Parameterized Complexity of Candidate Control in Elections and Related Digraph Problems

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Abstract. There are different ways for an external agent to influence the outcome of an election. We concentrate on "control" by adding or deleting candidates of an election. Our main focus is to investigate the parameterized complexity of various control problems for different voting systems. To this end, we introduce natural digraph problems that may be of independent interest. They help in determining the parameterized complexity of control for different voting systems including Llull, Copeland, and plurality votings. Devising several parameterized reductions, we provide a parameterized complexity overview of the digraph and control problems with respect to natural parameters.

1 Introduction and Preliminaries

The investigation of voting systems is an important field of interdisciplinary research. Besides obvious classical applications in political or other elections, voting systems also play an important role in multi-agent systems or rank aggregation. In addition to work that focuses on the problem to determine the winner of an election for different voting systems, there is a considerable amount of work investigating how an external agent or a group of voters can influence the election in favor or disfavor of a distinguished candidate. The studied scenarios are manipulation [3], electoral control [1, 6–8], lobbying [2], and bribery [6]. In this work, we investigate the parameterized complexity of some variants of electoral control and closely related digraph problems. Before describing our results, we introduce the considered problems.

Problem statements. An *election* (V, C) consists of a set V of n votes and a set C of m candidates. A *vote* is an ordered preference list containing all candidates. To *control* an election, an external agent, traditionally called *chair*, can change the voting procedure to reach certain goals. The considered types of control are adding, deleting, or partitioning candidates or voters [1, 8]. Further, one distinguishes between *constructive control* (CC), that is, the chair aims at making a distinguished candidate the winner, and *destructive control* (DC), that is, the chair wants to prevent a distinguished candidate from winning [8]. In this

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work, we focus on *candidate control*, that is, either deleting or adding candidates, for plurality and Copeland$^\alpha$ votings. In *plurality voting*, for every vote the candidate that is ranked first in the preference list gets one point. The *score* of a candidate is the total number of its points. A candidate with the highest score wins. Note that we still need the whole preference lists of the voters to see the effects of deleting or adding candidates. *Copeland$^\alpha$ voting* is based on pairwise comparisons between candidates: A candidate wins the pairwise head-to-head contest against another candidate if he is better positioned in more than half of the votes. The winner of a head-to-head contest is awarded one point and the loser receives no point. If the candidates are tied, both candidates get α points for $0 \leq \alpha \leq 1$. A *Copeland$^\alpha$* winner is a candidate with the highest score. Faliszewski et al. [6] devote their paper to the two important special cases $\alpha = 0$, denoted as *Copeland*, and $\alpha = 1$, denoted as *Llull*. Next, we introduce two digraph decision problems which are closely related to constructively controlling Copeland and Llull by deleting candidates.[1]

Max-Outdegree Deletion (MOD)
Given: A digraph $D = (W, A)$, a distinguished vertex $w_c \in W$, and an integer $k \geq 1$.
Question: Is there a subset $W' \subseteq W \setminus \{w_c\}$ of size at most k such that w_c is the only vertex that has maximum outdegree in $D[W \setminus W']$?

Analogously, we define Min-Indegree Deletion (MID), where one wants to make a distinguished vertex to be the only vertex with minimum indegree. The correspondence to elections is based on the fact that the relations between the candidates can be depicted by a digraph where the candidates are represented by the vertices and there is an arc from vertex c to vertex d iff the corresponding candidate c defeats the corresponding candidate d in the head-to-head contest. Then, the deletion of a vertex one-to-one corresponds to the deletion of a candidate in the election. Further, the Copeland score of a candidate c is exactly the number of the out-neighbors of the corresponding vertex v_c and the Llull score is the total number of vertices minus the number of in-neighbors of v_c.

Known results. A series of publications [1, 6–8] provides a complete picture of the classical computational complexity for four standard voting systems (approval, plurality, Condorcet, and Copeland$^\alpha$) for ten basic types of control.[2] Concerning candidate control, plurality and Copeland votings lead to NP-hardness results whereas all other voting systems are either immune or allow for polynomial-time solvability [6–8]. Regarding parameterized complexity, Faliszewski et al. [7] obtained some first results. They considered control of Copeland$^\alpha$ voting with respect to the parameters "number of candidates" and "number of votes". For candidate control they obtained fixed-parameter tractability with respect to the parameter "number of candidates". The parameterized complexity with respect to the parameter "number of votes" was left

[1] The digraph problems that are equivalent to adding candidates are omitted due to space restrictions.
[2] Besides the classification into P and NP-hard, a voting system can be classified as "immune" against a type of control if a non-winner can never be made a winner.

