Finding large degree-anonymous subgraphs is hard[☆]

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Abstract

A graph is said to be k-anonymous for an integer k, if for every vertex in the graph there are at least $k-1$ other vertices with the same degree. We examine the computational complexity of making a given undirected graph k-anonymous either through at most s vertex deletions or through at most s edge deletions; the corresponding problem variants are denoted by Anonym V-Del and Anonym E-Del.

We present a variety of hardness results, most of them hold for both problems. The two variants are intractable from the parameterized as well as from the approximation point of view. In particular, we show that both variants remain NP-hard on very restricted graph classes like trees even if $k = 2$. We further prove that both variants are W[1]-hard with respect to the combined parameter solutions size s and anonymity level k. With respect to approximability, we obtain hardness results showing that neither variant can be approximated in polynomial time within a factor better than $n^{1/2}$ (unless P=NP). Furthermore, for the optimization variants where the solution size s is given and the task is to maximize the anonymity level k, this inapproximability result even holds if we allow a running time of $f(s) \cdot n^{\mathcal{O}(1)}$ for any computable function f. On the positive side, we classify both problem variants as fixed-parameter tractable with respect to the combined parameter solution size s and maximum degree Δ.

Keywords: NP-hardness, approximation-hardness, W-hardness, graph algorithms

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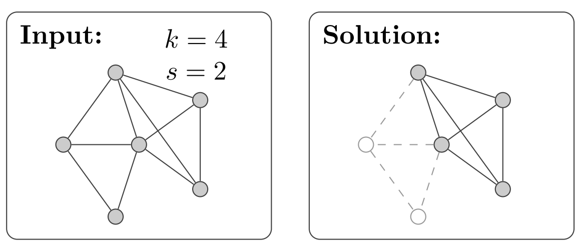
1. Introduction

With the enormously growing relevance of social networks, the protection of privacy when releasing underlying data sets has become an important and active field of research [3]. If a graph contains only few vertices with some distinguished feature, then this might allow the identification (and violation of privacy) of the underlying real-world entities with that particular feature. Hence, in order to ensure pretty good privacy and anonymity, every vertex should share its feature with many other vertices. In a landmark paper, Liu and Terzi [4] considered the vertex degrees as feature; see Wu et al. [3] for other features considered in the literature. Correspondingly, a graph is called k-anonymous if for each vertex there are at least $k-1$ other vertices of same degree. Therein, different values of k reflect different privacy demands and the natural computational task arises, given some fixed k, to perform few changes to a graph in order to make it k-anonymous. Liu and Terzi [4] proposed a heuristic algorithm for the task of making a graph k-anonymous by adding edges. We refer to Wu et al. [3] for a survey of anonymization models and a discussion about the pros and cons of the k-anonymity concept. Here, we study the vertex and edge deletion variants of Degree Anonymity. We start our investigations with the vertex deletion variant which is defined as follows.

Degree Anonymity by Vertex Deletion (Anonym V-Del)
Input: An undirected graph $G = (V, E)$ and two integers $k, s \in \mathbb{N}$.
Question: Is there a vertex subset $S \subseteq V$ of size at most s such that $G - S$ is k-anonymous?



Considering vertex deletions seems to be a promising approach on practical instances, especially on social networks. In these networks, the degree distribution of the underlying graphs often follows a so-called power law distribution [5], implying that there are only few high-degree vertices and most vertices are of moderate degree; this suggests that only few vertices have to be removed in order to obtain a k-anonymous graph. For instance, consider the DBLP co-author graph[4] (generated in Feb. 2012) with $\approx$ 715 thousand vertices corresponding to authors and $\approx$ 2.5 million edges indicating whenever two authors have a common

[4]The current dataset and a corresponding documentation are available online (http://dblp.uni-trier.de/xml/).

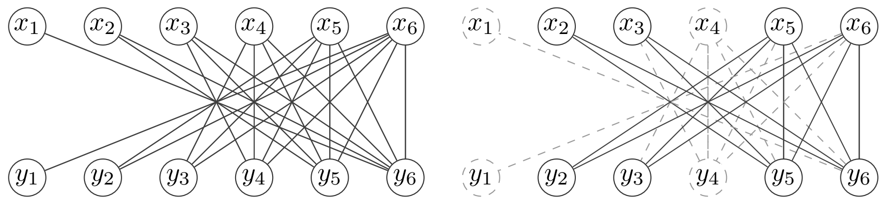


Figure 1: *Left:* A graph where a constant fraction of the vertices has to be removed in order to obtain a 3-anonymous graph. *Right:* A minimum size solution to make the graph on the left side 3-anonymous. See Example 2 for a detailed explanation.

scientific paper: This graph has maximum degree 804 but only 208 vertices are of degree larger than 208, whereas the average degree is 7. Interestingly, a heuristic that simply removes vertices violating the k-anonymous property shows that one has to remove at most 338 vertices to make it 5-anonymous and even to make it 10-anonymous requires at most 635 vertex deletions.

In Section 3, we will show that already the simple and highly specialized privacy model of Anonym V-Del is computationally hard from the parameterized as well as from the approximation point of view. A variety of hardness results holds even for very restricted graph classes, as for instance trees, cographs, and split graphs.

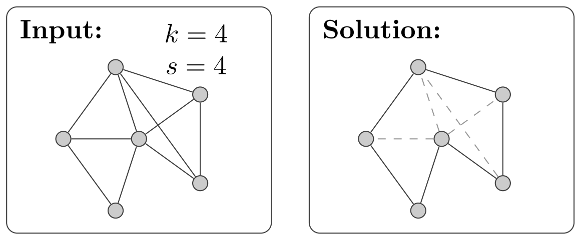
One reason of this hardness is that being k-anonymous is not a hereditary property: Simply deleting one vertex in a three-regular graph, that is, an n-anonymous graph, results in an only 3-anonymous graph. Another reason is shown in the following two examples illustrating that the number s of allowed removals and the anonymity level k are independent of each other, and that a small change in one of these parameter values might lead to a large jump of the other parameter value.

Example 1. Let G be a graph on $n \geq 5$ vertices that consists of two connected components: a clique of size $n-2$ and an isolated edge. This 2-anonymous graph cannot be transformed into a 3-anonymous graph by deleting only one vertex, however, deleting the two degree-one vertices makes it $(n-2)$-anonymous. Hence, by slightly increasing s from 1 to 2 the reachable anonymity level jumps from $k = 2$ to $k = n-2$.

Example 2. Let $G = (V, E)$ be a bipartite graph with the vertex sets $X := \{x_1, x_2, \ldots, x_\ell\}$ and $Y := \{y_1, y_2, \ldots, y_\ell\}$, $V = X \cup Y$, and there is an edge between x_i and y_j if $i + j > \ell$, see Figure 1 for a visualization. Clearly, x_i and y_i are of degree i implying that G is 2-anonymous. Since $N(x_i) \subseteq N(x_{i+1})$ for all i, deleting any subset of Y preserves the invariant $\deg(x_1) \leq \deg(x_2) \leq \ldots \leq \deg(x_\ell)$. As the previous argument is symmetric, one can observe that to make G 3-anonymous one has to remove $2/3$ of the "jumps" in the initial sequences $\deg(x_1) < \deg(x_2) < \ldots < \deg(x_\ell)$ and $\deg(y_1) < \deg(y_2) < \ldots < \deg(y_\ell)$. Since removing one vertex in X (Y) removes only one jump in the sequence of X (Y) and only one in Y (X), it follows that at least $2(\ell-1) \cdot 2/3 \cdot 1/2 \approx (2\ell)/3 = |V|/3$ vertices have to be deleted in order to get a 3-anonymous graph. Summarizing, by requiring anonymity level $k = 3$ instead of anonymity level $k = 2$, the number of vertices that need to be removed jumps from zero to a constant fraction of the vertices.

The second part of this work deals with the edge deletion variant which is defined as follows:

Degree Anonymity by Edge Deletion (Anonym E-Del)
Input: An undirected graph $G = (V, E)$ and two integers $k, s \in \mathbb{N}$.
Question: Is there an edge subset $S \subseteq E$ of size at most s such that $G - S$ is k-anonymous?



Considering social networks, their power law degree distribution suggests that the solution size in the edge deletion variant is significantly larger than in the vertex deletion variant. However, in the edge deletion variant the resulting graph contains, by definition, all vertices of the input graph, which might be important in some applications. Furthermore, deleting a vertex with high degree reduces the degree of many other vertices whereas deleting an edge reduces the degree of only two vertices. Hence, although requiring more edge deletions than vertex deletions, deleting edges might result in a graph that is actually "closer" to the original graph.

In Section 4, we transfer most hardness results from Anonym V-Del to Anonym E-Del, showing strong intractability results concerning parameterized complexity and approximability. Similarly to the vertex deletion variant, a small change in one of the two parameters k and s might lead to a large jump of the other parameter as demonstrated in the following two examples.

Example 3. Let G be an n-vertex cycle with two chords, that is, two additional edges within the cycle. As G contains four degree-three vertices and $n - 4$ degree-two vertices, G is 4-anonymous. Deleting one edge does not increase the anonymity level k; however, deleting the two chords results in an n-anonymous graph—a cycle. Hence, by slightly increasing s from one to two the reachable anonymity level jumps from $k = 4$ to $k = n$.

Example 4. Let $G = (V, E)$ be a disjoint union of a clique and an independent set, each containing $n/2$ vertices. Thus, G is $n/2$-anonymous. However, in order to obtain an $(n/2 + 1)$-anonymous graph, all edges have to be removed. Hence, by slightly increasing k from $n/2$ to $n/2 + 1$ the number of edges that have to be removed jumps from zero to $|E| = \binom{n/2}{2}$.

Related work. Hartung et al. [6] studied the Anonym E-Ins problem as proposed by Liu and Terzi [4]. Given a graph and two positive integers k and s, Anonym E-Ins asks whether there exists a set of at most s edges whose addition makes the graph k-anonymous. The main result of Hartung et al. [6] is a polynomial problem kernel with respect to the parameter maximum degree Δ of the input graph. Furthermore, they showed that an heuristic algorithm proposed by Liu and Terzi [4] is optimal for Anonym E-Ins solutions larger than Δ^4. Building on Liu and Terzi's work, Hartung et al. [7] enhanced their heuristic approach with the focus on improving lower and upper bounds on the solution size. Chester et al. [8] investigated the computational complexity of Anonym E-Ins and variants with edge labels. They showed NP-hardness for the considered variants and a polynomial time algorithm for bipartite graphs. Chester et al. [9] investigated the variant of adding vertices instead of edges; Bredereck et al. [10] provided first parameterized complexity results in this direction. Hartung and Talmon [11] studied the computational complexity of the edge contraction variant.

Concerning the vertex deletion variant, the work which is probably closest to ours is by Moser and Thilikos [12]. They studied the parameterized complexity of the Regular-Degree-d Vertex Deletion problem, where given an undirected graph G and an integer $s \in \mathbb{N}$, the task is to decide whether G can be made d-regular by at most s vertex deletions. Moser and Thilikos [12] showed that Regular-Degree-d Vertex Deletion can be solved in $\mathcal{O}(n(s+d)+(d+2)^s)$ time and presented a polynomial problem kernel of size $\mathcal{O}(sd(d+s)^2)$. Observe that for $k > n/2$ the problem of Anonym V-Del asks whether at most s vertices can be deleted to obtain a regular graph.

Our contributions. While every graph is trivially 1-anonymous, we will show that the combinatorial structure of 2-anonymous graphs is already rich and complicated: Anonym V-Del for $k = 2$ is NP-complete, even for strongly restricted graph classes like trees, interval graphs, split graphs, trivially perfect graphs, and bipartite permutation graphs. All these hardness results are established by means of a general framework. Furthermore, we show that Anonym V-Del is NP-complete even on graphs with maximum degree three.

On the positive side, we present (polynomial-time) dynamic programming approaches for Anonym V-Del on three graph classes: graphs of maximum degree two, cluster graphs, and threshold graphs. We frankly admit that these three graph classes carry an *extremely constraining* combinatorial structure: Anonym V-Del is such a vicious problem that without these heavily constraining structures there is basically no hope for polynomial-time algorithms. Figure 2 summarizes the considered graph classes and their containment relations.

For Anonym E-Del, we show NP-completeness on caterpillars and on graphs with maximum degree seven; this later result is in stark contrast with the fixed-parameter tractability of Anonym E-Ins with respect to the maximum degree Δ [6].

We analyze the parameterized complexity of Anonym V-Del and Anonym E-Del, see Table 1 for an overview. Once again, both problems show a difficult and challenging behavior: They are intractable with respect to each of the three

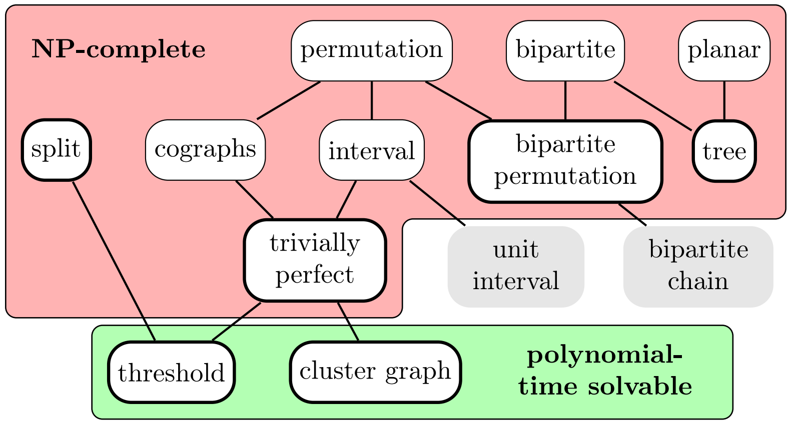


Figure 2: The complexity landscape of Anonym V-Del for various graph classes. The results for classes with thick frames are discussed in this work and they imply the results for classes with thin frames. The complexity of Anonym V-Del on unit interval graphs and on bipartite chain graphs remains open.

Table 1: Overview on the computational complexity classification of Anonym V-Del and Anonym E-Del.

