u_4, u_8\}$	-	-	-	+	-	-	-	+	+
$\{u_5, u_8\}$	-	-	-	-	+	-	-	+	+
$\{u_6, u_8\}$	-	-	-	-	-	+	-	+	+

Fig. 4. Illustration of the reduction from MAJORITY VERTEX COVER (MVC) to MAJAB. Top: The graph of the MVC instance. Vertices u_7 and u_8 cover seven out of eleven edges. Bottom: The corresponding MAJAB instance. The agenda is $(\{u_1, \ldots, u_8\}, 2)$. Seven out of eleven voters (as highlighted) are happy with the ballot $\{u_7, u_8, d_1\}$.

is covered. Finally, we show that the number of vertex proposals in Q is exactly k. Observe that ballot Q can have at most $k+2$ vertex proposals since otherwise no voter is happy: each voter supports at most two vertex proposals and $k-1$ dummy proposals, and hence, a solution cannot have more than k unsupported vertex proposals. Furthermore, there are at most $\binom{k+2}{2}$ happy voters such that both their vertex proposals are contained in Q. But since $s/2 \geq \binom{k+2}{2}$, there must be at least one happy voter j such that only one vertex proposal in his favorite ballot B_j is contained in Q. This implies that Q can only be of size $2k-1$, and hence, contains exactly k vertex proposals, since otherwise voter j is not happy. $\square$

As the most technical part of the proof of Theorem 2.5 we finally show that MAJAB is contained in W[1] for $b_{\max}$.

ALGORITHM 1: Nondeterministic part of program $\mathbb{P}$

```
Input: Proposal set P, the number n of voters, and table z where z[Q'] is the number of voters
       whose favorite ballots are supersets of Q'.
Guess a ballot Q ⊆ P with |Q+ ∩ Q| ≥ q+ and |Q| ≤ 2b_max;
Initialize z0 ← 0;
for s = min(b_max, |Q|) downto ⌈(|Q| + 1)/2⌉ do
    for each crucial subset Q' ⊆ Q of size s do
        z0 ← z0 + z[Q'];
        for each crucial subset Q'' ⊆ Q' do
            z[Q''] ← z[Q''] − z[Q'];
        end
    end
end
if z0 > n/2 then return 'yes'; else return 'no';
```

LEMMA 2.8. MAJORITYWISE ACCEPTED BALLOT *parameterized by the maximum size* $b_{\max}$ *of the favorite ballots is in W[1].*

PROOF. We use a characterization of W[1] [Flum and Grohe 2006, Theorem 6.22.] which states that a parameterized problem L with parameter k is in W[1] if and only if there is a *tail-nondeterministic* k*-restricted nondeterministic random access machine (NRAM)* program deciding L as follows: On each input (x, k), it

(a) performs at most $f(k) \cdot \text{poly}(|x|)$ steps with only the last $f(k)$ steps being nondeterministic,
(b) uses only the first $f(k) \cdot \text{poly}(|x|)$ registers, and
(c) has numbers of value at most $f(k) \cdot \text{poly}(|x|)$ in any register at any time

where f is a function only depending on the parameter k and poly is a polynomial function. (See Flum and Grohe [2006, Chapter 6] for further information about the machine characterization of problems in W[1].)

To show the W[1] containment, we describe a tail-nondeterministic $b_{\max}$-restricted NRAM program $\mathbb{P}$ to decide MAJAB. Given a MAJAB instance $I = (\mathcal{P}, \mathcal{V}, (Q_+, q_+))$ with $|\mathcal{P}| = m$, we say that a ballot $Q' \subseteq \mathcal{P}$ is *crucial* if it is a subset of the favorite ballot B_j for some voter $j \in \mathcal{V}$, that is, $\exists j \in \mathcal{V} : Q' \subseteq B_j$. The deterministic phase of $\mathbb{P}$ works as follows. For each crucial ballot Q' the program $\mathbb{P}$ counts and stores the number of voters whose favorite ballots are supersets of Q'; we denote this number as $z[Q']$ in the following. Since the maximum size of the favorite ballots is $b_{\max}$, table z can be filled in $f(b_{\max}) \cdot (nm)^c$ time where f is a function only depending on $b_{\max}$ and c is a constant. Every number in z has value at most n. The program $\mathbb{P}$ uses an additional counter z_0 to store the number of happy voters.

We remark that a straightforward implementation of table z would use $O(b_{\max} \cdot m^{b_{\max}})$ registers since a crucial ballot $Q' \subseteq \mathcal{P}$ can have up to $b_{\max}$ proposals. Furthermore, in the nondeterministic phase we assume that $\mathbb{P}$ can decide whether a ballot is crucial in $f(b_{\max})$ steps; we address both these issues at the end of the membership proof.

As for the nondeterministic phase of $\mathbb{P}$ (see Algorithm 1 for the pseudocode), note that any ballot making more than half of the voters happy contains at most $2b_{\max}$ proposals. Hence, the program $\mathbb{P}$ guesses a ballot Q with $|Q_+ \cap Q| \geq q_+$ and $|Q| \leq 2b_{\max}$. Checking whether Q is a solution works as follows.

(i) Take a *crucial* ballot $Q' \subseteq Q$ with $|Q'| > |Q|/2$ (note that $|Q'| \leq b_{\max}$),
(ii) increase the counter z_0 by $z[Q']$, and
(iii) decrease $z[Q'']$ for each *crucial* ballot $Q'' \subsetneq Q'$ with $|Q''| > |Q|/2$ by $z[Q']$.