Table 1. Parameterized complexity of Max-Outdegree Deletion and Min-Indegree Deletion.

	# deleted vertices k		maximum degree d		(k, d)	
	MOD	MID	MOD	MID	MOD	MID
general digraphs	W[2]-c	W[2]-c	NP-c for $d \geq 3$	FPT	FPT	FPT
acyclic digraphs	W[2]-c	P	NP-c for $d \geq 3$	P	FPT	P
tournaments	W[2]-c	W[2]-c	-	-	-	-

open. To the best of our knowledge, there is no previous work dealing with the newly introduced digraph problems MOD and MID.

Motivation. First, from the "control person's" point of view, it is interesting to find efficient strategies to reach his goal. There are legal scenarios as for example persuading additional players to participate in a sport competition in order to make the favorite player the winner. Parameterized complexity analysis is meaningful in this context. Second, the fact that a voting system is susceptible to control or manipulation can be considered as an undesirable property. Thus, the goal of most publications is to show that, if control is not impossible, it is at least computationally hard (often showing NP-hardness). Although NP-hardness is not a sufficient criterion, as it does not imply hardness on the practically relevant average case, it is plausible to investigate whether there are any hard instances at all. However, as also noted by Conitzer et al. [3], such hardness results lose relevance if there are efficient fixed-parameter algorithms for realistic settings.

Our contributions. We provide a first study of the two natural digraph problems MOD and MID and show that they are closely related to the considered control problems. In Section 2, we investigate the computational complexity of MOD and MID for several special graph classes and parameters providing a differentiated picture of their parameterized complexity including algorithms and intractability (Table 1). The main technical achievement of this part is to show that MOD and MID are W[2]-complete in tournaments. Some of the considered special cases and parameterizations of the digraph problems map to realistic voting scenarios with presumably small parameters. Based on these connections and by giving further parameterized reductions, in Section 3 we provide an overview of parameterized hardness results for control problems (Table 2). Regarding the structural parameter "number of votes", we answer an open question of Faliszewski et al. [7] for Llull and Copeland votings by showing that even for a constant number of voters candidate control remains NP-hard. Due to the lack of space, we defer many details and proofs to a full version.

Preliminaries. In an election, we can either seek for a *winner*, that is, if there are several candidates who are best in the election, then all of them win, or for a *unique winner*. Note that a unique winner does not always exist. We only consider the unique winner case, but all our results can be easily modified to work for the winner case as well. We focus on control by adding candidates (AC) or deleting candidates (DC). Then, for example, we can define the decision problems of constructively controlling a Copeland$^\alpha$ election as follows:

CC-DC-COPELAND$^\alpha$
Given: A set C of candidates, a set V of votes with preferences over C, a distinguished candidate $c \in C$, and an integer $k \geq 1$.
Question: Is there a subset $C' \subseteq C$ of size at most k such that c is (unique) Copeland$^\alpha$ winner in the election $(V, C \backslash C')$?

CC-AC-COPELAND$^\alpha$
Given: Two disjoint sets C, D of candidates, a set V of votes with preferences over $C \cup D$, a distinguished candidate $c \in C$, and an integer $k \geq 1$.
Question: Is there a subset $D' \subseteq D$ of size at most k such that c is (unique) Copeland$^\alpha$ winner in the election $(V, C \cup D')$?

The other problems are defined analogously (see for example [6, 8]). The *position* of a candidate a in a vote v is the number of candidates that are better than a in v plus one. That is, the leftmost (and best) candidate in v has position 1 and the rightmost has position m. Further, within every election we fix some arbitrary order over the candidates. Specifying a subset C' of candidates in a vote means that the candidates of C' are ordered with respect to that fixed order. An occurrence of $\overleftarrow{C'}$ in a vote means that the candidates of C' are ordered in reverse to that fixed order.