Parameter	Anonym V-Del	Anonym E-Del
k	NP-complete for $k = 2$ (Theorem 3)	NP-complete for $k = 2$ (Theorem 18)
(s, k)	W[2]-hard (Corollary 7)	W[1]-hard (Corollary 23)
Δ	NP-complete for $\Delta = 3$ (Theorem 2)	NP-complete for $\Delta = 7$ (Theorem 20)
(s, Δ)	FPT (Theorem 24)	
(k, Δ)	FPT (Corollary 27)	

(single) parameters s, k, and Δ. Even worse, they are intractable with respect to the combined parameter (s, k). The only positive parameterized results come with the combined parameters (Δ, s) and (Δ, k). The latter result is based on bounding the number s of deleted vertices in terms of Δ and k.

Finally, studying the approximability of the optimization problems naturally associated with Anonym E-Del or Anonym V-Del, we obtain hardness results showing that none of the considered problems can be approximated in polynomial time better than within a factor of $n^{1/2}$. Furthermore, for the optimization variants where the solution size s is given and the task is to maximize the anonymity level k, this inapproximability even holds if we allow a running time of $f(s)n^{\mathcal{O}(1)}$ for any computable f. Again, this result holds for the edge deletion and the vertex deletion variant, see Table 2 for an overview.

Table 2: Overview on the inapproximability of the optimization variants associated with Anonym V-Del and Anonym E-Del.

vertex deletion running time	Anonym Min-V-Del (fixed k, minimize s)	Max-Anonym V-Del (fixed s, maximize k)
polynomial time	no $n^{1-\varepsilon}$-approximation (Theorem 11)	no $n^{1/2-\varepsilon}$-approximation (Theorem 13)
$f(s) \cdot n^{O(1)}$	open	no $n^{1/2-\varepsilon}$-approximation (Theorem 12)

edge deletion running time	Anonym Min-E-Del (fixed k, minimize s)	Max-Anonym E-Del (fixed s, maximize k)
polynomial time	no $n^{1-\varepsilon}$-approximation (Theorem 21)	no $n^{1-\varepsilon}$-approximation (Theorem 20)
$f(s) \cdot n^{O(1)}$	open	no $n^{1-\varepsilon}$-approximation (Theorem 22)

Organization. We first introduce the necessary notation and concepts in Section 2. We then provide our results for Anonym V-Del in Section 3, starting with the NP-completeness results. To this end, we present in Subsection 3.1 a reduction showing NP-hardness on trees. This reduction serves in Subsection 3.2 as blueprint for a generic reduction yielding NP-hardness on several restricted graph classes. In Subsection 3.3, we then adjust this reduction in order to prove the inapproximability results for Anonym V-Del. We present the polynomial-time solvable cases of Anonym V-Del in Subsection 3.4. In Section 4, we transfer the central intractability results for Anonym V-Del to Anonym E-Del. In particular, we show in Subsection 4.1 that Anonym E-Del is NP-complete on caterpillars. In Subsection 4.2, we then give the inapproximability results. Finally, we show in Section 5 the fixed-parameter tractability of Anonym V-Del and Anonym E-Del with respect to the combined parameters (s, Δ) and (s, k).

2. Preliminaries

All graphs in this paper are undirected, loopless, and simple (that is, without multiple edges). Throughout we use n to denote the number of vertices in the considered graph. The maximum vertex degree of a graph $G = (V, E)$ is denoted by Δ_G. A vertex subset $S \subseteq V$ is called *k-deletion set* if $G[V \setminus S]$ is k-anonymous. For each vertex $v \in V$ we denote by $N_G(v)$ the set of neighbors of v and by $N_G[v] = N_G(v) \cup \{v\}$ the closed neighborhood. Correspondingly, for a vertex subset V' we set $N_G[V'] = \bigcup_{v \in V'} N_G[v]$ and $N_G(V') = N_G[V'] \setminus V'$. For $0 \leq \alpha \leq \Delta$, the *block of degree* α is the set $D_G(\alpha) \subseteq V$ of all vertices with degree α in G. Clearly, a graph is k-anonymous if and only if each block is either of size zero or at least k. We omit subscripts if the corresponding graph is clear from the context.

Parameterized Complexity. The concept of parameterized complexity was pioneered by Downey and Fellows [13] (see Flum and Grohe [14] and Niedermeier [15] for further monographs on parameterized complexity). Herein, a parameterized problem is called *fixed-parameter tractable* if there is an algorithm that decides any instance (I, p), consisting of the "classical" instance I and a parameter $p \in \mathbb{N}$, in $f(p) \cdot |I|^{O(1)}$ time, for some computable function f solely depending on p.

A core tool in the development of fixed-parameter algorithms is polynomial-time preprocessing by data reduction, called *kernelization*[5] [16, 17]. Here, the goal is to transform a given problem instance (I, k) in polynomial time into an equivalent instance (I', k') whose size is upper-bounded by a function of k. That is, (I, k) is a yes-instance if and only if (I', k'), $k' \leq g(k)$, and $|I'| \leq g(k)$ for some function g. Thus, such a transformation is a polynomial-time self-reduction with the constraint that the reduced instance is "small" (measured by $g(k)$). In case that such a transformation exists, I' is called *kernel* of size $g(k)$. Furthermore, if g is a polynomial, then it I' is called a *polynomial kernel*.

The parameterized complexity hierarchy is composed of the classes FPT $\subseteq$ W[1] $\subseteq$ W[2] $\subseteq \ldots \subseteq$ W[P]. A W[1]-hard problem is not fixed-parameter tractable (unless FPT = W[1]) and one can prove the W[1]-hardness by means of a *parameterized reduction* from a W[1]-hard problem. Such a reduction between two parameterized problems P and P' is a mapping of any instance (I, p) of P in $g(p) \cdot |I|^{O(1)}$ time (for some computable function g) into an instance (I', p') for P' such that $(I, p) \in P \Leftrightarrow (I', p') \in P$ and $p' \leq h(p)$ for some computable function h.

Approximation. Let Σ be a finite alphabet. Given an optimization problem $Q \subseteq \Sigma^*$ and an instance I of Q, we denote by $opt(I)$ the value of an optimum solution for I and by $val(I, S)$ the value of a feasible solution S of I. The *performance ratio* of S (or *approximation factor*) is $r(I, S) = \max\left\{\frac{val(I,S)}{opt(I)}, \frac{opt(I)}{val(I,S)}\right\}$. For a function ρ, an algorithm is a $\rho(n)$-*approximation*, if for every instance I of Q, it returns a solution S such that $r(I, S) \leq \rho(|I|)$. An optimization problem is $\rho(n)$-*approximable in polynomial time* if there exists a $\rho(n)$-approximation algorithm running in time $|I|^{O(1)}$ for any instance I. A parameterized optimization problem $Q \subseteq \Sigma^* \times \mathbb{N}$ is $\rho(n)$-*approximable in fpt-time w.r.t. the parameter* p if there exists a $\rho(n)$-approximation algorithm running in time $f(p) \cdot |I|^{O(1)}$ for any instance (I, p) and f is a computable function [18]. It is worth pointing that in this case, p is not related to the optimization value.

In this paper we use a gap-reduction between a decision problem and a minimization or maximization problem. A decision problem P is called *gap-reducible* to a maximization problem Q with gap $\rho \geq 1$ if there exists a polynomial-time computable function that maps any instance I of A to an instance I' of Q, while satisfying the following properties:

[5]It is well-known that a parameterized problem is fixed-parameter tractable if and only if it has a kernelization.

- if I is a yes-instance, then $opt(I') \geq \xi(|I'|) \cdot \rho(|I'|)$, and
- if I is a no-instance, then $opt(I') < \xi(|I'|)$,

where ξ and ρ are two computable functions. If A is NP-hard, then Q is not ρ-approximable in polynomial time, unless P = NP [19]. In this paper we also use a variant of this notion, called fpt gap-reduction.

Definition 1 (fpt gap-reduction)**.** A parameterized problem P is called *fpt gap-reducible* to a parameterized maximization problem Q with gap $\rho \geq 1$ if any instance (I, p) of P can be mapped to an instance (I', p') of Q in $f(p) \cdot |I|^{\mathcal{O}(1)}$ time while satisfying the following properties:

(i) $p' \leq g(p)$ for some computable function g,

(ii) if I is a yes-instance, then $\text{opt}(I') \geq \xi(|I'|) \cdot \rho(|I'|)$, and

(iii) if I is a no-instance, then $\text{opt}(I') < \xi(|I'|)$,

where ξ and ρ are two computable functions.

The interest of the fpt gap-reduction is the next result that follows from the previous definition:

Lemma 1. *If a parameterized problem P is $\mathcal{C}$-hard, fpt gap-reducible to a parameterized optimization problem Q with gap ρ, and Q is ρ-approximable in fpt-time, then FPT $= \mathcal{C}$, where $\mathcal{C}$ is any class of the W-hierarchy.*

Proof. We give a fixed-parameter algorithm for the parameterized problem P as follows: Since P is fpt gap-reducible to $Q = (\mathcal{I}, \text{sol}, \text{cost}, \max)$ with gap ρ, there exists an algorithm mapping the input (I, p) of P to an instance $(I', p') \in \mathcal{I}$ of Q in $f(p) \cdot |I|^{\mathcal{O}(1)}$ time such that the properties (i) to (iii) of Definition 1 are satisfied. We then apply the fixed-parameter ρ-approximation algorithm for Q on the instance (I', p'). Due to property (i), this algorithm runs in $g(p) \cdot |I|^{\mathcal{O}(1)}$ time for some computable function g. Let $x \in \text{sol}(I')$ be the solution produced by the fixed-parameter ρ-approximation algorithm for Q. Assume that (I, p) was a no-instance. Hence, we have $\text{cost}(x) \leq \text{opt}(I')$ and by property (iii) it follows that $\text{cost}(x) < \xi(I')$. Now assume that (I, p) was a yes-instance. Hence, we have $\text{opt}(I')/\text{cost}(x) \leq \rho(I')$ and thus $\text{cost}(x) \geq \text{opt}(I')/\rho(I')$. By property (ii) it follows that $\text{cost}(x) \geq \text{opt}(I')/\rho(I') \geq (\xi(I') \cdot \rho(I'))/\rho(I') = \xi(I')$. Hence, by distinguishing the two cases $\text{cost}(x) < \xi(I')$ and $\text{cost}(x) \geq \xi(I')$ we can decide the instance (I, p) of P in $(g(p) + f(p)) \cdot |I|^{\mathcal{O}(1)}$ time. Thus P is fixed-parameter tractable and since P is $\mathcal{C}$-hard, it follows that FPT $= \mathcal{C}$. □

3. Vertex Deletion

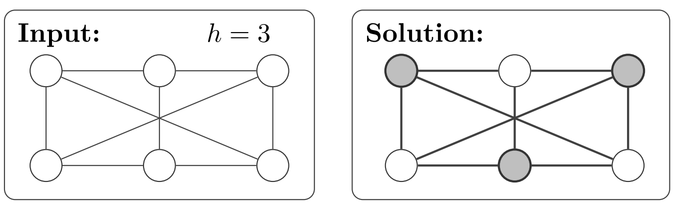
In this section, we provide various hardness results for ANONYM V-DEL on several restricted graph classes such as trees, split graphs, and trivially perfect graphs. In a first subsection (see Subsection 3.1), we show that Anonym V-Del remains NP-hard even on trees. Extracting the basic ideas of this result, subsequently we provide a generic reduction to show NP-hardness on trivially perfect graphs, bipartite permutation graphs, and split graphs (see Subsection 3.2) and strong inapproximability results for the two natural optimization problems associated with Anonym V-Del (see Subsection 3.3). We also identify several classes of graphs for which Anonym V-Del is polynomial-time solvable (see Subsection 3.4).

As a warm up, we first prove that Anonym V-Del is NP-complete on graphs with maximum degree three.

Theorem 2. Anonym V-Del *is NP-complete on graphs with maximum degree three.*

Proof. Since containment in NP is easy to see, we focus on showing NP-hardness. To this end, we give a reduction from the Vertex Cover problem which is known to be NP-complete even in three-regular graphs [20, GT1] and is formally defined as follows.

Vertex Cover [20, GT1]
Input: An undirected graph $G = (V, E)$ and $h \in \mathbb{N}$.
Question: Is there a vertex subset $V' \subseteq V$, $|V'| \leq h$, such that every edge has an endpoint in V'?



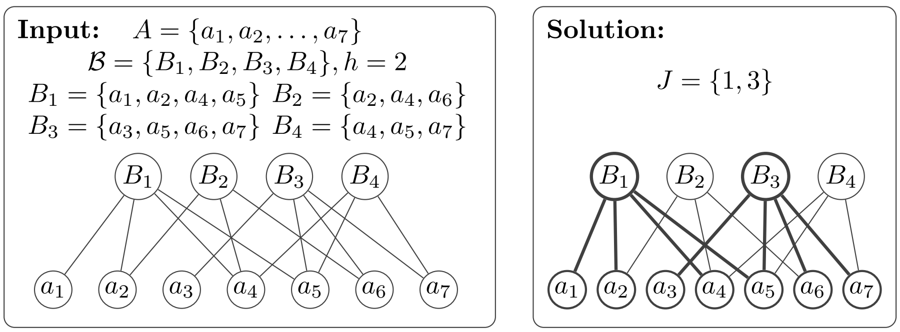
Given a Vertex Cover instance $(G = (V, E), h)$ with G being three-regular, start by copying G into a new graph G'. Finally, add $h + 1$ degree-zero vertices to G', set $s := h$, and $k := |V| + 1$.

If G contains a vertex cover V' of size h, then deleting V' in G' clearly results in an edgeless graph with $|V| + 1 = k$ vertices, implying that (G', s, k) is a yes-instance of Anonym V-Del. In the reverse direction, for any k-deletion set S, since $2k > n + h + 1$ and G' contains $s + 1$ degree-zero vertices, all vertices in $G' - S$ have degree zero. Thus, $S \cap V$ is a vertex cover in G. □

3.1. NP-Hardness on Trees

In this subsection, we show that Anonym V-Del remains NP-hard even on trees. This result and many further hardness results will be obtained using reductions from the NP-complete Set Cover problem, which is defined as follows:

Set Cover [20, SP5]
Input: A universe $A = \{a_1, a_2, \dots, a_\alpha\}$, a collection $\mathcal{B} = \{B_1, B_2, \dots, B_\beta\}$ of subsets of A, and $h \in \mathbb{N}$.
Question: Is there an index set $J \subseteq \{1, 2, \dots, \beta\}$ with $|J| \leq h$, such that $\bigcup_{j \in J} B_j = A$?