The program repeats Steps (i)–(iii), considering all crucial ballots $Q' \subseteq Q$ satisfying the condition in Step (i) ordered by decreasing size and starting with a Q' of size $\min(b_{\max}, |Q|)$. The number of steps needed in the nondeterministic part is in $O(4^{b_{\max}})$ since the number of sets Q' fulfilling (i) and the number of subsets $Q'' \subseteq Q'$ with $|Q''| > |Q|/2$ are both upper-bounded by $2^{b_{\max}}$.

Finally, the program decides whether I is a yes-instance by checking whether $z_0 \geq \lfloor n/2 \rfloor + 1$. To show the correctness of $\mathbb{P}$, it remains to show that z_0 indeed equals the number of voters happy with Q. Let the happy voters be $v_1, \ldots, v_r$. By accounting for every crucial subset $Q' \subseteq Q$ of size at least $\lceil (|Q|+1)/2 \rceil$ ordered by decreasing size and reducing the entry $z[Q'']$ of any subset $Q'' \subsetneq Q'$ by $z[Q']$, we count every happy voter exactly once. That is, we partition the happy voters $v_1, \ldots, v_r$ into subsets $V_1, \ldots, V_s$ such that the favorite ballots of any two voters from the same subset have the same intersection with Q and the subsets with lower indices have larger intersections with Q. Then, in the ith iteration, program $\mathbb{P}$ adds the size of set V_i to counter z_0. In this way, counter z_0 sums up to the number of voters happy with Q.

Let us now explain how to implement table z using only the first $f(b_{\max}) \cdot \text{poly}(|I|)$ registers and how to decide whether a ballot is crucial. For detecting crucial ballots we show that it is sufficient to be able to have certificates for crucial ballots and for "pseudo-crucial" ballots. To this end, we additionally store $z[Q'] = 0$ for every ballot Q' that is *pseudo-crucial*, that is, for every ballot Q' that is not crucial itself but contains a proposal whose (single) removal would yield a crucial ballot. This enlarges the table z by a factor of at most m (as all candidates for being pseudo-crucial can be obtained by adding one of the m proposals to some crucial ballot). Note that a ballot is non-crucial if and only if it has a subset that is pseudo-crucial, because we can assume without loss of generality that every singleton from $\mathcal{P}$ is crucial. Thus, a pseudo-crucial subset $Q'' \subseteq Q'$ is a certificate for Q' being non-crucial.

The table implementation is based on the fact that $\mathbb{P}$ actually uses only $f(b_{\max}) \cdot \text{poly}(|I|)$ entries. To ensure that we only use the first $f(b_{\max}) \cdot \text{poly}(|I|)$ registers we store the entries of table z in an unordered fashion in the first registers, but we augment the entry for each of such ballots Q' by additionally storing Q' itself; each of these augmentations has size at most $O(b_{\max} \cdot \log m)$ for encoding up to $b_{\max}$ proposals in each ballot Q'. Note that, for simplicity, we could just use a pair of registers for each entry. Throughout the filling of the table, this makes no difference as we have $f(b_{\max}) \cdot \text{poly}(|I|)$ time available and can afford sequential search for entries, if needed.

The essential idea for querying $z[Q']$ is to use nondeterminism to guess the position of a required table entry in the unordered sequence of entries and to check whether the augmented entry stored in this entry corresponds to Q'. The essential idea for deciding whether a ballot Q' is crucial is to simply guess and search for a certificate showing that the guess was correct.

More precisely, $\mathbb{P}$ does the following nondeterministic preprocessing. For each subset $Q' \subseteq Q$, program $\mathbb{P}$ guesses whether Q' is crucial. Then $\mathbb{P}$ guesses

(a) the position of the table entry $z[Q']$ in the registers if Q' is guessed crucial or
(b) the position of the table entry $z[Q'']$ for some pseudo-crucial subset $Q'' \subseteq Q'$ if Q' is guessed non-crucial.

For Case (a), program $\mathbb{P}$ checks whether the augmented entry corresponds to ballot Q' and $z[Q'] > 0$: If this is the case, then the guess was correct and $\mathbb{P}$ stores the position of the table entry for later access; otherwise, $\mathbb{P}$ returns 'no'. For Case (b), program $\mathbb{P}$ checks whether the augmented entry corresponds to some pseudo-crucial subset of $Q'' \subseteq Q'$ (including Q' itself) and whether $z[Q''] = 0$: If this is the case, then Q'' is pseudo-crucial which means that Q' is non-crucial and that the guess was correct; otherwise, $\mathbb{P}$ returns 'no'.

It is easy to see that if we have a yes-instance and an appropriate ballot Q, then there exist correct guesses such that for at least one set of guesses the machine will answer 'yes', as needed. If we have a no-instance and $\mathbb{P}$ returns 'yes', then one of the guesses was wrong. However, this is not possible since $\mathbb{P}$ validates the correctness of each single guess or returns 'no'. □

2.4. Further Parameterizations

In the previous subsections, we discussed parameters whose motivation is quite clear already from their definitions as all these parameters measure something which can be relatively small in realistic instances. Next, we discuss a parameter that may not immediately seem interesting from its definition. More precisely, we consider the "below guarantee parameter" [Mahajan and Raman 1999] $b_{\text{gap}} = \lceil (m+1)/2 \rceil - \min_{i \in \mathcal{V}} |B_i|$ measuring the distance to trivial yes-instances. To this end, observe that an instance of UNAAB or MAJAB is a yes-instance if the minimum size of the favorite ballots is at least $\lceil (m+1)/2 \rceil$ where m denotes the total number of proposals in $\mathcal{P}$. However, both problems become NP-complete when this minimum size is one less than the guarantee $\lceil (m+1)/2 \rceil$, even without hidden agenda. This implies that there is no hope for fixed-parameter tractability with respect to the "below guarantee parameter" b_{gap}.

