NP-Hardness and Fixed-Parameter Tractability of Realizing Degree Sequences with Directed Acyclic Graphs*

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Abstract

In graph realization problems one is given a degree sequence and the task is to decide whether there is a graph whose vertex degrees match the given sequence. This realization problem is known to be polynomial-time solvable when the graph is directed or undirected. In contrast, we show NP-completeness for the problem of realizing a given sequence of pairs of nonnegative integers (representing in- and outdegrees) with a *directed acyclic graph* (DAG), answering an open question of Berger and Müller-Hannemann. Furthermore, we classify the problem as fixed-parameter tractable with respect to the parameter "maximum degree". Investigating sparse and dense settings, we show that the problem remains NP-hard even if the realizing DAG (precisely, the underlying undirected graph) can be transformed into a clique (a tree) by adding (deleting) a constant fraction of the arcs. In contrast, if at most k arcs have to be inserted respectively removed to obtain a clique or a tree in the underlying undirected graph, then the problem becomes fixed-parameter tractable with respect to k.

1 Introduction

Berger and Müller-Hannemann [2] introduced the following graph realization problem:

DAG Realization
Input: A multiset $\mathcal{S} = \left\{ \binom{a_1}{b_1}, \ldots, \binom{a_n}{b_n} \right\}$ of pairs of nonnegative integers.
Question: Is there a directed acyclic graph (without parallel arcs and self-loops) that admits an ordering $v_1, \ldots, v_n$ of its vertex set such that for every $v_i \in V$ the indegree is a_i and the outdegree is b_i?

*An extended abstract of this paper appeared in *Proceedings of the 8th International Conference on Computability in Europe 2012 (CiE 2012)*, volume 7318 of LNCS, pages 283–292, Springer, 2012. Besides providing several omitted proof details, an important difference compared to the conference version is the proof of fixed-parameter tractability for the parameter "number of arcs minus number of vertices" (see Section 5).

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If a *degree sequence* $\mathcal{S}$ is a yes-instance, then $\mathcal{S}$ is called *realizable* and the corresponding directed acyclic graph (DAG for short) D is called a *realizing DAG* for $\mathcal{S}$. This problem arises in the context of randomly generating DAGs satisfying some prespecified degree constraints [1].

Our Contributions Berger and Müller-Hannemann [2] showed that DAG Realization is polynomial-time solvable for special types of degree sequences, but left the computational complexity of the general problem as their main open question. We answer this question by showing that DAG Realization is NP-complete. Moreover, on the positive side we classify DAG Realization as fixed-parameter tractable with respect to the parameter maximum degree $\Delta := \max\{a_1, b_1, \ldots, a_n, b_n\}$. Denoting by n the number of vertices and by m the number of arcs in a realizing DAG and assuming $\binom{0}{0} \notin \mathcal{S}$, Berger and Müller-Hannemann [3] showed that if $m < n$, then DAG Realization is polynomial-time solvable. We extend this result to less sparse and to dense settings by considering the four parameters $m-n+1$, m/n, $\binom{n}{2}-m$, and $\binom{n}{2}/m$. For the first and the third parameter $m-n+1$ and $\binom{n}{2}-m$ we prove fixed-parameter tractability, whereas for the second and fourth parameter m/n and $\binom{n}{2}/m$ we show NP-hardness for any constant parameter value larger than one.

Related Work It has been known for a long time that deciding whether a given degree sequence (a multiset of positive integers) is realizable with an *undirected graph* is polynomial-time solvable. There are characterizations for realizable degree sequences due to Erdős and Gallai [6] and algorithms by Havel [14] and Hakimi [13]. The problem variant where one asks whether there is a *directed* graph realizing the given degree sequence (a multiset of positive integer pairs), has also been intensively studied; see Chen [4], Fulkerson [10], Gale [11], and Ryser [23] for characterizations of realizable degree sequences and Kleitman and Wang [17] for polynomial-time algorithms. The problem of realizing degree sequences has also been studied in context of (loop-less) multigraphs, where the aim is to minimize or maximize the number of multi-edges [15].

Berger and Müller-Hannemann [1, 2, 3] investigated restricted variants of DAG Realization that are polynomial-time solvable and performed an extensive experimental study on the general problem.

Organization Our paper is structured as follows: We introduce the necessary notation in Section 2. Section 3 contains the proof of the NP-completeness of DAG Realization. Furthermore, we show that it remains NP-complete in case of any constant ratios $m/n > 1$ and $\binom{n}{2}/m > 1$. In Section 4 we show that DAG Realization is fixed-parameter tractable with respect to the parameter maximum degree Δ. Finally, in Section 5 we prove that DAG Realization is fixed-parameter tractable with respect to each of the parameters $m - n + 1$ and $\binom{n}{2} - m$.