If a Set Cover instance $I = (A, \mathcal{B}, h)$ contains such an index set J, then we refer to the set $\{B_j \mid j \in J\}$ as a *set cover* for I.

Reduction 1. The reduction showing NP-hardness of Anonym V-Del on trees is as follows: Let $(A, \mathcal{B}, h)$ be an instance of Set Cover. We assume without loss of generality that for each element $a \in A$ there exists a set $B \in \mathcal{B}$ with $a \in B$. Furthermore, we assume without loss of generality that each set $B \in \mathcal{B}$ occurs at least three times in $\mathcal{B}$. To decrease the amount of indices in the construction given below we introduce the function $f \colon \mathbb{N} \to \mathbb{N}$ with $f(i) = \alpha + (h+1)i$.

The reduction for trees is as follows, see Figure 3 for an example. Set $k := 2$ and $s := h$. To obtain an equivalent Anonym V-Del-instance (G, k, s), construct $G = (V, E)$ as follows: For each element $a_i \in A$ add an *element gadget* consisting of a star $K_{1,f(i)}$ with the center vertex $v(a_i)$. Denote with $V_A := \{v(a_1), v(a_2), \dots, v(a_\alpha)\}$ the set of all these center vertices.

For each set $B_j \in \mathcal{B}$ add a *set gadget* which is a tree rooted in a vertex $v(B_j)$. The root has $|B_j|$ child vertices where each element $a_i \in B_j$ corresponds to exactly one of the children of $v(B_j)$, denoted by $v(a_i, B_j)$. Additionally, we add to $v(a_i, B_j)$ exactly $f(i)$ degree-one neighbors. Hence, the set gadget is a tree of depth two rooted in $v(B_j)$. We denote with $V_{\mathcal{B}} := \{v(B_1), v(B_2), \dots, v(B_\beta)\}$ the set of all root vertices. Observe that, as each set $B_j \in \mathcal{B}$ occurs at least three times, the set gadgets are 2-anonymous. Finally, to end up with one tree instead of a forest, repeatedly add edges between any degree-one-vertices of different connected components.

Correctness of Reduction 1. Observe that for each element $a_i \in A$ the only vertex of degree $f(i)$ is $v(a_i)$ and there are no other vertices violating the 2-anonymous property. The key point in the construction is that, in order to get a 2-anonymous graph, one has to delete vertices of $V_{\mathcal{B}}$: Let $a_i \in A$ be an element and $v(B_j)$ a root vertex such that $a_i \in B_j$. By construction the child vertex $v(a_i, B_j)$ of $v(B_j)$ corresponds to a_i and therefore has $f(i)$ child vertices.

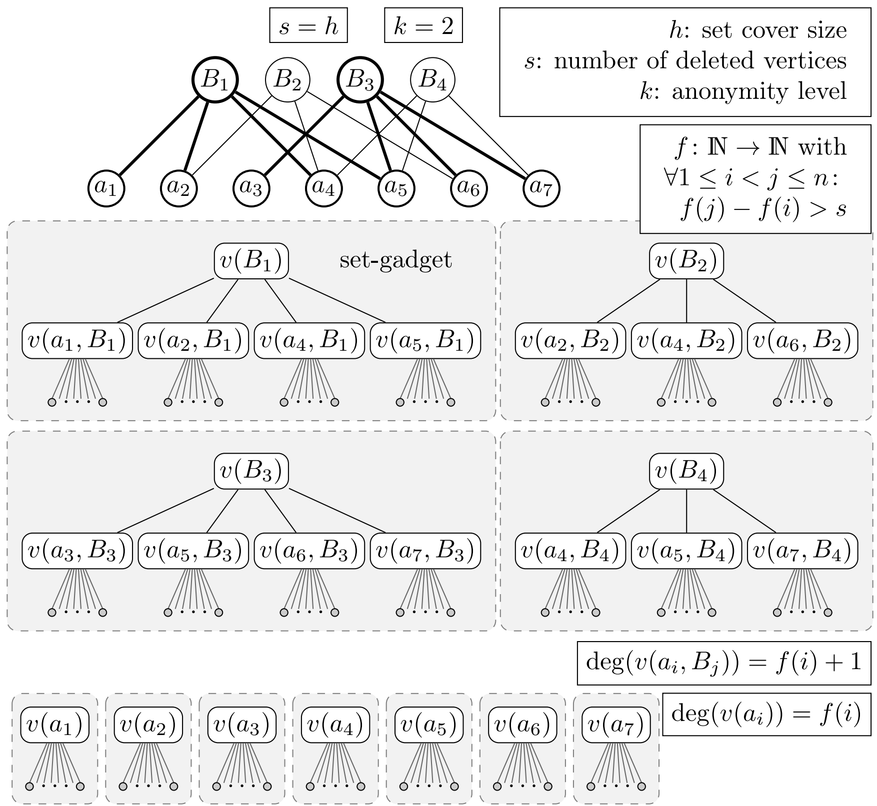


Figure 3: Example of the reduction for trees. Above the Set Cover instance with twelve sets (each set B_i , $i = 1, \ldots, 4$ appears three times) and seven elements is graphically displayed (for example, the set B_1 contains the elements a_1, a_2, a_4, and a_5, and $\{B_1, B_3\}$ forms a set cover). In our reduction, we assume without loss of generality that each set occurs at least times. However, to keep the figure clearly arranged, we omit these copies in the figure. Below are the four different set gadgets and the element gadgets are at the bottom of the picture. Observe that by the choice of f, the degrees of the vertices in the set-gadgets and vertex-gadgets are ensured to not interfere, even if s vertices are removed. The effect of these copies to the construction is that each of the four set-gadgets appears three times. Thus, deleting the vertices $v(B_1)$ and $v(B_3)$ makes the displayed graph 2-anonymous.

Thus, deleting $v(B_j)$ lowers the degree of $v(a_i, B_j)$ to $f(i)$ and, hence, $v(a_i)$ no longer violates the 2-anonymous property. Furthermore, as each set $B_j \in \mathcal{B}$ occurs at least three times, the vertices $V_{\mathcal{B}}$ are 2-anonymous. Hence, given a set cover one can construct a corresponding k-deletion set of the same size and, thus, if $(A, \mathcal{B}, h)$ is a yes-instance, then (G, k, s) is a yes-instance. The basic idea in the converse direction is that if there is a k-deletion set S, then, due to the choice of f, there is also a k-deletion set $S' \subseteq V_{\mathcal{B}}$ that is not larger than S. The formal proof which implies the following theorem will be given later (see Lemma 5), after introducing the generic reduction.

Theorem 3. Anonym V-Del *is NP-complete on trees even if* $k = 2$.

3.2. Generic Reduction

In this section, we generalize Reduction 1 given in the previous subsection. More specifically, we will define properties such that a graph G fulfilling them together with $s := h$ and $k := 2$ forms a yes-instance of Anonym V-Del if and only if the given Set Cover instance $(A, \mathcal{B}, h)$ is a yes-instance. Based on that, we then describe the construction of several graphs contained in different graph classes and fulfilling the properties. Formally, we require the constructed graph $G = (V, E)$ to fulfill the following:

1. Element-gadgets:
   (a) For each element $a_i \in A$ there is a corresponding vertex, denoted by $v(a_i)$, in G and the vertex set $V_A := \{v(a_1), v(a_2), \ldots, v(a_\alpha)\}$ is exactly the set of vertices not being 2-anonymous in G.
   (b) For each vertex $v \in V$ it holds that $|N[v] \cap V_A| \leq 1$.
2. Set-gadgets:
   (a) For each set $B_j \in \mathcal{B}$ there is a corresponding vertex $v(B_j)$ in G and for each element $a_i \in B_j$ the vertex $v(B_j)$ has a neighbor $v(a_i, B_j)$ with $\deg(v(a_i, B_j)) = \deg(v(a_i)) + 1$.
   Set $V_{\mathcal{B}} := \{v(B_1), v(B_2), \ldots, v(B_\beta)\}$ and $A_{B_j} := \{v(a_i, B_j) \mid a_i \in B_j\}$. Set $A_{\mathcal{B}} := \bigcup_{B_j \in \mathcal{B}} A_{B_j}$.
   (b) For all $B_j \in \mathcal{B}$ it holds that $N(A_{B_j}) \cap V_{\mathcal{B}} = \{v(B_j)\}$
   (c) For each vertex $v \in V$ there is a vertex $u \in V_{\mathcal{B}}$ such that $N(v) \cap A_{\mathcal{B}} \subseteq N(u)$.
3. Interaction between these gadgets:
   (a) The vertex subsets V_A, $V_{\mathcal{B}}$, and $A_{B_1}, A_{B_2}, \ldots, A_{B_\beta}$ are pairwise disjoint.
   (b) It holds that $N(V_A) \cap (V_{\mathcal{B}} \cup A_{\mathcal{B}}) = \emptyset$.
   (c) For each $D \subseteq V_{\mathcal{B}}$, $|D| \leq h$, the set of vertices violating the 2-anonymous property in $G - D$ is a subset of V_A.
   (d) Any two vertices $u \in V_A$ and $v \notin A_{\mathcal{B}}$ satisfy $|\deg(u) - \deg(v)| > s$.

It is not hard to verify that the graph constructed in the reduction in the previous paragraph has the above properties. Before proving the correctness of the generic reduction we make the following observation.

Observation 4. *For each* $D \subseteq V_{\mathcal{B}}$*,* $|D| \leq h$*, the set* $V_A \setminus \{v(a_i) \mid \exists v(B_j) \in D \colon a_i \in B_j\}$ *is exactly the set of vertices not being* 2*-anonymous in* $G - D$*.*

Proof. By Property 1a only the vertices in V_A are not 2-anonymous in G. Property 3c ensures that the set of vertices X violating the 2-anonymous property in $G - D$ is a subset of V_A.

Because of Property 3b ($N(V_A) \cap V_{\mathcal{B}} = \emptyset$) it holds that $\deg_G(v) = \deg_{G-D}(v)$ for all $v \in X$. Moreover, because $N(A_{B_j}) \cap V_{\mathcal{B}} = \{v(B_j)\}$ (Property 2b) it holds for all $B_j \in \mathcal{B}$ and all $v(a_i, B_j) \in A_{B_j}$ that $\deg_{G-D}(v(a_i, B_j)) = \deg_G(v(a_i, B_j)) - x$ where x is one if $v(B_j) \in D$ and otherwise zero. This implies with Property 2a that $X \subseteq V_A \setminus \{v(a_i) \mid \exists v(B_j) \in D \colon a_i \in B_j\}$.

By Property 3a it follows that $V_A \subseteq V \setminus D$. To show that $V_A \setminus \{v(a_i) \mid \exists v(B_j) \in D \colon a_i \in B_j\} \subseteq X$, assume by contradiction that there is a vertex $v(a_i) \in V_A \setminus X$ but for all $v(B_j) \in D$ it holds that $a_i \notin B_j$. By Property 3b it holds that $\deg_G(v(a_i)) = \deg_{G-D}(v(a_i))$ and hence by Property 3d it follows that there is some vertex $v \in A_{\mathcal{B}}$ with $\deg_{G-D}(v) = \deg_{G-D}(v(a_i))$. Thus, since $D \subseteq V_{\mathcal{B}}$ and $N(A_{B_j}) \cap V_{\mathcal{B}} = \{v(B_j)\}$ (Property 2b), by Property 2a it follows that there is some $v(B_j) \in D$ with $a_i \in B_j$, a contradiction. □

Lemma 5. *Let G be a graph satisfying Properties 1a to 3d for a given instance $(A, \mathcal{B}, h)$ of* Set Cover*. Then $(G, 2, h)$ is a yes-instance of* Anonym V-Del *if and only if $(A, \mathcal{B}, h)$ is a yes-instance of* Set Cover*.*

Proof. If there is a set cover $\mathcal{B}' \subseteq \mathcal{B}$, $|\mathcal{B}'| \le h$, such that $\bigcup_{B_j \in \mathcal{B}'} B_j = A$, then by Observation 4 the set $S = \{v(B_j) \mid B_j \in \mathcal{B}'\} \subseteq V_{\mathcal{B}}$, $|S| = |\mathcal{B}'|$, is a 2-deletion set for G. It remains to prove the reverse direction.

Let S be a 2-deletion set of size at most $s = h$ for $G = (V, E)$. We construct a set cover $\mathcal{B}'$, $|\mathcal{B}'| \le |S|$, for the Set Cover instance. First, initialize $\mathcal{B}' := \emptyset$. Then, consider each vertex $v \in S$: If $v = v(B_j) \in V_{\mathcal{B}}$, then add B_j to $\mathcal{B}'$ (Case 1). If $v \in N[V_A]$, then by Property 1b there is only one a_i such that $v \in N[v(a_i)]$ and we add any B_j with $a_i \in B_j$ to $\mathcal{B}'$ (Case 2). Finally, if $v \in N[A_{\mathcal{B}}]$, then by Property 2c there is a vertex $v(B_j) \in V_{\mathcal{B}}$ with $N(v) \cap A_{\mathcal{B}} \subseteq N(v(B_j))$ and we add B_j to $\mathcal{B}'$ (Case 3).