For a directed graph (digraph) $D = (W, A)$ and for a vertex $w \in W$, the set of *in-neighbors* of w is defined as $N_{in}(w) := \{u \in W \mid (u, w) \in A\}$ and the set of *out-neighbors* of w is given by $N_{out}(w) := \{u \in W \mid (w, u) \in A\}$. Moreover, the *indegree* (*outdegree*) of w is defined as $\text{indeg}(w) := |N_{in}(w)|$ ($\text{outdeg}(w) := |N_{out}(w)|$). Further, the *degree* is defined as $\deg(w) := \text{indeg}(w) + \text{outdeg}(w)$. In digraphs, we do not allow bidirected arcs and loops. An l-arc coloring $\mathcal{C} : A \rightarrow \{1, 2, \ldots, l\}$ is called *proper* if any two distinct arcs of the same color do not share a common vertex. A *tournament* is a digraph where, for every pair of vertices u and v, there is either (u, v) or (v, u) in the arc set.

A problem is called *fixed-parameter tractable (FPT)* with respect to a parameter k if it can be solved in $f(k) \cdot n^{O(1)}$ time, where n denotes the input size, and f is an arbitrary computable function. The first two levels of (presumable) parameterized intractability are captured by the complexity classes W[1] and W[2]. A *parameterized reduction* reduces a problem instance (I, k) in $f(k) \cdot |I|^{O(1)}$ time to an instance (I', k') such that (I, k) is a yes-instance if and only if (I', k') is a yes-instance and k' only depends on k but not on $|I|$.

As discussed in the introduction, there are parameterized reductions from MOD (MID) to CC-DC-COPELAND (CC-DC-LLULL) with respect to the parameters number of deleted vertices and candidates, respectively. The reverse parameterized reductions can be obtained by a simple construction of Faliszewski et al. [6]. Thus, we say that the problems are *FPT-equivalent*.

2 Parameterized Complexity of MOD and MID

This section is concerned with the parameterized complexity of MOD and MID with respect to the parameters "number of deleted vertices" k and "maximum

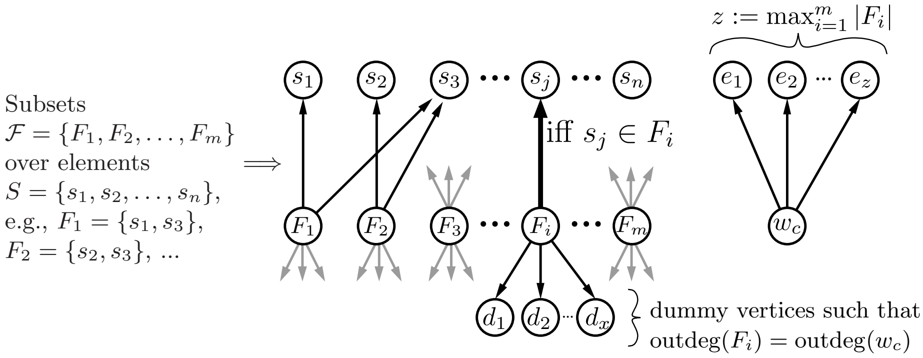

Fig. 1. Parameterized reduction from a Hitting Set-instance (left) to an MOD-instance (right). Deleting an "element-vertex" s_j in the digraph has the effect that for all "subset-vertices" corresponding to the subsets containing s_j the outdegree is decreased below the outdegree of the distinguished vertex w_c, that is, the corresponding subsets are "hit" in the Hitting Set-instance. Further, assume that there is a solution for the MOD-instance that contains a subset vertex F_i or one of its dummy neighbors. Then, instead of this vertex we can delete any subset-neighbor s_j of F_i. Based on these observations one can show that there is a hitting set of size k iff w_c can become vertex with maximum outdegree by deleting k vertices.

degree" d for different classes of graphs. Our results are summarized in Table 1. In the following, we only prove W[2]-hardness. Using the machinery of Downey and Fellows [4], it is not hard to also show containment in W[2].