2 Preliminaries

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$. We denote with $\uplus$ the multiset sum (e. g. $\{1, 1\} \uplus \{1, 2\} = \{1, 1, 1, 2\}$). The input of DAG Realization, a so-called *degree sequence*, is a multiset of non-negative integer pairs which are called *tuples*. Although in a multiset there is no ordering given, we stick to the term degree *sequence* as it is commonly used in the literature. For a degree sequence $\mathcal{S} = \left\{ \binom{a_1}{b_1}, \ldots, \binom{a_n}{b_n} \right\}$ we always assume that $\binom{0}{0} \notin \mathcal{S}$, that the maximum occurring value is at most $n-1$, and that $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$. For convenience we assume throughout this work that all $2n$ numbers in the degree sequences are given explicitly. Hence, the overall input size of the DAG Realization instance is $O(n \log \Delta)$ where $\Delta := \max\{a_1, b_1, \ldots, a_n, b_n\}$ is the largest number in the degree sequence. Note that by the above assumption we have $\Delta < n$ and, thus, the size of the input is also bounded by $O(n \log n)$.

We denote directed graphs by $D = (V, A)$ with vertex set V and arc set $A \subseteq V \times V$. For a vertex subset $V' \subseteq V$, the *subgraph induced by* V' is $D[V'] := \{V', (V' \times V') \cap A\}$. Let $n := |V|$ and $m := |A|$. Correspondingly, for a degree sequence $\mathcal{S} = \left\{ \binom{a_1}{b_1}, \ldots, \binom{a_n}{b_n} \right\}$ we set $m := \sum_{i=1}^{n} a_i$.

Two vertices in a directed graph $D = (V, A)$ are called *connected* if they are weakly connected, that is, in the underlying undirected graph there is a path between them. Correspondingly, *connected components* always refer to weakly connected components. For two vertices $v, u \in V$ with $(v, u) \in A$, we call v an *inneighbor* of u and u an *outneighbor* of v. We denote the indegree of a vertex $v \in V$ by $d^-(v)$ and the outdegree by $d^+(v)$. Correspondingly, for a degree sequence $\mathcal{S}$ and a tuple $s \in \mathcal{S}$ with $s = \binom{a}{b}$, we set $d^-(s) := a$ and $d^+(s) := b$. A vertex $v \in V$ or a tuple $\binom{a}{b} \in \mathcal{S}$ is called a *source* if $d^-(v) = 0 = a$ and it is called a *sink* if $d^+(v) = 0 = b$.

For a directed graph $D = (V, A)$, a *directed path* is a vertex sequence $v_1, \ldots, v_l$ such that for all $1 \leq i < l$ we have $(v_i, v_{i+1}) \in A$. If additionally $v_l = v_1$ holds, then it is a *directed cycle*. A directed graph $D = (V, A)$ is a *DAG* if it does not contain a directed cycle. Each DAG D admits a *topological ordering*, that is, an ordering of all its vertices $v_1, \ldots, v_n$ such that for all arcs $(v_i, v_j) \in A$ it holds that $i < j$.

A *transitive tournament* with n vertices is the (up to isomorphism) unique DAG that realizes the degree sequence $\{\binom{0}{n-1}, \binom{1}{n-2}, \ldots, \binom{n-1}{0}\}$. Equivalently, a transitive tournament is the only DAG with $\binom{n}{2}$ arcs and it admits only one topological ordering.

Next, we define a central notion of our work.

Definition 2.1. An ordered degree sequence $\sigma = \binom{a_1}{b_1}, \ldots, \binom{a_n}{b_n}$ is called *realizable degree ordering* if there is a realizing DAG D for σ that admits a topological ordering $v_1, \ldots, v_n$ that corresponds to σ, that is, a topological ordering such that $d^-(v_i) = a_i$ and $d^+(v_i) = b_i$ for all $1 \leq i \leq n$.

A realizable degree ordering *of a degree sequence* $\mathcal{S}$ refers to an ordering of $\mathcal{S}$ such that it fulfills Definition 2.1. Moreover, a realizing DAG for an ordered degree sequence σ always refers to a DAG admitting a topological ordering that corresponds to σ. Berger and Müller-Hannemann [3] proved that one can check in polynomial time whether a given ordering of a degree sequence is a realizable degree ordering.