We next prove that $\mathcal{B}'$ is indeed a set cover for the Set Cover instance $(A, \mathcal{B}, h)$. Assume towards a contradiction that $\mathcal{B}'$ is not a set cover, that is, there is an element $a_i \in A$ such that for each $B \in \mathcal{B}'$ we have $a_i \notin B$. Observe that in G, the vertex $v(a_i)$ violates the 2-anonymous property, that is, there is no other vertex with the degree of $v(a_i)$. Furthermore, from the construction of $\mathcal{B}'$ (see Case 2), it follows that $v(a_i) \notin S$ and that $\deg_G(v(a_i)) = \deg_{G-S}(v(a_i))$. Hence, there is a vertex $v \in V$ such that $\deg_G(v(a_i)) = \deg_{G-S}(v)$ and thus, by Property 3d, it follows that $v \in N[A_{\mathcal{B}}]$, that is, $v = v(a_i, B_j)$ for some B_j with $a_i \in B_j$. By Case 1 and Case 3 of the construction of $\mathcal{B}'$, this implies that $\mathcal{B}'$ contains a set B_j, a contradiction to the fact that there is an $a_i \in A$ such that for each $B \in \mathcal{B}'$ we have $a_i \notin B$. □

Using this generic reduction we now show NP-hardness on several graph classes which are defined as follows (see Brandstädt et al. [21]): *Trivially perfect graphs* are the $\{P_4, C_4\}$-free graphs, that is, they do not contain an induced path or cycle on four vertices. A graph G is a *bipartite permutation graph* if G is bipartite and does not contain an asteroidal triple (is *AT-free*). Three vertices of

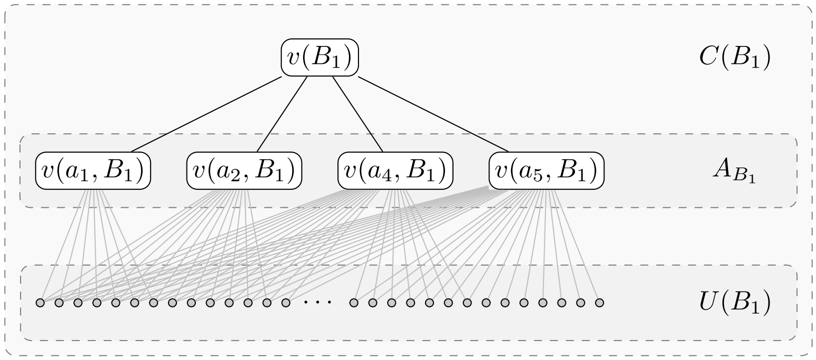


Figure 4: The set-gadget $C(B_1)$ for the constructed bipartite permutation graph. The given Set Cover instance is the same as in Figure 3 where $B_1 = \{a, a_2, a_4, a_5\}$.

a graph form an *asteroidal triple* if every two of them are connected by a path avoiding the neighborhood of the third. A graph is a *split graph* if it can be partitioned into a clique and an independent set.

Theorem 6. Anonym V-Del *is NP-complete on trivially perfect graphs, bipartite permutation graphs, and split graphs, even if* $k = 2$.

Proof. Since containment in NP is easy to see, we focus on showing NP-hardness. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_\beta\}$ be a collection of subsets of some universe $A = \{a_1, a_2, \ldots, a_\alpha\}$ which form together with some $h \in \mathbb{N}$ an instance of Set Cover. As in Reduction 1, we assume without loss of generality that for each element $a \in A$ there exists a set $B \in \mathcal{B}$ with $a \in B$. Furthermore, we assume without loss of generality that each set $B \in \mathcal{B}$ occurs at least three times in $\mathcal{B}$.

We first describe the reductions for each the three graph classes and then, due to the similarities in the constructed graphs, we show for all three graphs together that the above properties are satisfied. Let $f\colon \mathbb{N} \to \mathbb{N}$ be $f(i) = i(h+1) + \alpha$.

Bipartite permutation graphs: Analogously to the reduction for trees add for each set $a_i \in A$ an *element-gadget* consisting of star $K_{1,f(i)}$ with a center vertex denoted by $v(a_i)$. Clearly, a star is a bipartite permutation graph.

For each set $B_j \in \mathcal{B}$ we add a *set-gadget* as follows: First, add a vertex $v(B_j)$ to G. For each element $a_i \in B_j$ add a child vertex, denoted by $v(a_i, B_j)$, to $v(B_j)$. Let $i_{\max} := \max_{a_i \in B_j}\{i\}$, $\ell := |B_j|$, and $A_{B_j} := \{v(a_i, B_j) \mid a_i \in B_j\}$. Next, add the vertex set $U(B_j) := \{u_1(B_j), u_2(B_j), \ldots, u_{f(i_{\max})}(B_j)\}$ and for each $a_i \in B_j$ the edge set $\{(u_r(B_j), v(a_i, B_j)) \mid 1 \le r \le f(i)\}$.

Note that $\deg(v(a_i, B_j)) = \deg(v(a_i)) + 1$. Denote with $C(B_j)$ the set-gadget, that is, the connected component containing $v(B_j)$ which consists of the vertices $\{v(B_j)\} \cup A_{B_j} \cup U(B_j)$; see Figure 4 for an example.

Furthermore, observe that $N(u_1(B_j)) \supseteq N(u_2(B_j)) \supseteq \ldots \supseteq N(u_{i_\ell}(B_j))$ and thus, in contrast to the previous reduction for trees, $C(B_j)$ is AT-free.

Overall, the constructed graph is AT-free and clearly bipartite.

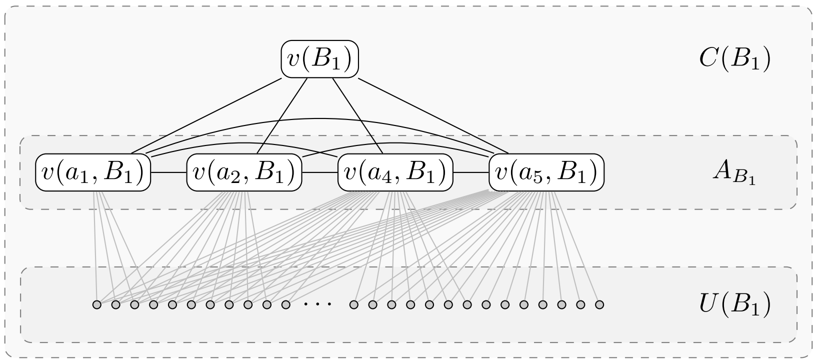


Figure 5: The set-gadget $C(B_1)$ for the constructed trivially perfect graphs. The given Set Cover instance is the same as in Figure 3 where $B_1 = \{a, a_2, a_4, a_5\}$.

Trivially perfect graphs: First, construct the graph as described above for the case of bipartite permutation graphs. Next for each $B_j \in \mathcal{B}$ apply the following changes to $C(B_j)$, see Figure 5 for an illustration: Add edges so that the vertices in A_{B_j} form a clique. To ensure that the degree of the vertices in A_{B_j} does not change by the previous "clique operation", remove the first $|B_j| - 1$ vertices from $U(B_j)$ which are all adjacent to each vertex in A_{B_j} due to the definition of f and $|B_j| \leq \alpha$.

Clearly, the star components containing the vertices from V_A are trivially perfect. Furthermore, note that each $C(B_j)$ is trivially perfect: since $\{v(B_j)\} \cup A_{B_j}$ is a clique, the remaining vertices in $U(B_j)$ form an independent set, and since $N(u_{|B_j|}(B_j)) \supseteq N(u_{|B_j|+1}(B_j)) \supseteq \ldots \supseteq N(u_{i_\ell}(B_j))$ it is easy to verify that $C(B_j)$ is indeed a threshold graph which is a special form of a trivially perfect graph [21].

Note that since the connected components containing the vertices in V_A are also threshold graphs, by the reduction above we have proven that Anonym V-Del is indeed NP-hard on graphs whose connected components are threshold graphs. However, in Theorem 17 we prove that Anonym V-Del is polynomial-time solvable on threshold graphs.

Split graphs: First, construct the graph G as described above for the case of bipartite permutation graphs. For each set $B_j \in \mathcal{B}$ set $W(B_j) := \{v(B_j)\} \cup U(B_j)$. Then, set $W_\mathcal{B} := \bigcup_{B \in \mathcal{B}} W(B)$. Finally, add edges to make the vertex subset $N(V_A) \cup W_\mathcal{B}$ to a clique. Observe that the remaining vertices form an independent set and, hence, the graph is a split graph.

Correctness: We now show that the constructed graphs satisfy Properties 1a to 3d. To this end, observe that, due to assumption that each set occurs three times in $\mathcal{B}$, each vertex in $C(B_j)$ is 2-anonymous. Hence, the vertices in V_A are exactly the ones that are not 2-anonymous. Thus, Property 1a is satisfied. Properties 2a and 3a are clearly satisfied. Since for each vertex $v(a_i) \in V_A$ the vertex set $N[v(a_i)]$ induces a star (a clique in the split graph case) and for each $j \neq i$ we have $N[v(a_i)] \cap N[v(a_j)] = \emptyset$, Property 1b is fulfilled. Observe that $A_{B_j} \subseteq N(v(B_j))$ for each $B_j \in \mathcal{B}$. Furthermore, the vertices in V_A and $V_{\mathcal{B}}$ are pairwise in different connected components in the case for trivially perfect graphs and bipartite permutation graphs. Thus, Properties 2b and 3b are fulfilled for these cases. For the case of split graphs, observe that we started with the construction for the bipartite permutation graphs and the vertices of V_A and $A_{\mathcal{B}}$ remained unchanged. Hence, Properties 2b and 3b are also fulfilled for the case of split graphs. In the constructed graphs for each $B_j, B_{j'} \in \mathcal{B}$, $j \neq j'$, we have $N(A_{B_j}) \cap N(A_{B_{j'}}) = \emptyset$. From this and $A_{B_j} \subseteq N(v(B_j))$, it follows that Property 2c is satisfied. Since $A_{\mathcal{B}} \subseteq N(V_{\mathcal{B}})$ this implies that Property 3c is fulfilled. Finally, since each vertex in $V \setminus (V_A \cup A_{\mathcal{B}})$ has degree at most α (at least $|N(V_A) \cup W_{\mathcal{B}}|$ in the split graph case), it follows from the definition of f that Property 3d is satisfied. □

Since Set Cover is W[2]-complete with respect to the solution size h [13] and the solution size s in the constructed instance was $s := h$, we have the following.

Corollary 7. Anonym V-Del *is W[2]-hard with respect to parameter s, even if $k = 2$ and if the input graph is a tree, a bipartite permutation graph, a split graph, or a trivially perfect graph.*

Set Cover is fixed-parameter tractable with respect to the combined parameter (α, h) [22] but does not admit a polynomial kernel with respect to (α, h) [23], unless NP $\subseteq$ coNP/poly. Observe that in all constructions for Theorem 6 except the one for split graphs we can bound s and Δ in a polynomial in α and h.

Corollary 8. Anonym V-Del *on trees, bipartite permutation graphs or trivially perfect graphs does not admit a polynomial kernel with respect to the combined parameter (k, s, Δ), unless NP $\subseteq$ coNP/poly.*

There are two natural optimization versions associated with Anonym V-Del: in one version (called Max-Anonym V-Del) the goal is to maximize the anonymity k subject to the constraint that the number s of deleted vertices does not exceed a given bound; in the other version (called Anonym Min-V-Del) the goal is to minimize the number s of deleted vertices subject to the constraint that the anonymity does not go below a certain given bound. As Set Cover is NP-hard to approximate within a ratio $o(\log n)$ [24, 25], the above reduction yields the following inapproximability result.

Corollary 9. Anonym Min-V-Del *cannot be approximated within a factor of $o(\log n)$ in polynomial-time, even if $k = 2$ and if the input graph is a tree, a bipartite permutation graph, a split graph, or a trivially perfect graph, unless P = NP.*

Since the above reduction gives NP-hardness for $k = 2$ and the input graph is 1-anonymous, we immediately get inapproximability within a factor of two for Max-Anonym V-Del.

Corollary 10. *For every $0 < \varepsilon < 1$,* Max-Anonym V-Del *cannot be approximated within a factor of $2 - \varepsilon$ in polynomial time, unless P = NP. Furthermore, if* Max-Anonym V-Del *admits for any $0 < \varepsilon \leq 1$ a fixed-parameter $(2 - \varepsilon)$-approximation algorithm with respect to parameter s, then FPT = W[2].*

In the next section, we show that we can strengthen these inapproximability results.

3.3. Inapproximability Results

Corollaries 9 and 10 give first lower bounds on the polynomial-time approximability of the two optimization problems associated to Anonym V-Del, namely Anonym Min-V-Del and Max-Anonym V-Del. For general graphs, these results, however, can be strengthened considerably in terms of the achievable approximation factor and, in case of Max-Anonym V-Del, also in terms of the allowed running time. Specifically, we prove that Anonym Min-V-Del is not $n^{1-\varepsilon}$-approximable in polynomial time, while Max-Anonym V-Del is not $n^{1/2-\varepsilon}$-approximable in fpt-time with respect to the parameter s, even on trees.

To this end, for the polynomial-time inapproximability of Anonym Min-V-Del, we slightly adjust the reduction given in the proof of Theorem 2.

Theorem 11. *For every $0 < \varepsilon \leq 1/2$,* Anonym Min-V-Del *is not $n^{1-\varepsilon}$-approximable in polynomial time, even on graphs with maximum degree three, unless P = NP.*

Proof. Let $0 < \varepsilon \leq 1$ be a constant. We establish a gap-reduction with gap $n^{1-\varepsilon}$ from the Vertex Cover problem which is known to be NP-complete even in three-regular graphs [20, GT1].

Given a Vertex Cover instance $(G = (V, E), h)$ we construct an instance $I' = (G' = (V', E'), k)$ of Anonym Min-V-Del. Start by copying G into a new graph G'. Next, set $x := \lceil n^{1/\varepsilon} \rceil - n + h$. Finally, add x degree-zero vertices to G' and set $k := n - h + x$. Denote by n' the number of vertices of G', thus $n' = n + x$.

We now show that if I is a yes-instance, then $\text{opt}(I') \geq h$ and if I is a no-instance, then $\text{opt}(I') = n + x$.

Suppose that G contains a vertex cover S of size h. Then, deleting S in G' clearly results in an edgeless graph with $n - h + x = k$ vertices, implying that $\text{opt}(I') \leq h$.

Suppose that G' contains a k-deletion set S of size at most $|V'| - 1$. Since $2k > n - h + x$ and G' contains $x > h$ degree-zero vertices, all vertices in $G' - S$ have degree zero. Furthermore, at least $k = n - h + x$ degree-zero vertices are contained in $G' - S$ and hence, $|S| \leq h$ and $S \cap V$ is a vertex cover in G. Thus, if G does not contain a vertex cover of size h, then $\text{opt}(I') = |V'| = n + x$.

We obtain a gap-reduction with the gap at least

$$\frac{n + x}{h} = \frac{\lceil n^{\frac{1}{\varepsilon}} \rceil + h}{h} = \frac{(\lceil n^{\frac{1}{\varepsilon}} \rceil + h)^{(\varepsilon + 1 - \varepsilon)}}{h} \geq \frac{n \cdot (\lceil n^{\frac{1}{\varepsilon}} \rceil + h)^{(1-\varepsilon)}}{h}$$
$$\geq (\lceil n^{\frac{1}{\varepsilon}} \rceil + h)^{(1-\varepsilon)} = (n + x)^{(1-\varepsilon)} = (n')^{1-\varepsilon}.$$

□

Next we show strong parameterized inapproximability results for Max-Anonym V-Del. To this end, we adjust Reduction 1 in order to obtain an fpt gap-reduction.