Theorem 1. Max-Outdegree Deletion *is W[2]-complete with respect to the parameter "number of deleted vertices" in acyclic digraphs and NP-complete in acyclic digraphs with maximum degree three.*

The W[2]-hardness can be shown by a parameterized reduction from the W[2]-complete Hitting Set (HS) problem [5]. Given a subset family $\mathcal{F} = \{F_1, F_2, \ldots, F_m\} \subseteq 2^S$ of a base set $S = \{s_1, s_2, \ldots, s_n\}$ and an integer $k \geq 1$, the Hitting Set problem asks to decide whether there exists a subset $S' \subseteq S$ of size at most k such that for every $1 \leq i \leq m$ we have $S' \cap F_i \neq \emptyset$. We defer the formal proof of Theorem 1 to the full version of this paper. Here, we only illustrate the basic construction (see Fig. 1). The resulting digraph of the MOD-instance is acyclic, which gives the first part of the theorem. The second part directly follows from the fact that Hitting Set is NP-complete even if every subset has size two and every element occurs in exactly three subsets (3X-2-Hitting Set).

Proposition 1. *a)* Min-Indegree Deletion *can be solved in polynomial time in acyclic digraphs. In general digraphs, it is fixed-parameter tractable with respect to the parameter "indegree of the distinguished vertex w_c".*

b) Max-Outdegree Deletion *is fixed-parameter tractable with respect to the combined parameters "*$\text{outdeg}(w_c)$*" and "number of deleted vertices k".*

Proof. (Sketch) a) First part (acyclic graphs): Since in acyclic graphs there always exists a vertex with indegree zero, w_c must have indegree zero to be the only minimum indegree vertex. Thus, one can iteratively delete all other vertices with indegree zero.

Second part (parameter $\mathrm{indeg}(w_c)$): If one knows for an MID-instance which in-neighbors of the distinguished vertex w_c are part of a minimum solution, then the problem becomes trivial: One can delete these vertices and extend the resulting partial solution to a minimum-cardinality solution. For this, one iteratively adds all vertices of indegree smaller than the (new) indegree of w_c to the solution since all vertices of indegree smaller than the distinguished vertex must be deleted. Hence, exhaustively trying all subsets of in-neighbors of w_c yields an algorithm with running time $O(2^{\mathrm{indeg}(w_c)} \cdot |W|^2)$.

b) Here, we give a simple branching strategy: Consider a vertex $u \in W \setminus \{w_c\}$ with outdegree at least $\mathrm{outdeg}(w_c)$. Furthermore, let $N \subseteq N_{out}(u)$ with $|N| = \mathrm{outdeg}(w_c)$. Then, we have to delete one of the vertices in $(N \cup \{u\}) \setminus \{w_c\}$, that is, we can branch into at most $\mathrm{outdeg}(w_c)+1$ cases. In each case, we can decrease the parameter k by one, leading to a search tree of size $O((\mathrm{outdeg}(w_c)+1)^k)$. □

The following theorem is based on a parameterized reduction from the W[2]-complete DOMINATING SET problem [5]. The basic idea is similar to the HITTING SET reduction (Fig. 1), but the details are quite involved.

Theorem 2. MAX-OUTDEGREE DELETION *and* MIN-INDEGREE DELETION *are W[2]-complete with respect to the parameter "number of deleted vertices" even in the case that the input graph is a tournament.*

3 Parameterized Complexity of Candidate Control

In this section, we turn our attention to elections. For candidate control in Llull and Copeland votings we show NP-hardness for a constant number of votes. Further, we provide parameterized intractability results with respect to the number of deleted/added candidates for plurality and Copeland$^\alpha$ votings.

Number of votes as parameter. In many election scenarios there is only a small number of votes. For example, consider a human resources department where few people are deciding which job applicant gets the employment. An open question of Faliszewski et al. [7] regards the parameterized complexity of Copeland$^\alpha$ elections with respect to the parameter "number of votes". We answer this question for Llull and Copeland.

Theorem 3. CC-DC-COPELAND *is NP-complete for six votes,* CC-AC-COPELAND *is NP-complete for eight votes,* CC-DC-LLULL *is NP-complete for ten votes, and* CC-AC-LLULL *is NP-complete for six votes.*

Proof. (Sketch) For all problems NP-membership is obvious. We only give the NP-hardness proof for CC-DC-COPELAND to demonstrate the basic idea.