THEOREM 2.9. *Every instance of* UNANIMOUSLY ACCEPTED BALLOT *or* MAJORITYWISE ACCEPTED BALLOT *where each voter i satisfies $|B_i| > m/2$ is a yes-instance.* UNANIMOUSLY ACCEPTED BALLOT *and* MAJORITYWISE ACCEPTED BALLOT *are NP-complete even without hidden agenda and when each voter i satisfies $|B_i| > m/2 - 1$.*

PROOF. As for the first statement, choosing $Q = \mathcal{P}$ makes every voter happy. To show the second statement, we many-one reduce from the NP-complete VERTEX COVER (VC) problem. Given an undirected graph $G = (U, E)$ and an integer $k \leq |U|$, VC asks whether there is a *vertex cover* of at most k vertices, that is, whether there is a set $U' \subseteq U$ with $|U'| \leq k$ and for each $e \in E$ we have $e \cap U' \neq \emptyset$.

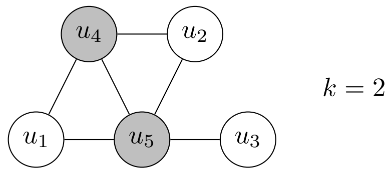
Let $I = ((U, E), k)$ with vertex set $U = \{u_1, \ldots, u_r\}$ and edge set $E = \{e_1, \ldots, e_s\}$ be a VC instance where we assume without loss of generality that $r \geq k+2$. We first reduce from it to an instance I' for UNAAB and then extend this reduced instance I' to an instance I'' for MAJAB.

Both instances I' and I'' have the same proposal set $\mathcal{P}$. It consists of one special proposal α, of all vertices in U, of k dummy proposals β_j ($1 \leq j \leq k$), and of $r-k$ additional dummy proposals $\gamma_{j'}$ ($1 \leq j' \leq r-k$). Thus, $|\mathcal{P}| = 2r + 1$.

Instance I' contains four types of voters: one voter v_0, one voter $\overline{v}_0$, s *edge voter*s, and $r-k$ *vertex hater*s. Voter v_0 supports proposal α and all the r dummy proposals. Voter $\overline{v}_0$ also supports proposal α, and all the vertices in U. For $1 \leq i \leq s$, the ith edge voter's favorite ballot A_i consists of the two vertices in e_i, of all the k dummy proposals β_j, and of $r-k-2$ arbitrarily chosen dummy proposals from $\{\gamma_1, \ldots, \gamma_{r-k}\}$. For $1 \leq i' \leq r-k$, the favorite ballot $B_{i'}$ of vertex hater i' consists of α and of all dummy proposals but $\gamma_{i'}$. In total, the number of voters in I' is $s + r - k + 2$, with each voter supporting at least $r = \lfloor |\mathcal{P}|/2 \rfloor$ proposals. Set $Q_+ := \emptyset$ and $q_+ := 0$. This reduction can be computed in polynomial time; it is illustrated with an example in Figure 5.

To show the reduction's correctness, we have to show that I has a vertex cover of size at most k if and only if there is a ballot $Q \subseteq \mathcal{P}$ that all the voters in I are happy with.

For the "only if" part, suppose that $U' \subseteq U$ with $|U'| \leq k$ is a vertex cover. We show that every voter is happy with $Q = \{\alpha\} \cup \{\beta_j \mid 1 \leq j \leq |U'|\} \cup U'$. First, the size of Q is $2|U'| + 1$. To make a voter happy, at least $|U'| + 1$ of his favorite proposals must be also in Q. Voters v_0, $\overline{v}_0$ and all vertex haters are happy with Q. For each $i \in \{1, \ldots, s\}$, $Q \cap A_i$ contains all dummy proposals β_j with $1 \leq j \leq |U'|$ and at least one vertex proposal $v_{j'}$



		u_1	u_2	u_3	u_4	u_5	α	β_1	β_2	γ_1	γ_2	γ_3
special voter	v_0	-	-	-	-	-	+	+	+	+	+	+
	$\overline{v}_0$	+	+	+	+	+	+	-	-	-	-	-
edge voter	$\{u_1, u_4\}$	+	-	-	+	-	-	+	+	+	-	-
	$\{u_1, u_5\}$	+	-	-	-	+	-	+	+	+	-	-
	$\{u_2, u_4\}$	-	+	-	+	-	-	+	+	+	-	-
	$\{u_2, u_5\}$	-	+	-	-	+	-	+	+	+	-	-
	$\{u_3, u_5\}$	-	-	-	+	-	+	+	+	+	-	-
	$\{u_4, u_5\}$	-	-	-	+	+	-	+	+	+	-	-
vertex hater	$1'$	-	-	-	-	-	+	+	+	-	+	+
	$2'$	-	-	-	-	-	+	+	+	+	-	+
	$3'$	-	-	-	-	-	+	+	+	+	+	-

Fig. 5. Illustration of the reduction from VERTEX COVER to UNAAB. Top: The graph of the VERTEX COVER instance. The vertices u_4 and u_5 cover all edges. Bottom: The corresponding UNAAB instance. All voters are happy with the ballot $\{u_4, u_5, \alpha, \beta_1, \beta_2\}$ (marked in gray).

with $v_{j'} \in e_i \cap U'$ since U' is a vertex cover. This sums up to at least $|U'| + 1$ proposals. Hence, every edge voter is also happy with Q.