Parameterized Complexity An instance of a *parameterized problem* consists of the "classical" problem instance I and a parameter $k \in \mathbb{N}$. It is called *fixed-*

parameter tractable if there is an algorithm that solves any instance (I, k) in $f(k) \cdot |I|^c$ time. Here, f is a computable function solely depending on k and $c \in \mathbb{N}$ is a constant independent from k and I. For a more detailed introduction to parameterized algorithmics we refer to the monographs [5, 8, 21].

3 NP-Completeness

In this section we first describe the construction of our reduction proving NP-hardness of DAG Realization and explain the idea of how it works. Then, we prove its correctness. Finally, we show that DAG Realization remains NP-hard for any constant ratio of ${}^{m}/{n}$ and of $\binom{n}{2}/m$.

We prove NP-hardness for DAG Realization by giving a polynomial-time many-to-one reduction from the strongly NP-hard 3-Partition problem [12, SP15]:

3-Partition
Input: A multiset $\mathcal{A} = \{a_1, \ldots, a_{3p}\}$ of $3p$ positive integers and an integer B with $\sum_{i=1}^{3p} a_i = pB$ and $\forall i\colon B/4 < a_i < B/2$.
Question: Is there a partition of the $3p$ integers from $\mathcal{A}$ into p disjoint triples such that in every triple the three elements sum up to B?

Construction Throughout this section let $(\mathcal{A}, B)$ be an instance of 3-Partition and let $\mathcal{S}_{\mathcal{A},B}$ be the degree sequence constructed as follows:

$$\mathcal{S}_{\mathcal{A},B} := X_0 \uplus X_1 \uplus \ldots \uplus X_p \uplus \left\{ \binom{a_1}{a_1}, \ldots, \binom{a_{3p}}{a_{3p}} \right\}.$$

We formally define the multisets X_i, $0 \leq i \leq p$, after giving the idea of the construction. For the description of the idea we need some notation: We set $X := \biguplus_{i=0}^{p} X_i$, we call a tuple in X an *x-tuple*, and all others are called *a-tuples*. In a realizing DAG the vertices realizing x-tuples are called *x-vertices* and the vertices realizing a-tuples are called *a-vertices*.

The intuition of the construction is that a DAG realizing $\mathcal{S}_{\mathcal{A},B}$ (if it exists) looks as follows (see Figure 1 for an illustration): The tuples of a multiset X_i, $0 \leq i \leq p$, form a "block" in a realizable degree ordering. These blocks are a skeletal structure in any realizable degree ordering and there are p "gaps" between them. The construction ensures that each gap is filled with a-vertices adjacent to the vertices in the blocks bordering the gap and, moreover, the indegrees and outdegrees of all a-vertices in a gap sum up to B. Hence, these p gaps require to partition the a-vertices into p sets where each set has in total in- and outdegree B and, thus, the p sets correspond to a solution for the 3-Partition instance that we reduce from. In the reverse direction, for each triple in a solution of a 3-Partition instance the corresponding a-vertices will be used to fill up one gap.

We next provide the remaining details of the reduction. The set X consists of the tuples $\{x_0, \ldots, x_{2Bp+2B-1}\}$ where, as indicated in Figure 1, each multiset X_i contains $2B$ tuples. More specifically, $X_i = \{x_{2Bi}, \ldots, x_{2Bi+2B-1}\}$ for all $i \in \{0, \ldots, p\}$. Subsequently, we describe properties that a DAG realizing $\mathcal{S}_{\mathcal{A},B}$ shall have. We later prove that any DAG realizing $\mathcal{S}_{\mathcal{A},B}$ indeed has these properties. To impose the required skeletal structure (having a gap between each pair of