The proof consists of two phases. The first phase is a reduction from 3X-2-Hitting Set to MOD as depicted in Fig. 1. The digraph D of a resulting MOD-instance (D, w_c, k) has maximum degree three and the underlying undirected graph of D is bipartite. More precisely, one partition consists of the subset-vertices and w_c, and the other partition consists of the element-vertices and the neighbors of w_c. As we reduce from 3X-2-Hitting Set, we do not have any further dummy vertices. In the second phase we show that D can be encoded by an election with only six votes by exploiting this special structure of D.

Now, we describe the second phase. Due to König [9] we know that a bipartite graph is Δ-edge-colorable, where Δ denotes the maximum degree of the graph. Moreover, a corresponding proper Δ-edge coloring can be computed in polynomial time. Thus, for D there exists a proper 3-arc-coloring $\mathcal{C} : A \to \{\mathcal{R}, \mathcal{G}, \mathcal{B}\}$. Note that in the underlying undirected graph of D the edges of the same color class form a matching, that is, two arcs of the same color do not share a common vertex. Hence, the coloring $\mathcal{C}$ partitions the arc set into three classes of independent arcs. We next describe how the arcs of graph D can be encoded in an election with six votes. Let $A_{\mathcal{R}} = \{(r_1, r'_1), \ldots, (r_l, r'_l)\}$ denote the arcs colored by $\mathcal{R}$. Furthermore, $W_{\overline{\mathcal{R}}}$ denotes the set of vertices that are not incident to any arc of $A_{\mathcal{R}}$. To encode $A_{\mathcal{R}}$, we add the two votes $r_1 > r'_1 > r_2 > r'_2 > \cdots > r_l > r'_l > W_{\overline{\mathcal{R}}}$ and $\overleftarrow{W_{\overline{\mathcal{R}}}} > r_l > r'_l > \cdots > r_2 > r'_2 > r_1 > r'_1$ to the election. In the same way we add two votes for the arcs colored by $\mathcal{B}$ and $\mathcal{G}$, respectively. The correctness of the construction follows from two observations. First, since the arcs of the same color do not share common endpoints, in every vote all vertices occur exactly once and we have a valid election. Second, consider an arc $(w', w'') \in A$ with $\mathcal{C}((w', w'')) = X$ for any color $X \in \{\mathcal{R}, \mathcal{B}, \mathcal{G}\}$. Then, w' defeats w'' in the votes $v_{X,1}$ and $v_{X,2}$ and ties with w'' in the remaining four votes. Moreover, since every arc occurs in exactly one color class, all arcs are encoded, and, since all other candidates are tied in every pair of the votes, we have ties between all other pairs of candidates.

In summary, in the constructed Copeland election a candidate c can become the unique winner by deleting k candidates iff in D the corresponding vertex w_c can become the maximum outdegree vertex by deleting k vertices. □

Number of deleted/added candidates as parameter. To control an election without raising suspicion one may add or delete only a limited number of candidates. Here, we investigate whether it is possible to obtain fixed-parameter algorithms under this assumption. More specifically, we consider the parameterized complexity of destructive and constructive control by adding or deleting a fixed number of candidates. Our results are summarized in Table 2. It turns out that all NP-complete problems are intractable from this parameterized point of view as well. This even holds true for plurality voting, which can be considered as the "easiest" voting system in terms of winner evaluation and for which the Manipulation problem can be solved optimally by a simple greedy strategy [3].

Copeland. For elections without ties in all pairwise head-to-head contests, CC-DC-Copeland$^\alpha$ coincides for all $0 \le \alpha \le 1$, since these problems only differ

Table 2. Results in boldface are new. The results for Copeland$^\alpha$ hold for all $0 \leq \alpha \leq 1$. The W[2]-hardness results for CC-AC-Plurality and DC-AC-Plurality follow from the NP-completeness proofs [1, 8]. The polynomial-time (P) results are from [6, 7].