For the "if" part, by applying Observation 1(ii) to the ballots of voters v_0 and $\overline{v}_0$, ballot Q must contain α, and furthermore, Q contains an equal number x of vertex proposals and dummy proposals. For each $i' \in \{1, \ldots, r-k\}$, ballot Q cannot contain dummy proposal $\gamma_{i'}$ since otherwise $|B_{i'} \cap Q| = x < \lfloor |Q|/2 \rfloor + 1$. Thus, vertex hater i' would not be happy. Therefore, the x dummy proposals must come from $\{\beta_1, \ldots, \beta_k\}$ and $x \leq k$. To make the ith edge voter happy, ballot Q must satisfy the condition $|Q \cap A_i| \geq x + 1$. But since no edge voter supports proposal α, ballot Q must contain at least one proposal $u_j \in A_i$. By definition of A_i, the corresponding vertex u_j is incident to edge e_i. This implies that the x vertices in Q form a vertex cover for (U, E).

Next, we extend instance I' to instance I'' for MAJAB by adding $r - k$ *vertex lovers* who have the same favorite ballot U, and s *edge-inverse voters* such that for $1 \leq i \leq s$, edge-inverse voter i's favorite ballot $C_i = (U \cup \{\gamma_1, \ldots, \gamma_{r-k}\}) \setminus A_i$. Thus, C_i and A_i are disjoint for all $1 \leq i \leq s$. In total, I'' has $2(s + r - k) + 2$ voters. Since each of the newly added voters supports exactly r proposals, the constraint that each voter's proposal set has at least $r = \lfloor |\mathcal{P}|/2 \rfloor$ holds. This extension can also be computed in polynomial time.

Now we show the correctness of the extended reduction, that is, I has a vertex cover of size at most k if and only if there is a ballot $Q \subseteq \mathcal{P}$ which more than half of the voters in I'' are happy with.

For the "only if" part, the ballot Q as constructed in the "only if" part above makes all voters in I' happy. This sums up to $s + r - k + 2$. Since I'' contains all the voters from I' and has $2(s+r-k)+2$ voters, this also means that more than half of the voters in I'' are happy with Q.

For the "if" part, for $1 \leq i \leq s$, the ith edge voter and the ith edge-inverse voter do not share a common favorite proposal. Furthermore, no vertex hater's favorite ballot intersects any vertex lover's favorite ballot. Hence, by applying Observation 1(i), any ballot can make at most s voters from the edge voters and the edge-inverse voters happy, and can make at most $r - k$ voters from the vertex haters and the newly constructed vertex lovers happy. But I'' has $2(s+r-k)+2$ voters. This means that in order to be a solution ballot for I'', Q must make both v_0 and $\overline{v}_0$ happy. By applying Observation 1(ii), Q must then contain α, and, furthermore, Q contains the same number x of vertex proposals and dummy proposals. The ballot Q cannot make any vertex lover happy since his favorite ballot and Q have an intersection of size x which is smaller than $\lfloor |Q|/2 \rfloor + 1$. Thus, Q needs to make all vertex haters happy. Then, Q cannot contain any dummy proposal $\gamma_{i'}$ since otherwise the vertex hater i' is not happy due to $|B_{i'} \cap Q| = x < \lfloor |Q|/2 \rfloor + 1$. Hence, Q contains x dummy proposals from $\{\beta_1, \ldots, \beta_k\}$ with $x \leq k$. Then, no edge-inverse voter is happy with Q since at most x proposals from his favorite ballot are in Q. This means that all edge voters must be happy with Q. To make the ith edge voter happy, Q must intersect with A_i in at least one vertex $u_j \in A_i$. By definition of A_i, the corresponding vertex u_j is incident to edge e_i. Thus, the x vertices in Q form a vertex cover for (U, E). □

We conclude this subsection with a brief discussion on the relation between the parameters "maximum size $b_{\max}$ of the favorite ballots" and "the size $h_{\max}$ of the maximum symmetric difference between any two favorite ballots". As the following proposition shows, for the cases without hidden agenda, the two parameters $h_{\max}$ and $b_{\max}$ are "equivalent" in terms of parameterized complexity theory: The fact that for two parameters x and y one has $x = \Theta(y)$ implies that the parameterization by x and the parameterization by y are in the same level of the W-hierarchy and yield the same parameterized hardness results.

PROPOSITION 2.10. *For any instance of* UNANIMOUSLY ACCEPTED BALLOT *or* MAJORITYWISE ACCEPTED BALLOT *it holds that* $h_{\max} \leq 2b_{\max}$*, where* $h_{\max}$ *denotes the size of the maximum symmetric difference between two favorite ballots and* $b_{\max}$ *denotes the maximum size of the given favorite ballots. Instances of* UNANIMOUSLY ACCEPTED BALLOT *or* MAJORITYWISE ACCEPTED BALLOT *are yes-instances if* $h_{\max} < b_{\max}/2$ *and* $Q_+ = \emptyset$*.*

PROOF. The first statement follows as $h_{\max}$ equals $\max_{i,j \in \mathcal{V}}(|B_i \setminus B_j| + |B_j \setminus B_i|)$, which is bounded by $2b_{\max}$. For the second statement, note that $h_{\max} < b_{\max}/2$ implies that every voter is happy with the favorite ballot B_ℓ of a voter ℓ with $|B_\ell| = b_{\max}$. To see this, consider some voter i . Now, $|B_i \cap B_\ell| = |B_\ell| - |B_\ell \setminus B_i| = b_{\max} - h_{\max} > b_{\max}/2$. □

3. COMBINATORIAL BOUNDS ON MINIMAL ACCEPTED BALLOTS

We say that a unanimously (resp. majoritywise) accepted ballot is *minimal* if no proper subset of it is also unanimously (resp. majoritywise) accepted. In this section, we investigate the largest possible size of a minimal unanimously accepted ballot for the situation with n voters and $Q_+ = \emptyset$. We derive (almost tight) upper and lower bounds on this quantity. From this bound, a similar result can be derived for majoritywise accepted ballots. Note that with a non-empty agenda set the size of a minimal unanimously (resp. majoritywise) accepted ballot cannot be expressed by a function only depending on n; it may additionally depend on q_+.