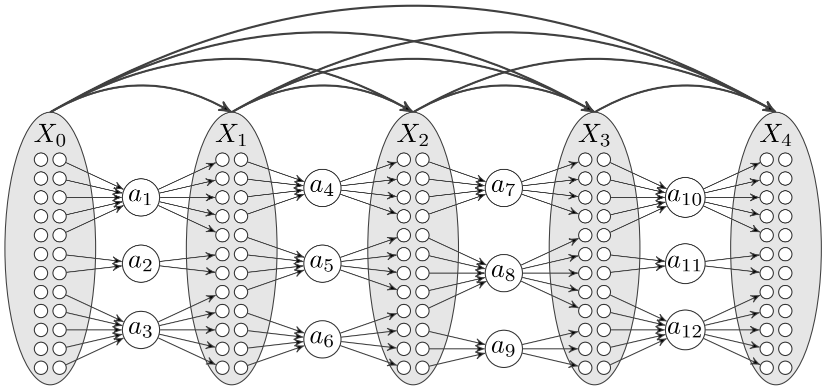

Figure 1: A schematic illustration of a DAG that realizes a degree sequence $\mathcal{S}_{\mathcal{A},B}$ that is constructed from a 3-Partition instance with $B = 12$ and $p = 4$. There are five blocks marked by the gray ellipses and four gaps between them. In each gap there are three a-vertices, altogether having in- and outdegree B. The sets X_i, $0 \leq i \leq p$, are partitioned into two parts of size B. The vertices in the left part (except for X_0) have B incoming arcs from the a-vertices that fill the gap between X_{i-1} and X_i. Correspondingly, the vertices in the right part (except in X_p) have B outgoing arcs to the a-vertices that fill the gap between X_i and X_{i+1}. The in- and outdegrees of the a-vertices in each triple sum up to B. The vertices in the gray ellipses induce a big transitive tournament where the first vertex with no inneighbors is the top-leftmost vertex and the vertex without outneighbors is the bottom-rightmost vertex. Here, the thick-drawn arcs on top indicate that each vertex in an ellipse has outgoing arcs to all vertices in the proceeding gray ellipses.

blocks X_i, X_{i+1}), the x-vertices shall induce a transitive tournament where the degrees in the topological ordering correspond to the order $x_0, \ldots, x_{2Bp+2B-1}$. Thus, denoting the x-vertices corresponding to X_i by $V(X_i)$, it follows that $V(X_i)$ precedes $V(X_{i+1})$ for all $i \in \{0, \ldots, p-1\}$. To ensure that each gap between $V(X_i)$ and $V(X_{i+1})$ is "filled up" by a-vertices with total indegree/outdegree B, we partition X_i into the "left" part $X_i^\ell = \{x_{2Bi}, \ldots, x_{2Bi+B-1}\}$ and the "right" part $X_i^r := \{x_{2Bi+B}, \ldots, x_{2Bi+2B-1}\}$. Then, each of the B right-part vertices in $V(X_i^r)$ is required to have, in addition to its outneighbors in $V(X)$, one a-vertex as an outneighbor. Symmetrically, each of the B left-part vertices in $V(X_i^\ell)$ has one additional a-vertex as an inneighbor. The "borders" of the skeletal structure, namely x-vertices corresponding to the left-part of X_0 and the right-part of X_p, are not supposed to have a-vertices as neighbors. The following definition of the tuple x_j from X_i captures the above mentioned properties:

$$x_j := \begin{pmatrix} j + \mathrm{in}(j) \\ |X| - 1 - j + \mathrm{out}(j) \end{pmatrix} \tag{1}$$

where $\text{in}(j)$ and $\text{out}(j)$ are defined as follows:

$$\text{in}(j) := \begin{cases} 1, & \text{if } x_j \in X_i^\ell \wedge i > 0 \\ 0, & \text{else} \end{cases} \qquad \text{out}(j) := \begin{cases} 1, & \text{if } x_j \in X_i^r \wedge i < p \\ 0, & \text{else.} \end{cases}$$

The construction above can be computed in polynomial time due to the strong NP-hardness of 3-Partition: The size of the constructed DAG Realization instance is upper-bounded by a polynomial in the values of the integers in $\mathcal{A}$. Since 3-Partition is strongly NP-hard, it remains NP-hard when the values of the integers in $\mathcal{A}$ are bounded by a polynomial in the input size. Hence, the size of the DAG Realization instance is polynomially bounded in the size of the 3-Partition instance.

Correctness Next, we prove the correctness of the construction given above. To start with the simpler case, we first show that if $(\mathcal{A}, B)$ is a yes-instance, then there is a realizing DAG for $S_{\mathcal{A},B}$.

Lemma 3.1. If $(\mathcal{A}, B)$ is a yes-instance of 3-Partition, then $\mathcal{S}_{\mathcal{A},B}$ is a yes-instance of DAG Realization.

Proof. We prove that if $(\mathcal{A}, B)$ is a yes-instance, then there exists a realizing DAG $D_{\mathcal{A},B}$ for $\mathcal{S}_{\mathcal{A},B}$ which is structured as depicted in Figure 1. Formally, let π be a permutation of the sequence $\mathcal{A}$ such that $a_{\pi(3i+1)} + a_{\pi(3i+2)} + a_{\pi(3i+3)} = B$ for all $0 \leq i < p$. Since $(\mathcal{A}, B)$ is a yes-instance of 3-Partition such a permutation exists. The x-vertices in $D_{\mathcal{A},B}$ induce a transitive tournament whose topological ordering corresponds to the ordering $x_0, x_1, \ldots, x_{2Bp+2B-1}$ of the x-tuples. Observe that the arcs in this transitive tournament ensure that the degree of the x-vertices is equal to Equation (1), except for the additive in- and out-terms.