	Copeland$^\alpha$		Plurality	
	CC	DC	CC	DC
Adding Candidates (AC)	**W[2]-c**	P	W[2]-h	W[2]-h
Deleting Candidates (DC)	**W[2]-c**	P	**W[2]-h**	**W[1]-h**

in the way ties are evaluated. As discussed in the introduction MOD and CC-DC-Copeland$^\alpha$ are FPT-equivalent. Using the same reductions one can show that MOD in tournaments is FPT-equivalent to CC-DC-Copeland$^\alpha$ without ties. Thus, the W[2]-hardness of CC-DC-Copeland$^\alpha$ without ties follows directly from Theorem 2.[3] For adding candidates we obtain W[2]-hardness using similar ideas.

Plurality. For plurality voting, the W[2]-hardness results for control by adding candidates follow from the reductions for the NP-hardness [1, 8]. In contrast, the reductions used to show NP-hardness for control by deleting candidates [1, 8] do not imply their parameterized hardness. Thus, we develop new parameterized reductions to show W[1]/W[2]-hardness.[4] For the constructive case we can show W[2]-hardness by a reduction from MOD. Note that the encoding of a MOD instance into a plurality election is more demanding than for Copeland voting and the other direction (encoding a plurality election into MOD) is not obvious.

Theorem 4. *Constructive control of plurality voting by deleting candidates is W[2]-hard with respect to the parameter "number of deleted candidates".*

Proof. (Sketch) We present a parameterized reduction from MOD. Given an MOD instance $(D = (W, A), w_c, k)$ with $W = \{w_1, w_2, \ldots, w_n\}$ and $w_c = w_1$, we construct an election (V, C) as follows: We have one candidate corresponding to every vertex, that is, $C' := \{c_i \mid w_i \in W\}$. The set of candidates C then consists of C' and an additional set F of "dummy" candidates (only used to "fill" positions that cannot be taken by other candidates in our construction). The set of votes V consists of two subsets V_1 and V_2. In V_1, for every $c_i \in C'$ we have outdeg(w_i) votes in which c_i is at the first position and with dummy candidates in the positions from 2 to $k+1$. Then, for every such vote, the remaining candidates follow in arbitrary order. In V_2, for every $c_i \in C'$ we have $|W|$ votes in which c_i is at the first position. For all candidates $c_j \neq c_i$ with $w_j \notin N_{in}(w_i)$, we observe that in exactly one of these $|W|$ votes c_j is at the second position. In all other of these votes, the second position is filled with a dummy candidate. Moreover, we add dummies to all positions from 3 to $k + 1$. Concerning the dummies, in V_1 and V_2 we ensure that every dummy candidate $f \in F$ has a position better than $k + 2$ only in one of the votes. This can be done such that the size of F

[3] Having no ties in the pairwise head-to-head contests is a realistic scenario. It is always the case for an odd number of votes and likely for a large number of votes. In contrast, the NP-hardness proofs of the considered problems rely on ties [6, 7].

[4] The class containment for all kinds of candidate control in plurality voting is open.

is less than $(k+1) \cdot |V|$. The dummies exclude the possibility of "accidently" getting candidates in the first position. Note that by deleting k candidates only a candidate that is at one of the first $k+1$ positions in a vote has the possibility to increase his plurality score. Further, by construction, the dummy candidates fulfill the following two conditions. First, the score of a dummy candidate can become at most one. Second, it does never make sense to delete a dummy as by this only other dummies can get into the first position of a vote. Next, we prove the correctness of the reduction.

Claim: Candidate c_1 can become the plurality winner of (V, C) by deleting k candidates iff w_1 can become the only maximum-degree vertex in D by deleting k vertices.

"$\Rightarrow$": Denote the set of deleted candidates by R. We show that after deleting the set of vertices $W_R := \{w_i \mid c_i \in R\}$ the vertex w_1 is the only vertex with maximum degree. Before deleting any candidates, for every candidate c_i we have $\text{score}(c_i) = \text{score}(c_1) + s_i$ with $s_i := \text{outdeg}(w_i) - \text{outdeg}(w_1)$. After deleting the candidates in R, candidate c_1 is the winner. Hence, for $i = 2, \ldots, |W|$ we must have either that $\text{score}(c_i) < \text{score}(c_1)$ or that c_i is deleted. For a non-deleted candidate c_i with $i > 1$ the difference between $\text{score}(c_i)$ and $\text{score}(c_1)$ must be decreased by at least $s_i + 1$. By construction, the only way to decrease the difference by one is to delete a candidate such that c_1 becomes first in one more vote and c_i does not increase its number of first positions. All candidates that can be deleted to achieve this correspond to vertices in $N_{in}(w_i) \setminus N_{in}(w_1)$. To improve upon c_i we must delete at least $s_i + 1$ candidates that fulfill this requirement. Hence, in D the outdegree of w_i is reduced to be less than the outdegree of w_1.