It is not hard to see that both upper and lower bounds come down to studying the case where the set $\mathcal{P}$ of all proposals already is a minimal accepted ballot: Such instances cannot have smaller solutions (giving a lower bound), and upper bounds directly carry over to $Q \subseteq \mathcal{P}$ by considering a restricted instance with $\mathcal{P}' := Q$. To make the question more amenable to combinatorial tools we translate it into a problem on a sequence of vectors with $\{-1, 1\}$-entries: Given n voters and m proposals we create m vectors $x_1, \ldots, x_m \in \{-1, 1\}^n$; the ith entry in vector x_j is 1 if the jth proposal is contained in the favorite ballot of voter i, else it is -1. In this formulation, a unanimously accepted ballot Q corresponds to a subset of the vectors whose vector sum is positive in each coordinate: Considering some voter i, for each proposal in $B_i \cap Q$ we incur 1, for each proposal in $Q \setminus B_i$ we incur -1. If $|B_i \cap Q| > |Q|/2$ then this gives a positive sum in coordinate i; the converse is true as well.

Let us normalize the question a little more. First of all, no minimal ballot can be of even size: Otherwise all coordinate sums would be even and hence each sum is at least 2; then however we may discard an arbitrary vector and still retain sums of at least 1 each. Secondly, it is clear that replacing $+1$ entries by -1 entries does not introduce additional subsequences with positive coordinate sums. Thus, we may restrict ourselves to the case where the coordinate sums over the minimal sequence of m vectors are all equal to 1 (the row sums over an odd number of $+1$ and -1 values are odd, and each replacement of a $+1$ by a -1 lowers the corresponding sum by 2).

Now, a collection of vectors is called a *minimal majority sequence of dimension* n (an n-mms for short) if all its coordinate-wise sums are 1 and no proper subsequence of the vectors has a positive sum in each coordinate. Note that an n-mms cannot contain a non-empty subsequence S whose sum is at most 0 in each coordinate, since otherwise the sum of the vectors that are in this n-mms but not in S must be positive in each coordinate—a contradiction to the minimality of an n-mms. Thus, the definition of an n-mms is equivalent to the condition that all its coordinate-wise sums are 1 and no non-empty subsequence has sum of at most 0 in each coordinate. The *length* of the sequence is the number m of its elements. Let $f(n)$ denote the maximum possible length of an n-mms. In this section, we show that $f(n) \approx n^{n/2+o(n)}$.

THEOREM 3.1. *The maximum possible length $f(n)$ of a minimal majority sequence of dimension n satisfies*

$$n^{n/2-o(n)} \leq f(n) \leq (n+1)^{(n+1)/2}.$$

The proof combines the main result of Alon and Vu [1997] with arguments from Linear Algebra, Geometry, and Discrepancy Theory. Before turning to the proof presented in Sections 3.1 and 3.2, let us give a corollary for the effect on UNANIMOUSLY ACCEPTED BALLOT and MAJORITYWISE ACCEPTED BALLOT.

COROLLARY 3.2. *Consider a* UNANIMOUSLY ACCEPTED BALLOT *instance with n voters. If there exists a unanimously accepted ballot, then there also exists one of size at most $(n+1)^{(n+1)/2}$. This bound is essentially tight, as there exist choices of accepted ballots such that any unanimously accepted ballot has size at least $n^{n/2-o(n)}$. For* MAJORITYWISE ACCEPTED BALLOT*, the corresponding upper and lower bounds are respectively $(t+1)^{(t+1)/2}$ and $t^{t/2-o(t)}$, where $t = \lceil (n+1)/2 \rceil$ denotes the majority threshold.*

PROOF. As the correspondence between favorite ballots and vector sequences has been thoroughly discussed above for the unanimous case, we now concentrate on the majority case.

To see the lower bound for the majority case, we start from a lower bound example for the unanimous case with t old voters and a minimum accepted ballot size of $t^{t/2-o(t)}$, and we add $n-t<n/2$ new voters with empty favorite ballots to it. Note that the resulting instance has a total of n voters and that its majority threshold indeed is t. Then any majoritywise accepted ballot must be unanimously accepted by the t old voters, so that the minimum majoritywise accepted ballot has size at least $t^{t/2-o(t)}$.

For the upper bound, consider any majoritywise accepted ballot Q for n voters and consider any minimal majority of t voters that (amongst themselves) unanimously accept this ballot. Then any other unanimously accepted ballot for these voters is also majoritywise accepted by all n voters, so that we get the desired upper bound of $(t+1)^{(t+1)/2}$ on the size of Q. □

In the remainder of this section, we prove Theorem 3.1 by first showing the upper bound (Section 3.1) and then the lower bound (Section 3.2).