Each a-tuple $\binom{a_i}{a_i}$ is realized in $D_{\mathcal{A},B}$ by a vertex called u_i. The a-vertex $u_{\pi(3i+1)}$ ($u_{\pi(3i+2)}, u_{\pi(3i+3)}$ resp.) has an incoming arc from each of the first $a_{\pi(3i+1)}$ (next $a_{\pi(3i+2)}$, last $a_{\pi(3i+3)}$) vertices in $V(X_i^r)$ and an outgoing arc to each of the first $a_{\pi(3i+1)}$ (next $a_{\pi(3i+2)}$, last $a_{\pi(3i+3)}$) vertices in $V(X_{i+1}^\ell)$. In this way, each a-vertex u_i has an in- and outdegree of a_i. Furthermore, as $a_{\pi(3i+1)} + a_{\pi(3i+2)} + a_{\pi(3i+3)} = B$, all vertices in $V(X_i^\ell)$ (for $i > 0$) have an incoming arc from an a-vertex and all vertices in $V(X_i^r)$ (for $i < p$) have an outgoing arc to an a-vertex. □

To show the reverse direction, we first need some lemmas.

Lemma 3.2. In any DAG realizing $\mathcal{S}_{\mathcal{A},B}$, the a-vertices form an independent set and the x-vertices induce a transitive tournament.

Proof. The number $d^-(X)$ of incoming arcs to all x-vertices is

$$\begin{aligned} d^-(X) &= \sum_{j=0}^{|X|-1} d^-(x_j) \overset{(1)}{=} \sum_{j=0}^{|X|-1} j + \text{in}(j) \\ &= pB + \sum_{j=0}^{|X|-1} j = pB + \binom{|X|}{2}. \end{aligned}$$

Moreover, the number $d^+(X)$ of outgoing arcs from all x-vertices is

$$\begin{aligned} d^+(X) &= \sum_{j=0}^{|X|-1} d^+(x_j) \overset{(1)}{=} \sum_{j=0}^{|X|-1} |X| - 1 - j + \mathrm{out}(j) \\ &= \sum_{j=0}^{|X|-1} j + \mathrm{out}(j) = pB + \sum_{j=0}^{|X|-1} j = pB + \binom{|X|}{2}. \end{aligned}$$

Hence, $d^-(X) = d^+(X)$ and since there can be at most $\binom{|X|}{2}$ arcs between two x-vertices, it follows that there are at least $d^-(X) - \binom{|X|}{2} = d^+(X) - \binom{|X|}{2} = pB$ arcs from an x-vertex to an a-vertex and vice versa. Together with $\sum_{i=1}^{3p} a_i = pB$, it follows that each a-vertex is adjacent only to x-vertices. Thus, in any realizing DAG the a-vertices form an independent set and the number of arcs between the x-vertices is exactly $\binom{|X|}{2}$. Hence, the x-vertices form a transitive tournament. □

Next we show that for a realizable degree sequence $\mathcal{S}_{\mathcal{A},B}$ there exists a realizable degree ordering in which the x-vertices are ordered correspondingly to $x_0, x_1, \ldots, x_{2Bp+2B-1}$. To this end, we need some further notations: We use the *opposed order* $\leq_{\mathrm{opp}}$ for the tuples of a degree sequence as introduced by Berger and Müller-Hannemann [2]:

Definition 3.3. For two tuples $\binom{a}{b}$ and $\binom{a'}{b'}$ it holds that $\binom{a}{b} \leq_{\mathrm{opp}} \binom{a'}{b'} \iff a \leq a' \wedge b \geq b'$.

Note that there might be tuples in a degree sequence that are not comparable with respect to the opposed order (for example $\binom{1}{1}$ and $\binom{2}{2}$). However, Berger and Müller-Hannemann [2] proved that one can always reorder a realizable degree ordering with respect to the opposed order.