"$\Leftarrow$": Let $T \subseteq D$ denote the solution for MOD. We can show ("reverse" to the other direction) that by deleting the set of candidates $C_T := \{c_i \mid w_i \in T\}$ candidate c_1 becomes a plurality winner. □

In contrast to Copeland$^\alpha$ elections, for plurality elections destructive control by deleting candidates is NP-hard [8]. We show that it is even W[1]-hard by presenting a parameterized reduction from the W[1]-complete Clique problem [5]. Given an undirected graph $G = (W, E)$ and a positive integer k, the Clique problem asks to decide whether G contains a complete subgraph of size at least k.

Theorem 5. *Destructive control of plurality voting by deleting candidates is W[1]-hard with respect to the parameter "number of deleted candidates".*

Proof. Given a Clique instance $(G = (W, E), k)$, we construct an election as follows: The set of candidates is $C := C_W \uplus C_E \uplus \{c, w\} \uplus D$ with $C_W := \{c_u \mid u \in W\}$, $C_E := \{c_{uv} \mid \{u, v\} \in E\}$, and a set of dummy candidates D. In the following, the candidates in C_W and C_E are called *vertex candidates* and *edge candidates*, respectively. Further, we construct the votes in a way such that w is the candidate that we like to prevent from winning, c is the only candidate that can beat w, and D contains dummy candidates that can gain a score of at most one. In the set of votes V we have for every vertex $u \in W$ and for each incident edge $\{u, v\} \in E$ one vote of the type $c_u > c_{uv} > c > \ldots$, that is, there are $2 \cdot |E|$

votes of this type, two for every edge. Additionally, V contains $|W| + k \cdot (k-1)$ votes in which w is at the first position and $|W| + 1$ votes in which c is at the first position. In all votes, the remaining free positions between 2 and $k + \binom{k}{2} + 1$ are filled with dummies such that every dummy occurs in at most one vote at a position better than $k + \binom{k}{2} + 2$. This can be done using less than $|V| \cdot (k + \binom{k}{2} + 1)$ dummy candidates. In every vote the candidates that do not occur in this vote at a position less than $(k + \binom{k}{2} + 1)$ follow in arbitrary order.

Claim: Graph G contains a clique K of size k iff candidate c can become plurality winner by deleting $k' := k + \binom{k}{2}$ candidates.

"$\Rightarrow$": Delete the $k + \binom{k}{2}$ candidates that correspond to the vertices and edges of K. Then, for every of the $\binom{k}{2}$ deleted edge candidates we also deleted the two vertex candidates that correspond to the endpoints of this edge. Therefore, for every of the $\binom{k}{2}$ edges candidate c gets in the first position in two more votes. Hence, the score of candidate c is increased by $2 \cdot \binom{k}{2} = k \cdot (k-1)$ and the score of candidate w is not affected. Thus, the total score of w is $|W| + k \cdot (k-1)$ and the total score of c is $|W| + k \cdot (k-1) + 1$; therefore, w is defeated by c.
"$\Leftarrow$": Note that, by construction, we cannot decrease the score of w and we cannot increase the score of a vertex candidate (which is at most $|W| - 1$). Further, by the deletion of at most k' candidates the score of a dummy candidate can become at most one, and the score of an edge candidate can become at most two. Hence, c is the only candidate that can prevent w from winning. Furthermore, as the deletion of at most k' dummies never moves c into a first position, we can assume that the solution deletes only edge and vertex candidates.

We omit the proof that the only way to increase the score of c by at least $k \cdot (k-1)$ is to choose edge and vertex candidates that correspond to the vertices and edges of a clique of size k. $\square$

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