3.1. Upper bound

One way to obtain an upper bound on $f(n)$ is to apply a known result of Sevastyanov [1978]. It asserts that any sequence of vectors whose sum is the zero vector, where the vectors lie in an arbitrary n-dimensional normed space R and each of them has norm at most 1, can be permuted so that all initial sums of the permuted sequence are of norm at most n. We will apply this result using the maximum norm l_∞. Thus, given an n-mms $v_1,\ldots,v_m\in\{-1,1\}^n$, append to it the vector $-\mathbf{1}$ where $\mathbf{1}$ is the all 1 vector of length n to get a zero-sum sequence of $m+1$ vectors in R^n, where the ℓ_∞ norm of each vector is 1. By the above mentioned result there is a permutation $u_1,u_2,\ldots,u_{m+1}$ of these vectors so that the ℓ_∞-norm of each initial sum $\sum_{i=1}^{j}u_i$ is at most n. If $m+1>(2n+1)^n$ then, by the pigeonhole principle, some two distinct initial sums are equal. It follows that summing over the set of vectors appearing only in the longer sequence gives 0. This implies a proper subsequence of the original mms with sum either $\mathbf{0}$ (if this set does not include the vector $-\mathbf{1}$), or $\mathbf{1}$ (if it does). In both cases, this contradicts the assumption that the original sequence is an mms. This shows that $f(n)\le(2n+1)^n$. See Alon and Berman [1986] for a similar argument.

The proof of the stronger upper bound stated in Theorem 3.1 is similar to that of a result of Huckeman, Jurkat, and Shapley (cf. [Graver 1973]) and is based on some simple facts from convex geometry. The details follow.

Let $u_1,u_2,\ldots,u_t$ be a sequence of pairwise distinct nonzero vectors in R^n, and let $C=C(u_1,\ldots,u_t)$ denote the *cone*

$$C=\left\{(x_1,x_2,\ldots,x_t)\mid x_i\ge 0\text{ and }\sum_{i=1}^{t}u_ix_i=0\right\}.$$

A point $x=(x_1,x_2,\ldots,x_t)\in C$ is called *basic* if the vectors $\{u_i\mid i\in I\}$ are affinely independent, where I is the support of x, i.e., the set $I=\{i\mid x_i>0\}$. Note that in this case $|I|\le n+1$, and the nonzero coordinates of the vector $(x_1,x_2,\ldots,x_t)$ form a solution of the system $\sum_{i\in I}u_ix_i=0$, which has a unique solution, up to a constant factor.

LEMMA 3. *Any vector $x=(x_1,x_2,\ldots,x_t)\in C$ is a positive linear combination of fewer than ℓ basic points, where $\ell\le t$ is the size of the support of x.*

PROOF. We apply induction on ℓ. The assertion is clear for $\ell \leq 2$. Indeed, there is no vector in C with support of size smaller than 2, as the vectors u_i are distinct non-zero vectors. Moreover, every vector of C with support of size 2 is basic, since any two distinct vectors u_i are affinely independent. Assuming the lemma holds for all $\ell' < \ell$ we prove it for ℓ. By assumption, $\mathbf{0}$ lies in the convex hull of the vectors $\{u_i \mid x_i > 0\}$. Let p be the minimum dimension of a facet of this convex hull that includes $\mathbf{0}$. The extreme points of this facet are $p+1$ of the vectors u_i, which are affinely independent. Therefore there is a basic point of the cone whose support is the set of these $p+1$ vectors. We can now subtract from x an appropriate positive multiple of this basic point, keeping x nonnegative and reducing the size of its support by at least 1. This completes the proof of the lemma. □

LEMMA 4. *Suppose that $u_i \in \{-1, 1\}^n$ for all i. Let y be an integral basic point of C, and suppose that the greatest common divisor of the nonzero coordinates of y is 1. Then the ℓ_1-norm of y, i.e. the sum of entries of y, is bounded from above by $2^{1-n}(n+1)^{(n+1)/2}$.*

PROOF. Let $p+1$ ($\leq n+1$) be the size of the support of y. Then, by Cramer's rule there is a p by $p+1$ matrix C with $\{-1, 1\}$-entries so that the ith nonzero entry of y is given by

$$y_i = \frac{(-1)^{i+1} \det(C_i)}{\mathbf{gcd}[\det(C_1), \det(C_2), \ldots, \det(C_{p+1})]}, \tag{6}$$

where C_i is the matrix obtained from C by omitting column number i. Each of the determinants $\det(C_i)$ is the determinant of a p by p matrix with $\{-1, 1\}$-entries. By adding the first row to each of the others we get a matrix in which all rows but the first are divisible by 2, hence the determinant is divisible by 2^{p-1}, implying that the denominator of (6) is at least 2^{p-1}. In addition, by appending to C the all-1-vectors of length $p+1$ as a first row we get a $p+1$ by $p+1$ matrix with $\{-1, 1\}$-entries whose determinant is the ℓ_1-norm of y. The assertion of the lemma thus follows from Hadamard's inequality [Hadamard 1893]. □