Lemma 3.4. ([2, Corollary 3]) Let $\mathcal{S}$ be a realizable degree sequence. Then, there exists a realizable degree ordering $\binom{a_1}{b_1}, \ldots, \binom{a_n}{b_n}$ of $\mathcal{S}$ such that for all $1 \leq i, j \leq n$ with $\binom{a_i}{b_i} \leq_{\mathrm{opp}} \binom{a_j}{b_j}$ and $\binom{a_i}{b_i} \neq \binom{a_j}{b_j}$, it holds that $i < j$.

As a consequence of Lemma 3.4, if there are two tuples t_1, t_2 in a realizable degree sequence $\mathcal{S}_{\mathcal{A},B}$ such that $t_1 \leq_{\mathrm{opp}} t_2$ and $t_1 \neq t_2$, then we can always assume that there is a realizable degree ordering where the tuple t_1 is ahead of t_2. Furthermore, observe that if $t_1 = t_2$ and there is a realizable degree ordering where t_1 is ahead of t_2, then there is also a realizable degree ordering where t_2 is ahead of t_1 (just exchange these two identical tuples). We use this fact to prove the next lemma.

Lemma 3.5. If $\mathcal{S}_{\mathcal{A},B}$ is realizable, then there exists a realizable degree ordering σ of $\mathcal{S}_{\mathcal{A},B}$ such that in σ tuple x_i is ahead of x_j for all $0 \leq i < j < |X|$.

Proof. To prove that such a realizable degree ordering σ exists, by Lemma 3.4 and the above discussion, it is sufficient to show for all $0 \leq i < j < |X|$ that $x_i \leq_{\mathrm{opp}}$

x_j. This can be verified easily as $\text{in}(k), \text{out}(k) \in \{0,1\}$ for all $0 \le k < |X|$:

$$\begin{aligned} d^-(x_i) - d^-(x_j) &\overset{(1)}{=} i + \text{in}(i) - j - \text{in}(j) \le i + 1 - j \le 0 \\ d^+(x_i) - d^+(x_j) &\overset{(1)}{=} |X| - 1 - i + \text{out}(i) - (|X| - 1 - j + \text{out}(j)) \\ &= j + \text{out}(i) - i - \text{out}(j) \ge j - i - 1 \ge 0. \end{aligned}$$

□

With Lemmas 3.2 and 3.5 we can prove the next lemma, which completes the proof of the correctness of our reduction.

Lemma 3.6. If $\mathcal{S}_{\mathcal{A},B}$ is a yes-instance of DAG Realization, then $(\mathcal{A}, B)$ is a yes-instance of 3-Partition.

Proof. Let $D_{\mathcal{A},B}$ be a realizing DAG of a degree sequence $\mathcal{S}_{\mathcal{A},B}$ with a topological ordering ϕ. Let v_j be the x-vertex realizing x_j. Denoting for each vertex w by $\text{pos}_\phi(w)$ the position of w in ϕ, by Lemma 3.5 we may assume that that $\text{pos}_\phi(v_j) < \text{pos}_\phi(v_k)$ for all $0 \le j < k < |X|$. From Lemma 3.2 it follows that the x-vertices induce a transitive tournament in $D_{\mathcal{A},B}$. Hence, it follows from the in- and outdegrees of the x-vertices (Equation (1)) that each x-vertex in $V(X_i^r)$, $0 \le i < p$, has exactly one outgoing arc to an a-vertex and each x-vertex in $V(X_i^\ell)$, $0 < i \le p$, has exactly one incoming arc from an a-vertex.

The fact that all B vertices from $V(X_0^r)$ have one outgoing arc to an a-vertex, all B vertices in $V(X_1^\ell)$ have one incoming arc from an a-vertex, and the a-vertices form an independent set (Lemma 3.2), imply that between $\text{pos}_\phi(v_B)$ (first vertex from $V(X_0^r)$) and $\text{pos}_\phi(v_{2B+B-1})$ (last vertex from $V(X_1^\ell)$) there are a-vertices in ϕ whose total in- and out-degree sum up to B. (From the definition of 3-Partition, it follows that this must be exactly three a-vertices.) As these three a-vertices have all their outneighbors in X_1^ℓ, by the same argumentation it follows that for each pair $V(X_i)$ and $V(X_{i+1})$, $0 \le i < p$, between the first vertex from $V(X_i^r)$ and the last vertex from $V(X_{i+1}^\ell)$ there must three a-vertices whose total in- and outdegree sum up to B. Thus, they provide a p-partition of $\mathcal{A}$ into triples each summing up to B and, thus, imply that $(\mathcal{A}, B)$ is a yes-instance of 3-Partition. □

Our construction together with Lemma 3.1 and Lemma 3.6 yields the NP-hardness of DAG Realization. Containment in NP follows from the fact that it can be checked in polynomial time whether a degree ordering is realizable or not [3]. Together, this implies the central theorem of this section.