Theorem 12. *For every $0 < \varepsilon \leq 1/2$, Max-Anonym V-Del is not fixed-parameter $n^{1/2-\varepsilon}$-approximable with respect to parameter s, even on trees, unless FPT = W[2].*

Proof. Let $0 < \varepsilon \leq 1/2$ be a constant. We provide an fpt gap-reduction with gap $n^{1/2-\varepsilon}$ from the W[2]-hard Set Cover problem [13] parameterized by the solution size h. Let $I = (A, \mathcal{B}, h)$ be an instance of Set Cover. We assume without loss of generality that for each element $a_i \in A$ there exists a set $B_j \in \mathcal{B}$ with $a_i \in B_j$. Let $f\colon \mathbb{N} \to \mathbb{N}$ be $f(i) = (h+4)i$. Set $t := \lceil (\alpha\beta)^{(1-2\varepsilon)/(2\varepsilon)} \rceil$. We will aim for making the constructed graph t-anonymous.

The instance I' of Max-Anonym V-Del is defined by $s = h$ and a graph $G = (V, E)$ constructed as follows: For each element $a_i \in A$ add a star $K_{1,f(i)}$ with the center vertex $v(a_i)$. Denote with $V_A = \{v(a_1), v(a_2), \ldots, v(a_\alpha)\}$ the set of all these center vertices. Furthermore, for each element $a_i \in A$ add t stars $K_{1,f(i)+1}$.

For each set $B_j \in \mathcal{B}$ add a set-gadget which will consist of a tree rooted in a vertex $v(B_j)$, see Figure 6 for an illustration. The root has $|B_j| \cdot t$ child vertices where each element $a_i \in B_j$ corresponds to exactly t of these children, denoted by $v_1(a_i, B_j), v_2(a_i, B_j), \ldots, v_t(a_i, B_j)$. Additionally, for each $\ell \in \{1, 2, \ldots, t\}$ we add to $v_\ell(a_i, B_j)$ exactly $f(i)$ degree-one neighbors. Hence, the set gadget is a tree of depth two rooted in $v(B_j)$. To ensure that the root $v(B_j)$ does not violate the t-anonymous property we add t stars $K_{1,\deg(v(B_j))}$. We denote with $V_{\mathcal{B}} = \{v(B_1), v(B_2), \ldots, v(B_\beta)\}$ the set of all root vertices. Finally, to end up with one tree instead of a forest, repeatedly add edges between any degree-one-vertices of different connected components. Denoting by n the number of vertices in G it holds that

$$n \leq \underbrace{t\beta\alpha^2}_{\text{vertices for elements}} + \underbrace{t(\beta\alpha)^2 + t\beta\alpha}_{\text{vertices for sets}} < (t\beta\alpha)^2.$$

We now show that if I is a yes-instance, then $\mathrm{opt}(I') \geq t$ and if I is a no-instance, then $\mathrm{opt}(I') = 1$.

Suppose that I has a set cover of size h. Observe that for each element $a_i \in A$ the only vertex of degree $f(i)$ is $v(a_i)$, and there are no other vertices violating the t-anonymous property. The key point in the construction is that, in order to get a t-anonymous graph, one has to delete vertices of $V_{\mathcal{B}}$. Indeed, let $a_i \in A$ be an element and $v(a_i)$ a root vertex such that $a_i \in B_j$. By construction, for each $1 \leq \ell \leq t$ the child vertex $v_\ell(a_i, B_j)$ of $v(B_j)$ has $f(i)$ child vertices and hence a degree of $f(i)+1$. Thus, deleting $v(B_j)$ lowers the degree of each $v_\ell(a_i, B_j)$ to $f(i)$ and, hence, $v(a_i)$ no longer violates the t-anonymous property. Hence, given a set cover of size h one can construct a corresponding t-deletion set for G.

Conversely, we show that if there exists a 2-deletion set of size at most h in G, then $(A, \mathcal{B}, h)$ is a yes-instance of Set Cover. Let $S \subseteq V$ be a 2-deletion

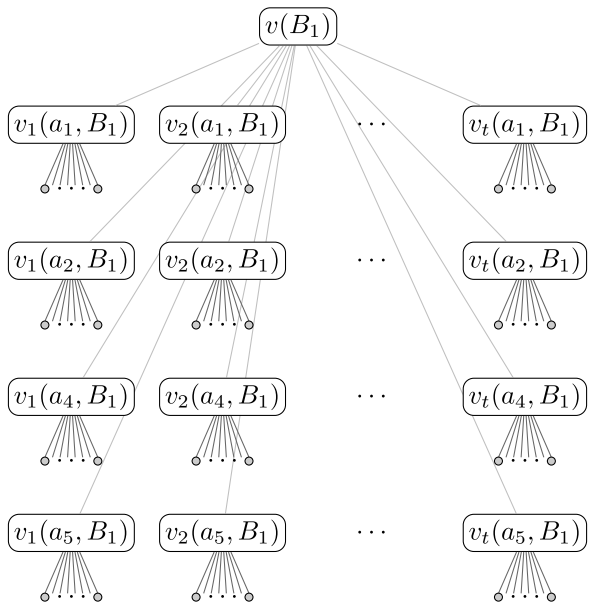


Figure 6: The set-gadget for the set B_1 in the fpt gap-reduction of Theorem 12. The given Set Cover instance is the same as in Figure 3 where $B_1 = \{a, a_2, a_4, a_5\}$. The fpt gap-reduction is an extension of Reduction 1 (depicted in Figure 3). The main difference is that the fpt gap-reduction introduces a lot of copies of certain vertices to increase the anonymity level. This can be seen in the set-gadget above: While in Reduction 1 only one vertex corresponds to the combination (a_1, B_1) with $a_1 \in B_1$, namely $v(a_1, B_1)$, in the fpt gap-reduction t vertices correspond to the combination, namely $v_1(a_1, B_1), v_2(a_1, B_1), \dots, v_t(a_1, B_1)$.

set of size at most h. We construct a set cover $\mathcal{B}'$ of size at most $|S|$ as follows. First, initialize $\mathcal{B}' := \emptyset$. Then, add for each vertex $v(B_j) \in S \cap V_\mathcal{B}$ the set B_j to $\mathcal{B}'$ (Step 1). Next, as long as there is an element a_i with $a_i \notin \bigcup_{B \in \mathcal{B}'} B$, add a set B_j with $a_i \in B_j$ to $\mathcal{B}'$ (Step 2). It is clear that $\mathcal{B}'$ is a set cover for $(A, \mathcal{B}, h)$. It remains to prove that $|\mathcal{B}'| \leq |S|$. To this end, partition the set S into $S_1 \cap S_2$ where S_1 contains exactly the vertices in $V_\mathcal{B}$, that is $S_1 := S \cap V_\mathcal{B}$ and $S_2 := S \setminus S_1$. Observe that the number of sets added to $\mathcal{B}'$ in Step 1 is exactly $|S_1|$. Furthermore, observe that all vertices in V_A violate the 2-anonymous property and each of these vertices is a center of an isolated star with more than two leaves. Since the only vertices in G that are adjacent to more than one vertex of degree at least three are the vertices in $V_\mathcal{B}$, it follows, each vertex in S_2 "fixes" for at most one vertex in V_A the 2-anonymous property. Hence, the number of sets added in Step 2 is at most $|S_2|$. Thus $|\mathcal{B}'| \leq |S|$ and $(A, \mathcal{B}, h)$ is a yes-instance of Set Cover.

We obtain an fpt gap-reduction with the gap

$$\begin{aligned} t &= (t^2)^{1/2+\varepsilon-\varepsilon} = t^{2\varepsilon}(t^2)^{1/2-\varepsilon} = (\alpha\beta)^{1-2\varepsilon}(t^2)^{1/2-\varepsilon} \\ &= (\alpha^2\beta^2)^{1/2-\varepsilon}(t^2)^{1/2-\varepsilon} = (\alpha^2\beta^2 t^2)^{1/2-\varepsilon} > n^{1/2-\varepsilon} \end{aligned}$$

since $n < t^2\alpha^2\beta^2$. Thus, the statement of the theorem follows from Lemma 1. $\square$

Since the fpt gap-reduction provided in the proof of Theorem 12 can be constructed in polynomial time and since Set Cover is NP-complete, we also obtain polynomial-time inapproximability under the stronger assumption P = NP.

Theorem 13. *For every $0 < \varepsilon \leq 1/2$,* Max-Anonym V-Del *is not $n^{1/2-\varepsilon}$-approximable in polynomial time, even on trees, unless P = NP.*

3.4. Polynomially-Time Solvable Cases

We complement our intractability results for Anonym V-Del from the previous sections by showing that Anonym V-Del is polynomial-time solvable on graphs with maximum degree two, on graphs that are disjoint unions of cliques, and on threshold graphs.

3.4.1. Graphs with Maximum Degree Two

In contrast to graphs of maximum degree three (see Theorem 2), we observe that Anonym V-Del is polynomial-time solvable on graphs of maximum degree two. Note that a graph of maximum degree two is just a collection of paths and cycles. Given five integers d_0, d_1, d_2, x, y, it is easy to decide whether it is possible to remove x vertices from a path of length y (respectively, from a cycle of length y) such that there survive precisely d_0 vertices of degree zero, d_1 vertices of degree one, and d_2 vertices of degree two. A straight-forward dynamic programming approach based on this observation leads to the following.

Theorem 14. *On graphs of maximum degree two,* Anonym V-Del *is polynomial-time solvable.*

3.4.2. Disjoint Union of Cliques

Note that Anonym V-Del is trivial on cliques: either the clique size is at least k, or otherwise one has to delete all the vertices. The following theorem shows that polynomial-time solvability also carries over to the case where the graph is the disjoint union of several cliques, that is, a cluster graph. A graph is a cluster graph if and only if it does not contain the 3-vertex path P_3 as an induced subgraph.

Theorem 15. *On a cluster graph G with maximum degree Δ,* Anonym V-Del *can be solved in $\mathcal{O}(n^2\Delta)$ time.*

Proof. Note that removing any number of vertices from a cluster graph yields another cluster graph. For an integer $c \geq 1$, we denote by $\#_{\text{comp}}(G, c)$ the number of components of size c in G. For integers $x, y \geq 1$, we denote by $G(x, y)$ the graph that consists of all components of G of size up to x, together with y new components (cliques) of size exactly x.

We design a dynamic program that solves Anonym V-Del for all such graphs $G(x, y)$. We denote by $f(x, y)$ the smallest possible number of vertices whose removal from $G(x, y)$ yields a k-anonymous graph, and we store all these values in the dynamic programming table. In the initialization phase of the dynamic program we handle the cases with $x = 1$. Note that the graph $G(1, y)$ consists of $t := \#_{\text{comp}}(G, 1)+y$ isolated vertices. Then $f(1, y) = 0$ whenever $t \geq k$, and $f(1, y) = t$ whenever $t < k$.

The cases with $x \geq 2$ are handled as follows. Consider a graph $G(x, y)$ that contains $t := \#_{\text{comp}}(G, x) + y$ components of size x. A k-anonymous subgraph of $G(x, y)$ will contain a certain number z of these components, while from each of the remaining $t - z$ components (at least) one vertex is to be removed; note that this requires $x \cdot z \geq k$ whenever $z \neq 0$. This yields the formula

$$f(x, y) \;=\; \min \{ f(x-1, t-z) + t - z \;\mid\; z = 0 \text{ or } k/x \leq z \leq t \} .$$

As the largest clique in G contains $\Delta+1$ vertices, the dynamic programming table has $\mathcal{O}(n\Delta)$ entries. We precompute all the values $\#_{\text{comp}}$, and then determine every value $f(x, y)$ in $\mathcal{O}(n)$ time per entry. All in all, this yields the claimed running time of $\mathcal{O}(n^2\Delta)$. The final answer for the graph G is given by $f(\Delta + 1, 0)$. □

3.4.3. Threshold Graphs

We recall that a graph $G = (V, E)$ is a *threshold graph* if there are positive real vertex weights $w(v)$ for $v \in V$ such that $\{v_1, v_2\} \in E$ if and only if $w(v_1)+w(v_2) \geq 1$; see Brandstädt et al. [21] for more information. Without loss of generality we will assume that the vertex weights satisfy the following conditions:

- The vertex weights are pairwise distinct, and satisfy $0 < w(v) < 1$.
- Any $v_1, v_2 \in V$ satisfy $w(v_1) + w(v_2) \neq 1$; in particular $w(v_1) \neq 1/2$.

Note that the closed neighborhoods in a threshold graph are totally ordered by inclusion: whenever $w(v_1) < w(v_2)$, then $N_G[v_1] \subseteq N_G[v_2]$ and consequently $\deg(v_1) \leq \deg(v_2)$.

Lemma 16. *Let $U \subseteq V$ be a subset of vertices with $|U| \geq 2$, let $w_{\min} = \min_{u \in U} w(u)$ and $w_{\max} = \max_{u \in U} w(u)$, and let $u_0, u_1 \in U$ be the vertices with $w(u_0) = w_{\min}$ and $w(u_1) = w_{\max}$. All vertices in U have identical degree if and only if there is no vertex $v \in V \setminus \{u_0, u_1\}$ with $1 - w_{\max} < w(v) < 1 - w_{\min}$.*

Proof. Note that all vertices in U have identical degree if and only if $N_G[u_0] = N_G[u_1]$. The latter condition in turn holds if and only if there is no vertex v in the graph (with $v \neq u_0$ and $v \neq u_1$) that is adjacent to u_1 but not to u_0, and this is equivalent to the stated condition $1 - w_{\max} < w(v) < 1 - w_{\min}$. □

Theorem 17. Anonym V-Del *on threshold graphs is solvable in $\mathcal{O}(n^2)$ time.*

Proof. We provide a dynamic program to solve the problem in the claimed running time. To this end, we first need some further notation: Recall that a block $B_G(d)$ of degree d contains all degree-d vertices of G. Now consider some block $B_G(d)$ of constant degree d in an optimal subgraph for Anonym

V-Del, and let $u_0, u_1 \in B_G(d)$ and $w_{\min}$ and $w_{\max}$ be defined as in the lemma. The *territory* of this block is defined as the union of the two closed intervals $[w_{\min}, w_{\max}]$ and $[1 - w_{\max}, 1 - w_{\min}]$; note that these two intervals will overlap if $w_{\min} < 1/2 < w_{\max}$. The *canonical superset* $U^* \subseteq V$ consists of u_0 and u_1, together with all vertices $v \in V$ that satisfy $w_{\min} \leq w(v) \leq w_{\max}$ but not $1 - w_{\max} < w(v) < 1 - w_{\min}$. One message of Lemma 16 is that distinct blocks in an optimal subgraph must have disjoint territories. Another message of the Lemma 16 is that we may as well replace every block $B_G(d)$ by its canonical superset U^*: By adding these vertices, the degree in every block either remains the same or is uniformly increased by $|U^*| - |B_G(d)|$. And if the territories of distinct blocks were disjoint before the replacement, then they will also be disjoint after the replacement. In other words, such a replacement does not violate k-anonymity but simplifies the combinatorial structure of the considered subgraph.