PROOF OF THEOREM 3.1, UPPER BOUND. Let $v_1, v_2, \ldots, v_m$ be an n-mms of length $m = f(n)$. Then $\sum_i v_i - \mathbf{1} = \mathbf{0}$, where $\mathbf{1}$ is the all-1-vector of length n, and no proper subsequence of the sequence $S = (v_1, v_2, \ldots, v_m, -\mathbf{1})$ has zero sum (again, depending on inclusion of $-\mathbf{1}$, this would give a subsequence of $v_1, \ldots, v_m$ with sum $\mathbf{0}$ or the vector $\mathbf{1}$). Let $u_1, u_2, \ldots, u_t$ be a sequence consisting of all distinct vectors in the sequence S, and let x_i be the number of times u_i appears in this sequence. Then $t \leq 2^{n-1}$, since S cannot contain a vector and its inverse, and the vector $x = (x_1, x_2 \ldots, x_t)$ belongs to the cone $C(u_1, u_2, \ldots, u_t)$. By Lemma 3 this vector is a positive linear combination of less than t basic integral points of the cone. Note that each such coefficient is at most 1, since otherwise the corresponding basic integral point provides a proper subsequence of S with zero sum, a contradiction. By Lemma 4 the ℓ_1-norm of each integral basic point is at most $2^{1-n}(n+1)^{(n+1)/2}$. Therefore, the ℓ_1-norm of x, which is the length of S, that is, $m+1$, is at most $t \cdot 2^{1-n}(n+1)^{(n+1)/2} \leq (n+1)^{(n+1)/2}$, completing the proof. □

3.2. Lower bound

We use the following result by Alon and Vu [1997].

LEMMA 5. *For every k there exists a nonsingular k by k matrix A with $\{-1, 1\}$-entries, so that the maximum entry of the inverse of A is at least $k^{k/2-o(k)}$. This is tight up to the $o(k)$ term.*

We will also use the following result by Spencer [1985].

LEMMA 6. *For any k by $k+1$ matrix A with $\{-1,1\}$-entries, one can change the sign of some of the rows of A to get a matrix A' so that the absolute value of the sum of entries in each column of A' is less than $6\sqrt{k}$. This is tight up to the constant 6.*

PROOF OF THEOREM 3.1, LOWER BOUND. By Lemma 5 and Cramer's rule (and by permuting rows and columns, if needed), there is a k by k matrix A with $\{-1,1\}$-entries, so that the ratio between $\det(A_{11})$ and $\det(A)$ is at least $k^{k/2-o(k)}$, where A_{ij} is the submatrix of A obtained by deleting row number i and column number j. Add to A a column c with $\{-1,1\}$-entries, where $c_i = \text{sign}(\det(A_{i1}))$. Then, by Cramer's rule, the unique solution x of $Ax = c$ has

$$x_1 = \frac{\sum_i |\det(A_{i1})|}{\det(A)} \geq \frac{\det(A_{11})}{\det(A)} \geq k^{k/2-o(k)}.$$

This implies that the unique positive integral linear dependence among the columns of the extended k by $k+1$ matrix consisting of A and c, in which the greatest common divisor of the entries is 1, has ℓ_1 norm at least $k^{k/2-o(k)}$, since there is an integral solution in which the first coordinate is at least $\det(A_{11})$ and the last is $\det(A)$. By changing the signs of some of these columns, if needed, we get $k+1$ column vectors in $\{-1,1\}^k$ so that there is an integral positive linear combination of them with coefficients $x_1, x_2, \ldots x_{k+1}$ that sums to zero, the greatest common divisor of its coefficients x_i is 1, and its ℓ_1 norm $x_1 + x_2 + \ldots + x_{k+1}$ is at least $k^{k/2-o(k)}$. Moreover, since any linear combination of these columns that sums to zero is a multiple of this one, there is no nontrivial integral nonnegative combination of these vectors that sums to zero in which the coefficients $y_1, y_2, \ldots, y_{k+1}$ satisfy $0 \leq y_i \leq x_i$ for all i and $\sum y_i < \sum x_i$.

Consider the k by $k+1$ matrix whose columns are the above vectors. By Lemma 6 one can change the signs of some of the rows of this matrix to ensure that the absolute value of the sum of entries in each column is smaller than $6\sqrt{k}$. Denote the resulting column vectors by $w_1, \ldots, w_{k+1}$ and note that $\sum x_i w_i = 0$ for the integers x_i as before, and if $\sum y_i w_i = 0$ for nonnegative integers $y_i \leq x_i$ then either $y_i = 0$ for all i or $y_i = x_i$ for all i. Put $t = \sum_i x_i$ and let B be the k by t matrix obtained by picking each column w_i exactly x_i times. Then the sum of columns of B is the zero vector, and no proper subset of the columns has zero sum.

Next we add a (relatively small) number of rows to B to get a matrix in which the sum of every row and the sum of every column is zero. To do so, let $s \leq 6\sqrt{k}$ be an integer so that $s + k$ is even and so that the absolute value of the sum of any column of B is at most s.

LEMMA 7. *There is a matrix B' obtained from B by adding to it s rows of $\{-1,1\}$-entries, so that the sum of every row and every column of B' is 0.*

PROOF. Let s_j be the sum of entries of the jth column of B. Note that $\sum s_j = 0$ (as the sum of columns of B is the zero vector), that $|s_j| \leq s$ for all j, and that s_j has the same parity as s. We have to show that there is an s by t matrix with $\{-1,1\}$-entries, in which the sum of elements in column number j is $-s_j$ and the sum of elements in every row is 0.