Theorem 3.7. DAG Realization is NP-complete.

Theorem 3.7 proves that unless P = NP DAG Realization is not solvable in polynomial time. This motivates a parameterized complexity analysis of the problem meaning to perform a more fine-grained complexity analysis with respect to various parameters [7, 19, 22]. Thereby, the general target is to identify certain "quantities" or parameters whose restriction allows the problem to be solved efficiently. In this respect, Berger and Müller-Hannemann [2] already identified special cases of DAG Realization that can be solved in polynomial time; for instance, this is the case when the degree sequence can be linearly ordered with respect to the opposed order. Moreover, Berger and Müller-Hannemann [3] showed that DAG Realization is polynomial-time solvable when the number m of arcs is less than the number n of vertices. Since in our NP-hardness proof construction, the resulting DAG Realization instance contains $\Theta(n^2)$ arcs, the natural question then is whether more general "sparse" instances allow for polynomial-time algorithms. Unfortunately, the next theorem proves that, unless P = NP, this is not the case. Furthermore, it shows that although the problem is easy if $m = \binom{n}{2}$ (the only realizing DAG is a transitive tournament), the problem becomes hard if m is "almost" $\binom{n}{2}$.

Theorem 3.8. For every constant $\ell > 1$, DAG Realization remains NP-complete even if $m < \ell n$ or if $m > \binom{n}{2} \cdot \ell^{-1}$.

Proof. We prove both results with simple padding arguments. To this end, let $\ell > 1$ be some constant and let $\mathcal{S} = \left\{ \binom{a_1}{b_1}, \dots, \binom{a_n}{b_n} \right\}$ be an arbitrary instance of DAG Realization. We will modify $\mathcal{S}$ in order to obtain an instance with the desired ratios between m and n. We denote the modified instance with $\mathcal{S}'$ which is initialized as a copy of $\mathcal{S}$.

First, we describe the modifications for the case $m > \binom{n}{2} \cdot \ell^{-1}$: We repeatedly add universal sources one after the other to $\mathcal{S}'$, that is, we add a tuple $\binom{0}{n'}$ to $\mathcal{S}'$ and increase the first component of the other tuples of $\mathcal{S}'$ by one, that is, replace $\binom{a}{b} \in \mathcal{S}'$ by $\binom{a+1}{b}$. Notice that once a universal source is added to $\mathcal{S}'$ all former sources become non-sources; in particular, if a second universal source is added, then the first added universal source is no longer a source. We keep adding universal sources until $m' > \binom{n'}{2} \cdot \ell^{-1}$. Observe that the condition is fulfilled after adding polynomially many universal sources.

Observe, that adding a universal source to a DAG Realization instance results in an equivalent instance: The vertex realizing the universal source has to have outgoing arcs to each other vertex in any realizing DAG . Thus, for every realizing DAG for the new instance the subgraph induced by all vertices not realizing the universal sink forms a realizing DAG for the original instance. Conversely, any realizing DAG for the old instance can be easily extended to a realizing DAG for the new instance. Thus, the constructed instance $\mathcal{S}'$ is a yes-instance if and only if $\mathcal{S}$ is a yes-instance. Hence, DAG Realization remains NP-complete even if $m > \binom{n}{2} \cdot \ell^{-1}$.

Second, for the case $m < \ell n$ we do the following: We add some number of $\binom{0}{0}$-tuples to $\mathcal{S}'$ and then add one universal source turning the $\binom{0}{0}$-tuples into $\binom{1}{0}$-sinks. By choosing the appropriate amount of $\binom{0}{0}$-tuples the modified instance $\mathcal{S}'$ satisfies $m' < \ell n'$. Furthermore, as adding a $\binom{0}{0}$-tuple results in an equivalent instance, it follows that the constructed instance $\mathcal{S}'$ is a yes-instance if and only if $\mathcal{S}$ is a yes-instance. Hence, DAG Realization remains NP-complete even if $m < \ell n$. $\square$

Complementing Theorem 3.8, we show in Section 5 that DAG Realization is fixed-parameter tractable with respect to each of the parameters $m - n + 1$ and $\binom{n}{2} - m$. Besides the above mentioned parameters, the maximum degree Δ is unbounded in our NP-hardness proof and is hence a good candidate for further investigations. Indeed, we show in the next section that DAG Realization is linear-time solvable for constant maximum degree.