This suggests the following dynamic programming approach. For every real number r with $0 \leq r \leq 1/2$, we consider the threshold graph G_r that is induced by the vertices $v \in V$ with $r \leq w(v) \leq 1 - r$; note that the only crucial values for r are the $\mathcal{O}(n)$ values $w(v)$ and $1 - w(v)$ that fall between the bounds 0 and $1/2$. The goal is to compute for every graph G_r a largest k-anonymous subgraph. We start our computations with $r = 1/2$ and work downwards towards $r = 0$.

The initialization step of the dynamic program handles subgraphs that consist of a single block whose territory contains the number $1/2$. Such a block will either be empty, or it is a canonical superset specified by two values $w_{\min}$ and $w_{\max}$. All in all, this only yields a polynomial number of cases to handle. In the main computation phase of the dynamic program, we consider a general graph G_r and check all possibilities for the outermost block, which is the block whose territory is farthest away from the center point $1/2$. Since this territory is the union of two intervals $[r, q]$ and $[1 - q, 1 - r]$, we may simply check all possibilities for the interval boundary q, and then combine the corresponding block with the (previously computed) largest k-anonymous subgraph for graph G_q. Since there is only a linear number $\mathcal{O}(n)$ of candidate values for q, the largest k-anonymous subgraph of G_r can be found in linear time. □

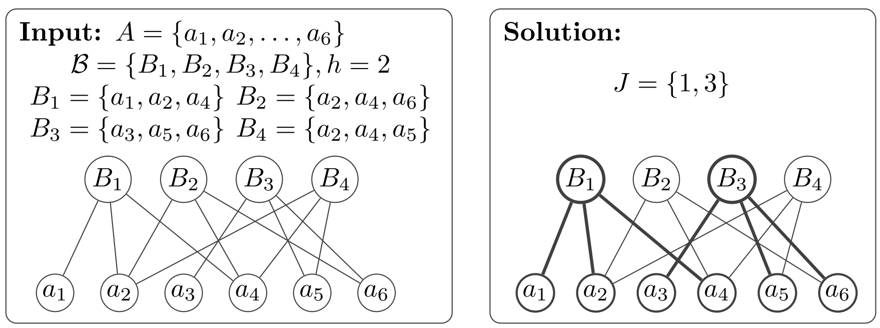
4. Edge Deletion

In this section, we transfer the central intractability results from Section 3 to the setting where instead of vertices edges are removed; see Section 1 for a discussion about vertex deletions versus edge deletions. To this end, we first show in Subsection 4.1 that Anonym E-Del is NP-complete on caterpillars, a subclass of trees. Compared to the NP-completeness of Anonym V-Del on trees (see Subsection 3.1) this gives a slightly stronger intractability result for Anonym E-Del. The employed reduction is, however, more complicated than the one given in Subsection 3.1 and we could not come up with a general reduction scheme as provided in the vertex deletion case in Subsection 3.2. We then provide in Subsection 4.2 polynomial-time inapproximability results for Anonym Min-E-Del and Max-Anonym E-Del for bounded-degree graphs and parameterized inapproximability results for Max-Anonym E-Del on general graphs.

4.1. NP-Hardness on Caterpillars

In this section, we establish a polynomial-time reduction from the NP-complete Exact Cover by 3-Sets problem, which is defined as follows:

Exact Cover by 3-Sets [20, SP2]
Input: A universe $A = \{a_1, a_2, \ldots, a_{3h}\}$, a collection $\mathcal{B} = \{B_1, B_2, \ldots, B_\beta\}$ of 3-element sets over A, and $h \in \mathbb{N}$.
Question: Is there an index set $J \subseteq \{1, 2, \ldots, \beta\}$ with $|J| = h$, such that $\bigcup_{j \in J} B_j = A$?



If an Exact Cover by 3-Sets instance $I = (A, \mathcal{B}, h)$ contains such an index set J, then we refer to the set $\{B_j \mid j \in J\}$ as an *exact cover* for I.

The reduction in the following proof, showing that Anonym E-Del is NP-complete on caterpillars, is an adaption of the reduction provided in Subsection 3.1. A caterpillar is a tree that has a dominating path [21], that is, a caterpillar is a tree such that deleting all leaves results in a path.

Theorem 18. Anonym E-Del *is NP-complete on caterpillars, even if* $k = 2$.

Proof. Since containment in NP is easy to see, we focus on showing NP-hardness. To this end, we provide a polynomial-time reduction from Exact Cover by 3-Sets. Let $I = (A, \mathcal{B}, h)$ be an instance of Exact Cover by 3-Sets. We assume without loss of generality that for each element $a_i \in A$ there exists a set $B_j \in \mathcal{B}$ with $a_i \in B_j$. Let $f \colon \mathbb{N} \to \mathbb{N}$ be $f(i) = (2h + 3)i$.

The instance I' of Anonym E-Del is defined on a graph $G = (V, E)$ constructed as follows. For each element $a_i \in A$ add a star $K_{1,f(i)}$ with the center vertex $v(a_i)$. Denote with $V_A = \{v(a_1), v(a_2), \ldots, v(a_{3h})\}$ the set of all these center vertices. Furthermore, for each element $a_i \in A$ add two stars $K_{1,f(i)+1}$ and two stars $K_{1,f(i)+2}$.

For each set $B_j \in \mathcal{B}$ with $B_j = \{a_{j_1}, a_{j_2}, a_{j_3}\}$ add a *set-gadget* containing the stars $K_{1,f(j_1)}$, $K_{1,f(j_2)}$, and $K_{1,f(j_3)}$. See Figure 7 for the difference of the set-gadget in this reduction and the reduction in Subsection 3.1. Denote with $v(a_{j_1}, B_j)$, $v(a_{j_2}, B_j)$, and $v(a_{j_3}, B_j)$ the center vertices of these stars and denote with $V_{\mathcal{B}}$ the set of all these center vertices, formally $V_{\mathcal{B}} = \{v(a_i, B_j) \mid$

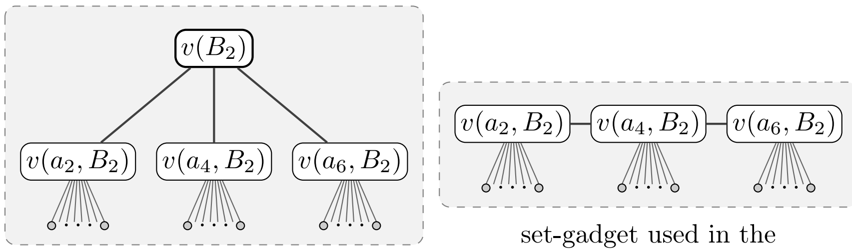


Figure 7: The difference between the set-gadgets used in Reduction 1 and the reduction showing NP-hardness of Anonym E-Del on caterpillars. Deleting the vertex $v(B_2)$ corresponds to deleting the two edges $\{v(a_2, B_2), v(a_4, B_2)\}$ and $\{v(a_4, B_2), v(a_6, B_2)\}$.

$1 \leq i \leq 3h \wedge 1 \leq j \leq \beta \wedge a_i \in B_j\}$. Next, add the edges $\{v(a_{j_1}, B_j), v(a_{j_2}, B_j)\}$ and $\{v(a_{j_2}, B_j), v(a_{j_3}, B_j)\}$ to E. Observe that $\deg(v(a_{j_1}, B_j)) = f(j_1) + 1$, $\deg(v(a_{j_2}, B_j)) = f(j_2) + 2$, and $\deg(v(a_{j_3}, B_j)) = f(j_3) + 1$. To end up with one caterpillar instead of a forest of caterpillars, do the following:

1. Take two different connected components (caterpillars) C_1 and C_2, let v_1 be an endpoint of a dominating path in C_1, and let v_2 be an endpoint of a dominating path in C_2, such that $\deg_G(v_1) = \deg_G(v_2) = 1$.

2. Then, add the edge $\{v_1, v_2\}$ to reduce the number of connected components by one.

3. If there exists more than one connected component, goto Step 1.

The resulting graph is clearly a caterpillar. We complete the construction of I' by setting $s = 2h$ and $k = 2$.

We now prove that I is a yes-instance of Exact Cover by 3-Sets if and only if $I' = (G, k, s)$ is a yes-instance of Anonym E-Del.

"$\Rightarrow$:" Let $\mathcal{B}' \subseteq \mathcal{B}$ be an exact cover of size h. Then we construct a 2-deletion set $S \subseteq E$ of size $2h$ as follows: For each set $B_j \in \mathcal{B}'$ with $B_j = \{a_{j_1}, a_{j_2}, a_{j_3}\}$ insert the edges $\{v(a_{j_1}, B_j), v(a_{j_2}, B_j)\}$ and $\{v(a_{j_2}, B_j), v(a_{j_3}, B_j)\}$ into S. First, observe that $|S| = 2h$. Next, we show that S is indeed a 2-deletion set. Suppose towards a contradiction that there exists a vertex $v \in V$ such that there is no further vertex of the same degree in $G - S$. Then, by construction of G, it follows that $v = v(a_i) \in V_A$ for some $i \in \{1, 2, \ldots, 3h\}$ and, by construction of S, it follows that $a_i \notin \bigcup_{B_j \in \mathcal{B}'} B_j$, a contradiction.

"$\Leftarrow$:" Let S be a 2-deletion set of edges of size at most $2h$. Observe that the only vertices in G that violate the 2-anonymous property are the vertices in V_A. Furthermore, for each $a_i \in A$ there is exactly one vertex in G with a degree d between $f(i) - 2h \leq d \leq f(i)$, namely $v(a_i)$. Since S is a 2-deletion set, it follows that for each $v(a_i) \in V_A$ there is a vertex $v \in V(S)$ having the same degree as $v(a_i)$ in $G - S$. Since $|V_A| = 3h$ and $|\deg(v(a_i)) - \deg(v(a_{i'}))| > 2h$ for all $i, i' \in \{1, 2, \ldots, 3h\}$, it follows that $|V(S)| \geq 3h$. For the further argumentation we need some notation. A vertex $v \in V$ is a *type-ℓ* vertex, $\ell \in \mathbb{N}$, if there exists a vertex $v(a_i) \in V_A$ such that $\deg_G(v) = \deg_G(v(a_i)) + \ell$. Now, observe that in G the type-1 vertices are all pairwise non-adjacent and have pairwise disjoint neighborhood sets. Thus, $V(S)$ contains at most $2h$ type-1 vertices. Furthermore, since $|V(S)| \geq 3h$, this implies that $V(S)$ contains exactly $2h$ type-1 vertices and exactly h type-2 vertices and that $|V(S)| = 3h$. Thus, for each edge in S it follows that one endpoint is a type-1 vertex and the other endpoint is a type-2 vertex. Note that the only edges fulfilling this requirement are the ones making two vertices in $V_{\mathcal{B}}$ adjacent and, thus, $V(S) \subseteq V_{\mathcal{B}}$. Thus, each type-2 vertex of $V(S)$ is contained in some set-gadget. Denote with $\mathcal{B}'$ the set of h sets corresponding to the set-gadgets that contain the h type-2 vertices in $V(S)$. We now prove that $\mathcal{B}'$ is an exact cover. Suppose towards a contradiction that there is an element $a_i \notin \bigcup_{B_j \in \mathcal{B}'} B_j$. This implies, that no vertex $v(a_i, B_j)$ such that $j \in \{1, 2, \ldots, n\}$ and $a_i \in B_j$ is contained in $V(S)$. However, as $V(S) \subseteq V_{\mathcal{B}}$, this means that $v(a_i)$ has a unique degree in $G - S$, a contradiction to the fact that S is a 2-deletion set. Finally, since $|\mathcal{B}'| = h$, $\bigcup_{B_j \in \mathcal{B}'} B_j = A$, each set contains exactly three elements, and $|A| = 3h$, it follows that no element is covered twice. Hence, $\mathcal{B}'$ is an exact cover and, thus, I is a yes-instance. □

Note that Exact Cover by 3-Sets is fixed-parameter tractable with respect to the solution size h: There is a simple polynomial kernel which can be obtained by removing for each set all copies from the collection $\mathcal{B}$. After this deletion of the copies, the number of sets in the collection is bounded by $|\mathcal{B}| \leq |A|^3 = (3h)^3$.

Hence, we cannot state an equivalent of Corollary 9. However, since we established NP-completeness for $k = 2$, we obtain the following equivalent of the polynomial-time inapproximability result in Corollary 10.

Corollary 19. *For every $0 < \varepsilon < 1$,* Max-Anonym V-Del *on caterpillars cannot be approximated within a factor of $2-\varepsilon$ in polynomial time, unless P = NP.*

4.2. Inapproximability Results

As in Subsection 3.3, we can state strong inapproximability results for Anonym Min-E-Del and Max-Anonym E-Del. We remark that these inapproximability results transfer modulo the bounded-degree restriction to Anonym Min-E-Ins and Max-Anonym E-Ins, since the edge insertion variant is equivalent to the edge deletion variant in the complement graph.

Two very similar gap-reductions from Exact Cover by 3-Sets yield that Max-Anonym E-Del as well as Anonym Min-E-Del are not $n^{1-\varepsilon}$-approximable in polynomial-time on bounded degree graphs.