One possible proof of that is to apply the well known Gale-Ryser theorem (cf. [West 2001, Theorem 4.3.18]) which gives a necessary and sufficient condition for the existence of a $\{0,1\}$ matrix with prescribed row and column sums. Another possibility is to describe explicitly a cyclic procedure for completing the new entries of B'. The shortest proof seems, however, to consider, among all s by t matrices M with $\{-1,1\}$-entries in which the sum of entries in column number j is $-s_j$, the one in which the sum of squares of row-sums is minimum. If this sum of squares is 0 we are done, else, there is a row with positive sum, and a row with negative sum (as the sum of all row sums is zero). Let these two rows be rows numbers p and q. Then there is some column j so that entry $M_{p,j} = 1$ and entry $M_{q,j} = -1$. Swapping those to $M_{p,j} = -1$ and $M_{q,j} = 1$ reduces the sum of squares (note that every row-sum is even so if it is positive it is at least 2), contradicting the minimality. This completes the proof of the lemma. □

We can now define our $n = (k + s)$-mms. Put $m = t + 1$, and let the first $t - 1$ vectors $v_1, v_2, \ldots, v_{t-1}$ in the sequence be the columns of B', without the last one. This last vector has $+1$ in a set J of $n/2$ coordinates, and -1 in its complement. Therefore, the sum of the vectors $v_1, \ldots, v_{t-1}$ is -1 in the coordinates of J and $+1$ in the other coordinates. We now add two vectors v_t and v_{t+1} to our sequence. Both vectors have $+1$ in the coordinates of J, whereas in the other coordinates one of them has -1 and one has $+1$, ensuring that each of them has at least one $+1$ in these coordinates. Note that the sum of coordinates of v_t is strictly positive, and so is the sum of coordinates of v_{t+1}. Note also that the sum of all $t + 1$ vectors v_i is the all 1 vector.

It is not difficult to check that there is no non-empty subsequence of the sequence v_i for which the sum in every entry is at most 0. Indeed, suppose there is such a subsequence. As the sum of coordinates of every vector v_i for $i < t$ is 0, and the sum of coordinates of v_t and of v_{t+1} is positive, this subsequence can contain neither v_t nor v_{t+1}. Thus, the sum of its elements must be the zero vector, because the sum of its coordinates is 0 and they are all non-positive. However, if this is the case, then the sum of the vectors of length k obtained from the vectors in the subsequence by taking the first k coordinates of each vector is also the (k-dimensional) zero vector. This contradicts the property of the matrix B, showing that indeed $v_1, \ldots, v_{t+1}$ is an n-mms.

Note that $n = k + O(\sqrt{k})$ and thus $m = t + 1 = k^{k/2-o(k)} = n^{n/2-o(n)}$. Note also that the definition easily implies that $f(n+1) \geq f(n)$ for every n, and therefore it is enough to prove the lower bound for a sufficiently dense set of values of n (for example, all large even numbers) to get a similar estimate for every n. This completes the proof. □

4. CONCLUSION AND OPEN QUESTIONS

We have introduced problems in computational social choice which model the task of finding a bundle of proposals that is accepted by a society of voters while containing a specific number of proposal from an agenda set. This can be seen as an extension of a special case of approval voting for committee election by threshold functions as introduced by Fishburn and Pekeč [2004] (see also [Kilgour 2010; Kilgour and Marshall 2012]). We studied the computational complexity of our problems and revealed both (fixed-parameter) tractable and intractable special cases. Furthermore, we started an analysis of their combinatorial properties. We conclude this paper with a few challenges for future research.

First, recall that in Theorem 2.10 we stated upper bounds on $h_{\max}$ (the size of the maximum symmetric difference between two favorite ballots) in terms of linear functions in $b_{\max}$ (the maximum ballot size of voters). Hence, parameterized hardness results with respect to $b_{\max}$ transfer to the parameterization by $h_{\max}$. In the case of nonempty agenda, that is, $q_+ \geq 1$, however, we have no good lower bounds for $h_{\max}$ in terms of $b_{\max}$. Thus, it remains to classify the parameterized computational complexity of both UNANIMOUSLY ACCEPTED BALLOT and MAJORITYWISE ACCEPTED BALLOT using parameter $h_{\max}$. Notably, in the cases without hidden agenda the parameters $h_{\max}$ and $b_{\max}$ are linearly related so that the same parameterized complexity results will hold for both parameterizations.

Second, with respect to the parameter h (the size of the solution ballot Q), we established W[2]-hardness for MAJORITYWISE ACCEPTED BALLOT even without hidden agenda, but we left open the precise location of this problem in the parameterized complexity hierarchy. It might be W[2]-complete, but all we currently know is that it is contained in W[2] (Maj), a class presumably larger than W[2] [Fellows et al. 2010]. See also a recent survey [Bredereck et al. 2014a] for a more detailed description of this challenge.

Third, the combinatorial bounds from Section 3 do not hold for instances with non-empty agenda, since such bounds cannot be independent of $|Q_+|$. For cases with non-empty agenda there are similar bounds with an extra factor of $|Q_+|$. A detailed analysis could be part of investigations of weighted variants of our problems. In this regard, weights on the voters, weights on the proposals, or weights on the acceptance threshold of the voters seem to be well-motivated.

Fourth, can we avoid Integer Linear Programs for showing fixed-parameter tractability with respect to the parameter number n of votes and provide direct combinatorial algorithms beating the ILP-based running times? In this context, the exponential lower bound on the number of proposals in ballots accepted by society from Section 3 might be relevant.

Fifth, it remains a puzzling open question whether MAJORITYWISE ACCEPTED BALLOT parameterized by $b_{\max}$ is fixed-parameter tractable when the agenda set is empty—we could only show W[1]-hardness for instances with a non-empty agenda set.

Finally, our studies and results basically focused on "single parameterizations", yielding several hardness results. In future studies, in the spirit of multivariate complexity analysis [Fellows et al. 2013; Niedermeier 2010] one may try to extend the range of tractable cases by studying parameter combinations.

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