Theorem 20. *For every $0 < \varepsilon \leq 1$,* Max-Anonym E-Del *is not $n^{1-\varepsilon}$-approximable in polynomial time, even on bounded degree graphs, unless P = NP.*

Proof. Let $0 < \varepsilon \leq 1$ be a constant. We provide a gap-reduction with gap $n^{1-\varepsilon}$ from Exact Cover by 3-Sets which remains NP-complete even when no element occurs in more than three subsets [20, SP2]. For these instances we have $h \leq \beta \leq 3h$.

Let $I = (A, \mathcal{B}, h)$ be an instance of Exact Cover by 3-Sets where no element occurs in more than three subsets. Construct an instance $I' = (G, s)$ of Max-Anonym E-Del as follows. The graph $G = (V, E)$ contains an *element-vertex* $v(a_i)$ for each element a_i from A and a *set-vertex* $v(B_j)$ for each subset B_j from $\mathcal{B}$. There is an edge in G between $v(a_i)$ and $v(B_j)$ if B_j contains a_i. For each vertex $v(B_j)$ add four degree-one vertices that are adjacent to $v(B_j)$, thus the degree of each vertex $v(B_j)$ is seven. For each vertex $v(a_i)$ add up to three degree-one vertices that are adjacent to $v(a_i)$ such that the degree of $v(a_i)$ is three (observe that each element occurs in at most three sets). Set $x := \lceil (6h17^{1-\varepsilon})^{1/\varepsilon} \rceil$. Next, add x stars $K_{1,7}$ and x stars $K_{1,4}$ to G. If the number of degree-one vertices is odd, then add one further star $K_{1,7}$ to G to ensure that the number of degree-one vertices is even. Now, add a perfect matching on the degree-one vertices to increase their degrees to two. Finally set $s := 3h$. Thus the graph G has $\beta + x$ or $\beta + x + 1$ degree-seven vertices, x degree-four vertices, $3h$ degree-three vertices, and between $4\beta + 11x$ and $4\beta + 9h + 11x + 7$ degree-two vertices. Hence, G is $3h$-anonymous. Overall, G is a graph with maximum degree seven and at most $12x + 12h + 5\beta + 7$ vertices. Observe, that $x \geq 6h \geq 2\beta$ and thus $|V| \leq 17x$.

We now show that if I is a yes-instance, then $\text{opt}(I') \geq x$ and if I is a no-instance, then $\text{opt}(I') \leq 6h$.

Suppose that I contains an exact cover $\mathcal{B}' \subseteq \mathcal{B}$ of size h. Then removing from G the $3h$ edges between $v(B_j) \in \mathcal{B}'$ and $v(a_i) \in A$, we obtain an x-anonymous graph G', since all vertices from the block of degree three from G are in G' in the block of degree two.

Suppose that $S \subseteq E$ is a $(6h+1)$-deletion set of size $|S| \leq s = 3h$, that is, $G - S$ is $(6h+1)$-anonymous. First, observe that $V(S)$ does not contain a vertex having degree two in G: Since $|S| \leq 3h$, at most $6h$ degree-two vertices can be contained in $V(S)$. Since $G - S$ is $(6h+1)$-anonymous and G does not contain any degree-zero or degree-one vertices, this implies that $V(S)$ does not contain any degree-two vertex. Next, observe that the only edges in G that have no degree-two vertex as endpoint are edges with one set-vertex and one element-vertex as endpoints. Since each set-vertex is, by construction, adjacent to at most three element-vertices, this implies that all set-vertices in $G - S$ have degree at least four. Furthermore, since the $3h$ element-vertices are the only vertices in G having degree three and S is a $(6h+1)$-deletion set, this implies that $V(S)$ contains all element-vertices. Hence, $|S| = 3h$ and each element-vertex is incident to exactly one edge in S. Observe that G contains no vertex of degree five or six. Since S is a $(6h+1)$-deletion set, this implies that each set-vertex in $V(S)$ has degree four in $G - S$ and is incident to exactly three edges in S. Hence, $V(S)$ contains exactly h set-vertices and the corresponding sets form an exact cover of size h for I. Thus, if I does not contain any exact cover of size h, then there exists no $(6h+1)$-deletion set of size h for G and, hence, $\text{opt}(I') \leq 6h$.

Thus we obtain a gap-reduction with the gap

$$\frac{x}{6h} = \frac{x^{\varepsilon} x^{1-\varepsilon}}{6h} = \frac{6h \cdot 17^{1-\varepsilon} \cdot x^{1-\varepsilon}}{3h} \geq (17x)^{1-\varepsilon} \geq |V|^{1-\varepsilon}.$$

□

Adjusting the gap-reduction above a little bit yields the following result.

Theorem 21. *For every $0 < \varepsilon \leq 1$, Anonym Min-E-Del is not $n^{1-\varepsilon}$-approximable in polynomial time, even on bounded degree graphs, unless $P = NP$.*

Proof. Let $0 < \varepsilon \leq 1$ be a constant. We provide a gap-reduction with gap $n^{1-\varepsilon}$ from Exact Cover by 3-Sets to Anonym Min-E-Del. This reduction is very similar to the gap-reduction provided in the proof of Theorem 20. Let $I = (A, \mathcal{B}, h)$ be an instance of Exact Cover by 3-Sets where no element occurs in more than three subsets. We provide an instance $I' = (G, k)$ of Anonym Min-E-Del where the graph is constructed as in the proof of Theorem 20 and $k := x$.

We now show that if I is a yes-instance then $\mathrm{opt}(I') = 3h$ and if I is a no-instance then $\mathrm{opt}(I') \geq x/2$.

Suppose that I contains an exact cover $\mathcal{B}' \subseteq \mathcal{B}$ of size h. Then removing from G the $3h$ edges between $v(B_j) \in \mathcal{B}'$ and $v(a_i) \in A$, we obtain a k-anonymous graph G', since all vertices from the block of degree three from G are in G' in the block of degree two.

Suppose that G has a k-deletion set S of size at most $x/2 - 1$. First, observe that $V(S)$ does not contain a vertex having degree two in G: Since $|S| \leq x/2 - 1$, at most $x - 2$ degree-two vertices can be contained in $V(S)$. Since $G - S$ is k-anonymous, $k = x$, and G does not contain any degree-zero or degree-one vertex, this implies that $V(S)$ does not contain any degree-two vertex. Next, observe that the only edges in G that have no degree-two vertex as endpoint are edges with one set-vertex and one element-vertex as endpoints. Since each set-vertex is, by construction, adjacent to at most three element-vertices, this implies that all set-vertices in $G - S$ have degree at least four. Furthermore, since the $3h$ element-vertices are the only vertices in G having degree three and S is a k-deletion set with $k = x > 3h$, this implies that $V(S)$ contains all element-vertices. Furthermore, as G does not contain any degree-zero or degree-two vertex, it follows that each element-vertex is incident to exactly one edge in S. Observe that G contains no vertex of degree five or six. Since S is a k-deletion set of size at most $x/2 - 1$, this implies that each set-vertex in $V(S)$ has degree four in $G - S$ and is incident to exactly three edges in S. Hence, $V(S)$ contains exactly h set-vertices and the corresponding sets form an exact cover of size h for I. Thus, if I does not contain any exact cover of size h, then there exists no k-deletion set of size $x/2 - 1$ for G and, hence, $\mathrm{opt}(I') \geq x/2$.

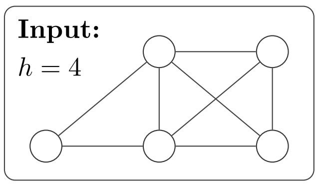
Thus we obtain a gap-reduction with the gap at least $x/(2 \cdot 3h) \geq |V|^{1-\varepsilon}$ (see the proof of Theorem 20 for intermediate steps in the inequality). ☐

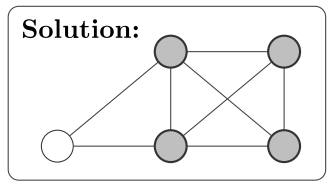
Similarly to Max-Anonym V-Del, we now show strong inapproximability of Max-Anonym E-Del, even when allowing fpt-time instead of polynomial time. Note that, in contrast to the vertex deletion case in Subsection 3.3, we obtain the same inapproximability result as in the minimization variant in terms of the approximation factor. Unlike the previous reductions and the reductions in Subsection 3.3, we reduce from the W[1]-complete Clique problem, thus building on a slightly stronger assumption.

Theorem 22. *For every $0 < \varepsilon \leq 1$,* Max-Anonym E-Del *is not fixed-parameter $n^{1-\varepsilon}$-approximable with respect to parameter s, unless FPT = W[1].*

Proof. Let $0 < \varepsilon \leq 1$ be a constant. We provide an fpt gap-reduction with gap $n^{1-\varepsilon}$ from the W[1]-complete Clique problem [13] parameterized by the solution size h.

Clique [20, GT19]
Input: An undirected graph $G = (V, E)$ and an integer $h \in \mathbb{N}$.
Question: Is there a subset $V' \subseteq V$ of at least h pairwise adjacent vertices?





Let $I = (G, h)$ be an instance of Clique. Assume without loss of generality that $\Delta_G + 2h + 1 \leq n$, where $n = |V|$. If this is not the case, then one can add isolated vertices to G until the bound holds.

We construct an instance $I' = (G' = (V', E'), s)$ of Max-Anonym E-Del as follows: First, copy G into G'. Then, add a vertex u and connect it to the n vertices in G'. Next, for each vertex $v \in V$ add to G' degree-one vertices that are adjacent only to v such that $\deg_{G'}(v) = n - h$. This is always possible since we assumed $\Delta_G + 2h + 1 \leq n$. Observe that in this way at most $n(n-h)$ degree-one vertices are added. Now, set $x := \lceil (4n)^{3/\varepsilon} \rceil$ and add cliques with two, $n - 2h + 1$, and $n - h + 1$ vertices such that after adding these cliques the number of degree-d vertices in G', for each $d \in \{1, n - 2h, n - h\}$, is between $x + h$ and $x + h + n$, that is, $x + h \leq |B_{G'}(d)| \leq x + h + n$; recall that $B_{G'}(d)$ is the set of vertices having degree d in G'. After inserting these cliques, the graph consists of four blocks: of degree one, $n - h$, $n - 2h$, and n, where the first three blocks are roughly of the same size (between $x + h$ and $x + h + n$ vertices) and the last block of degree n contains exactly one vertex. To finish the construction, set $s := \binom{h}{2} + h$.

Now we show that if I is a yes-instance, then $\mathrm{opt}(I') \geq x$, and if I is a no-instance, then $\mathrm{opt}(I') < 2s$.

Suppose that I contains a clique $C \subseteq V$ of size h. Then, deleting the $\binom{h}{2}$ edges within C and the h edges between the vertices in C and u does not exceed the budget s and results in an x-anonymous graph G'': Since h edges incident to u are deleted, it follows that $\deg_{G''}(u) = n - h$. Furthermore, for each clique-vertex $v \in C$ also h incident edges are deleted ($h - 1$ edges to other clique-vertices and the edge to u), thus it follows that $\deg_{G''}(v) = n - 2h$. Since the degrees of the remaining vertices remain unchanged, and $|B_{G'}(n - h)| \geq x + h$, it follows that each of the three blocks in G'' has size at least x. Hence, G'' is x-anonymous.

For the reverse direction, suppose that there is a $2s$-deletion set S of size at most s in G'. Since u is the only vertex in G' with degree n, and all other vertices in G' have degree at most $n - h$, it follows that S contains at least h edges that are incident to u. Since $N_{G'}(u) = V$, it follows that the degree of at least h vertices of the block $B_{G'}(n-h)$ is decreased by one. Denote these vertices by C. Since $|S| \leq s$ and h edges incident to u are contained in S, it follows that at most $2s-h+1$ vertices are incident to an edge in S. Furthermore, since S is a $2s$-deletion set, it follows that the vertices in C have in $G'-S$ either degree one or degree $n-2h$. Thus, by deleting the at most $\binom{h}{2}$ remaining edges in S, the degree of each of the h vertices in C is decreased by at least $h-1$. Hence, these $\binom{h}{2}$ edges in S form a clique on the vertices in C and thus I is a yes-instance. Therefore, it follows that if I is a no-instance, then there is no $2s$-deletion set of size s in G'' and hence $\mathrm{opt}(I') < 2s$.

Altogether, we obtain a gap-reduction with the gap at least $x/(2s)$. Set $n' := |V'|$. By construction we have $3x \leq n' \leq n^2+3x+3h+3n+1$. By the choice of x it follows that $x > n'/4$, since

$$\frac{n'}{4} \leq \frac{1}{4}(n^2+3x+3h+3n+1) = x + \underbrace{\frac{1}{4}(n^2+3h+3n+1-x)}_{<0} < x.$$

Hence the gap is

$$\frac{x}{2s} > \frac{(n')^{1-\varepsilon+\varepsilon}}{4(h^2+h)} \geq n'^{1-\varepsilon}\frac{(n')^{\varepsilon}}{8h^2} > (n')^{1-\varepsilon}\frac{x^{\varepsilon}}{8n^2} = (n')^{1-\varepsilon}\frac{(4n)^{3\varepsilon/\varepsilon}}{8n^2} > (n')^{1-\varepsilon}.$$

Thus, the statement of the theorem follows from Lemma 1. □

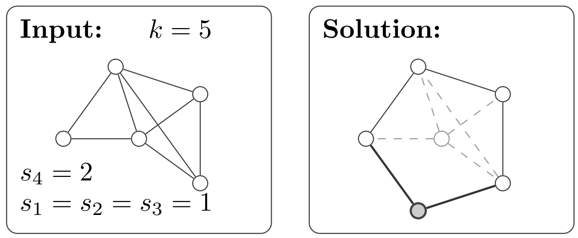
Note that the reduction above also shows that Anonym E-Del is W[1]-hard with respect to the combined parameter (s,k): It is shown that if the input graph G contains a clique of size h, then there exists an x-deletion set S of size $s = \binom{h}{2}+h$ in G'. Since $x > 2s$ it follows that S is also a $2s$-deletion set of size s. We also proved that if G' contains a $2s$-deletion set of size s, then there exists a size-h clique in G. Hence, we obtain the following: (G,h) is a yes-instance of Clique if and only if $(G', 2s, s)$ is a yes-instance of Anonym E-Del. Thus, we arrive at the following corollary.

Corollary 23. Anonym E-Del *is W[1]-hard with respect to the combined parameter* (s,k).

5. Fixed-Parameter Tractable Cases

Theorem 2 and Corollaries 7 and 23 show that Anonym E-Del and Anonym E-Del are fixed-parameter intractable for the each of single parameters s, k, and Δ as well as for the combined parameter (s,k). Here we show fixed-parameter tractability with respect to the combined parameter (s,Δ) for the following general problem variant where one might insert and delete specified numbers of vertices and edges.

Degree Anonymity Editing (Anonym-Edt)
Input: An undirected graph $G = (V, E)$ and five positive integers s_1, s_2, s_3, s_4, and k.
Question: Is it possible to obtain a graph $G' = (V', E')$ from G using at most s_1 vertex deletions, s_2 vertex insertions, s_3 edge deletions, and s_4 edge insertions such that G' is k-anonymous?



Observe that here we require that the inserted vertices have degree zero and we have to "pay" for making the inserted vertices adjacent to the existing ones. In particular, if $s_4 = 0$, then all inserted vertices are isolated in the target graph. Note that there are other models where the added vertices can be made adjacent to an arbitrary number of vertices [9, 10]. Our ideas, however, do not directly transfer to this variant.

For convenience, we set $s := s_1 + s_2 + s_3 + s_4$ to be the number of allowed graph modification operations.

Theorem 24. Anonym-Edt *is fixed-parameter tractable with respect to the combined parameter* (s, Δ).

Proof. Let $I = (G = (V, E), k, s_1, s_2, s_3, s_4)$ be an instance of Anonym-Edt. In the following we describe an algorithm finding a solution if it exists. Intuitively, the algorithm first guesses a "solution structure" and then checks whether the graph modification operations associated to this solution structure can be performed in G. A solution structure is a graph S with at most $s(\Delta + 1)$ vertices where

1. each vertex is equipped with an color from $\{0, 1, \ldots, \Delta\}$ indicating the degree of the vertex in G and
2. each edge and each vertex is marked either as "to be deleted", "to be inserted", or "not to be changed" such that
   (a) all edges incident to a vertex marked as "to be inserted" are also marked as "to be inserted",
   (b) at most s_1 vertices and at most s_3 edges are marked as "to be deleted", and
   (c) at most s_2 vertices and at most s_4 edges are marked as "to be inserted".

The intuition behind this definition is that a solution structure S contains all graph modification operations in a solution *and* the vertices that are affected by the modification operations, that is, the vertices whose degree is changed when performing these modification operations. Observe that any solution for I defines such a solution structure with at most $s(\Delta+1)$ vertices as each graph modification affects at most $\Delta+1$ vertices. This bound is tight in the sense that deleting a vertex v affects v and its up to Δ neighbors. Furthermore, observe that once given such a solution structure, we can check in polynomial time whether performing the marked edge/vertex insertions/deletions results in a k-anonymous graph G' since the coloring of the vertex indicates the degrees of the vertices that are affected by the graph modification operations.

Our algorithm works as follows: First it branches into all possibilities for the solution structure S. In each branch it checks whether performing the graph modification operations indicated by the marks in S indeed result in a k-anonymous graph. If yes, then the algorithm checks whether the graph modification operations associated to S can be performed in G. To this end, all edges and vertices marked as "to be inserted" are removed from S and the marks at the remaining vertices and edges are also removed and the resulting "cleaned" graph is called S'. Finally the algorithm tries to find S' as an induced subgraph of G such that the vertex degrees coincide with the vertex-coloring in S'. If the algorithm succeeds and finds S' as an induced subgraph, then the graph modification operations encoded in S can be performed which proves that I is a yes-instance. If the algorithm fails in every branch, then, due to the exhaustive search over all possibilities for S, it follows that I is a no-instance. Thus, the algorithm is correct.

As to the running time: There are $s(\Delta+1)$ possibilities for the number of vertices in the solution structure. Hence, there are at most $s(\Delta+1)\cdot 2^{\binom{s(\Delta+1)}{2}} < 2^{(s(\Delta+1))^2}$ graphs with $s(\Delta+1)$ vertices. Furthermore, there are at most $(\Delta+1)^{s(\Delta+1)}$ possibilities to equip the vertices with colors $\{0,1,\ldots,\Delta\}$ and $3^{s(\Delta+1)+\binom{s(\Delta+1)}{2}}$ possibilities to mark the vertices and edges.

Overall, the algorithm branches into $2^{\mathcal{O}((s\Delta)^2)}$ possibilities for the solution structure S. As mentioned above, checking whether performing the graph modification operations indicated by S indeed results in a k-anonymous graph can be done in polynomial time.

Next, the algorithm checks for each S that may lead to a k-anonymous graph whether the cleaned graph S' occurs as an induced subgraph in G such that degree constraints given by the vertex coloring are fulfilled. Observe that since our input graph G has maximum degree Δ it also has a local tree-width of at most Δ [26]. Thus, for finding S' as induced subgraph, we can use a general result of Frick and Grohe [26, Theorem 1.2] showing that deciding whether a graph H of local tree-width at most ℓ satisfies a property ϕ definable in first-order logic is fixed-parameter tractable with respect to the combined parameter $(|\phi|,\ell)$. The subgraph isomorphism problem can be solved with this result on graphs with bounded local tree-width [26]. Thus it remains to specify the part of the formula ϕ that ensures the degree constraints. To this end, Frick and Grohe [26] gave as example the formula

$$x \in V \land \neg\exists y \exists z(\neg(y=z) \land (x,y)\in E \land (x,z)\in E)$$

to express that a vertex $x \in V$ has degree at most one. This formula can be extended to express that $x \in V$ has degree at most ℓ for some $1 \leq \ell \leq \Delta$ and the size of the formula is upper-bounded in a function of Δ. Similarly, removing the first negation symbol yields the statement that $x \in V$ has a degree of at least two (degree at least $\ell + 1$ in the extended version). Hence, we can express the degree constraints and the formula size is still bounded by a function of s and Δ (as there are up to $s(\Delta + 1)$ vertices in S'). Hence, applying the results of Frick and Grohe [26] shows that the overall algorithm runs in fpt-time with respect to (s, Δ). □

We remark that Theorem 24 is a mere classification result. We claim without proof by slightly adapting the color-coding approach used by Cai et al. [27] and Golovach [28] one can obtain a running time of $2^{(s\Delta)^{\mathcal{O}(1)}} n^{\mathcal{O}(1)}$: The idea is to randomly color the vertices in the graph with green and red. Then the subgraph $G' = (V', E')$ we are looking for is with probability $2^{(\Delta+1)|V'|}$ completely contained within the green vertices and $N_G(V \setminus V')$ are colored red. By brute-force, one can determine in $\mathcal{O}(|V'|!)$ whether a green component fits with a connected component of the sought subgraph such that the degree constraints are fulfilled. Thus, using a knapsack dynamic program over the green components, one can compute the whole subgraph G' in the claimed running time. As the running time would be still impractical, we refrain from giving a formal proof.

Next, we show that considering Anonym V-Del we can assume that $s < f(\Delta, k)$ for some function f. This implies that the above fixed-parameter tractability results transfers to the parameter (k, Δ).

Lemma 25. *For every yes-instance $(G = (V, E), k, s)$ of* Anonym V-Del *with Δ denoting the maximum degree of G, there is a subset $S \subseteq V$ with $|S| < 2^{\Delta+1}\Delta^3 k$ such that $G - S$ is k-anonymous.*

Proof. Let $(G = (V, E), k, s)$ be a yes-instance of Anonym V-Del and let $S \subseteq V$ be a k-deletion set. We show that if $|S| \geq 2^\Delta \Delta^3 2k$, then we get a smaller k-deletion set by removing a subset of k vertices from S.

Let $D = \{0, 1, \ldots, \Delta\}$ be the set of possible vertex degrees in $G - S$. We say a vertex $v \in S$ is of type (D', d') with $D' \subseteq D$ and $0 \leq d' \leq \Delta$ if $D' = \{\deg_{G-S}(v') \mid v' \in N_{G-S}(v)\}$ and $d' = \deg_{G[(V \setminus S) \cup \{v\}]}(v)$. If $|S| \geq 2^\Delta \Delta^3 2k$, then S contains a set S' of $\Delta^2 \cdot 2k$ vertices which are of the type (D', d') for some $D' \subseteq D$ and $0 \leq d' \leq \Delta$. Note that each vertex has at most Δ vertices in its first and at most $\Delta(\Delta - 1)$ vertices in its second neighborhood. Hence, there must be a set $S'' \subset S'$ of $2k$ independent vertices with pairwise disjoint neighborhoods. Let $S_+, S_- \subseteq S''$ be any two sets of size k each such that $S_+ \cup S_- = S''$. Consider the graphs $G_1 = G - S$ and $G_2 = G - (S \setminus S_+)$, that is, S_+ is the subset of vertices from S'' that remains in G_2 and S_- is the subset of vertices from S'' that is not in G_2.

We show that if G_1 is k-anonymous then G_2 is also k-anonymous. Every vertex from S_+ has degree d' in G_2 because S_+ is an independent set. Since $|S_+| = k$, there are at least k vertices of degree d', that is, the vertices from S_+ are k-anonymous. Every vertex v that is in G_1 and in G_2 satisfies that either $\deg_{G_2}(v) = \deg_{G_1}(v)$ or $\deg_{G_2}(v) = \deg_{G_1}(v) + 1$, because the vertices from S_+ have pairwise disjoint neighborhoods. Now, there are two cases for $d'' = \deg_{G_1}(v)$: If $d'' \notin D'$, then $\deg_{G_2}(v) = d''$. Furthermore, there are at least as many vertices of degree d'' in G_1 as in G_2, because no vertex from S_+ is adjacent to any vertex of degree d'' in G_1. If $d'' \in D'$, then a vertex with degree d'' in G_1 may have degree $d''+1$ in G_2 because it is adjacent to some vertex in S_+. However, since the vertices from S_+ have pairwise disjoint neighborhoods, for each of the k vertices from S_+ there is at least one vertex that has degree d'' in G_1 and degree $d''+1$ in G_2 Furthermore, for each of the k vertices from S_- there is at least one vertex that has degree d'' in G_1 and G_2. In each case, there are at least k vertices with degree $\deg_{G_2}(v)$ in G_2. Thus, G_2 is k-anonymous. □

By combining Theorem 24 and Lemma 25 we obtain fixed-parameter tractability with respect to the combined parameter (k, Δ). For an instance (G, k, s) of Anonym V-Del simply run the algorithm from Theorem 24 on the instance $(G, k, \min\{s, 2^\Delta \Delta^3 2k\})$.

The ideas behind Lemma 25 can be easily transferred to the edge deletion variant.

Lemma 26. *For every yes-instance $(G = (V, E), k, s)$ of* Anonym E-Del *with Δ denoting the maximum degree of G there is a subset $S \subseteq E$ with $|S| < 2\Delta^3 2k$ such that $G - S$ is k-anonymous.*

Proof. Let $(G = (V, E), k, s)$ be a yes-instance of Anonym E-Del and let $S \subseteq E$ be a k-deletion set. We show that if $|S| \geq 2\Delta^3 2k$, then we get a smaller k-deletion set by removing a subset of k edges from S.

We say an edge $e = \{u, v\} \in S$ is of type (d_1, d_2) with $1 \leq d_1, d_2 \leq \Delta$ if $d_1 = \deg_{G-S}(u)$ and $d_2 = \deg_{G-S}(v)$. If $|S| \geq 2\Delta^3 2k$, then S contains a set S' of $2\Delta \cdot 2k$ edges which are of the type (d_1, d_2) for some $0 \leq d_1, d_2 \leq \Delta$. Since each vertex has, by definition of Δ, at most Δ neighbors, there must be a set $S'' \subset S'$ of $2k$ pairwise disjoint edges. Let $S_+ \subseteq S''$ be a set of size k. Now, similarly to proof of Lemma 25, it follows that $G - (S \setminus S_+)$ is also k-anonymous as it contains at least k vertices of degree d_1, $d_1 + 1$, d_2, and $d_2 + 1$, respectively and the other vertices remain untouched. □

By combining Theorem 24 and Lemma 26 we also obtain fixed-parameter tractability for Anonym E-Del with respect to the parameter (k, Δ). Thus, we arrive at the following classification result.

Corollary 27. Anonym V-Del *and* Anonym E-Del *are fixed-parameter tractable with respect to the combined parameter (k, Δ).*

6. Conclusion

In this work, we provided a thorough overview on the computational complexity of the Degree Anonymity problem when considering vertex or edge deletions. We obtained various hardness results from the viewpoints of approximation and parameterized complexity, even in restricted graph classes. Besides this large amount of hardness results we obtained a few positive results (polynomial-time solvable cases) on highly structured graph classes.

Despite this (in terms of algorithmic tractability) discouraging picture of the computational complexity, a number of open questions remains that still may raise hope for broader positive results. In particular, these questions are:

1. Are Anonym Min-E-Del or Anonym Min-V-Del constant-factor approximable in polynomial time when k is a constant?
2. Are the two optimization variants of Anonym E-Edt constant-factor approximable in polynomial time?
3. What is the complexity of Anonym V-Del on unit interval graphs and on bipartite chain graphs?
4. Do all our NP-completeness results for Anonym V-Del on special graph classes (see Subsection 3.2) also carry over to Anonym E-Del?

Despite serious efforts, we failed to extend the polynomial-time inapproximability results for Anonym Min-E-Del and Anonym Min-V-Del to exclude approximation algorithms running in fpt-time with respect to the parameter k. The reason is that all our gap-reductions relied on k being in the order of n. This restriction made it easy to control the possibilities for the solutions in the constructed graph, but leaves Question 1 as challenge for future research. Question 2 seems to be closely related to Question 1 as we failed to answer both questions for the same reason: The variant of editing edges allows to "repair" a suboptimal decisions by reverting the degree of a vertex with one further operation (edge deletion or insertion). In the case of edge deletions with constant values of k it might be possible to "repair" suboptimal decision by decreasing the degrees of just a few other vertices. We found no way of dealing even with one of these two possibilities to repair suboptimal decisions. As to Question 4, our findings so far support the conjecture that the hardness results mostly transfer, but the reductions to prove this will become messy.

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