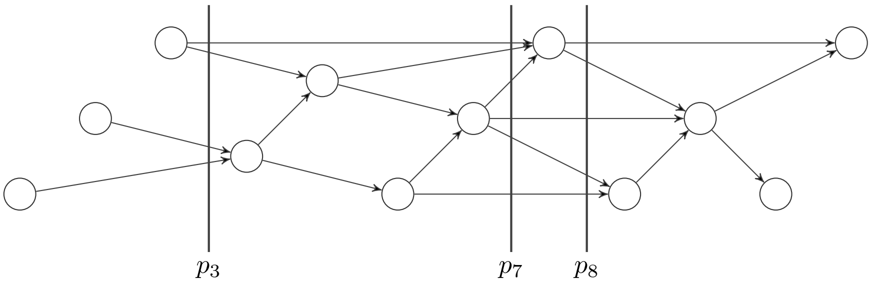

Figure 2: A realizing DAG for the degree sequence $\mathcal{S} = \{\binom{0}{1}, \binom{0}{1}, \binom{0}{2}, \binom{2}{2}, \binom{2}{2}, \binom{1}{2}, \binom{2}{3}, \binom{3}{2}, \binom{2}{1}, \binom{3}{2}, \binom{1}{0}, \binom{2}{0}\}$. The highlighted potentials are as follows: $p_3 = (3,1,0)$, $p_7 = (4,1,1)$, and $p_8 = (3,2,0)$. Their values (number of arcs crossing the line indicating the potential) are: $\omega(p_3) = 4$, $\omega(p_7) = 6$, and $\omega(p_8) = 5$.

4 Fixed-Parameter Tractability with Respect to Maximum Degree Δ

Let $\Delta := \max\{a_1, b_1, \dots, a_n, b_n\}$ denote the maximum degree in a degree sequence. In this section we show that DAG Realization is fixed-parameter tractable with respect to the parameter Δ. To this end, we assume that Δ is some fixed value and all degree sequences considered in this section have a maximum degree of Δ.

A high-level description of our approach is as follows: First, we show that we can reorder any realizable degree ordering for our given input instance $\mathcal{S}$ so that it has a certain structure. Second, our algorithm branches into all possible orderings of the tuples in $\mathcal{S}$ satisfying this structure and returns "yes" if in at least one branch the considered ordering is indeed a realizable degree ordering, otherwise it returns "no". Herein, we will distinguish two cases for the sought structures. In both cases, however, we use the same reordering operations and their description makes up the major part of Section 4.1. To describe how we reorder realizable degree orderings, we need the following central definition.

Definition 4.1. Let $\phi = v_1, v_2, \dots, v_n$ be a topological ordering of a DAG D. For all $0 \le i \le n$, the *potential* at position i is a vector $p_i^\phi \in \mathbb{N}^\Delta$ where $p_i^\phi[\ell]$ for $1 \le \ell \le \Delta$ is the number of vertices in the subsequence $v_1, \dots, v_i$ that have in D at least ℓ neighbors in the subsequence $v_{i+1}, \dots, v_n$. The *value* of the potential p_i^ϕ is $\omega(p_i^\phi) := \sum_{\ell=1}^{\Delta} p_i^\phi[\ell]$.

See Figure 2 for an example. If the DAG D and the topological ordering ϕ are clear from the context, then we write p_i instead of p_i^ϕ. We denote with 0^Δ the potential of value zero, for example, it holds that $p_0 = p_n = 0^\Delta$. To indicate the role of potentials in the reordering operation, consider a topological ordering $\phi = v_1, \dots, v_n$ where at two positions $0 < i < j < n$ the potentials are equal, that is $p_i^\phi = p_j^\phi$. We will show in Section 4.1 that we can cut out the vertices $v_{i+1}, \dots, v_j$ and rewire the arcs to obtain another DAG without changing the degrees of the remaining vertices or their relative position in the topological ordering.

In order to give some intuition about potentials, in the rest of this introductory part of Section 4, we provide some general observations. First, observe that the value of a potential $\omega(p_i)$ is just the number of arcs with tail in $\{v_1, \ldots, v_i\}$ and head in $\{v_{i+1}, \ldots, v_n\}$; for example in Figure 2 four arcs "cross" the third position and hence $\omega(p_3) = 4$. Since the number of arcs is determined by the degrees of the vertices, the value of the potential at position $i+1$ can be determined from the potential at position i and the degree of the vertex v_{i+1} at position $i+1$: As v_{i+1} "absorbs" $d^-(v_{i+1})$ arcs and "contributes" $d^+(v_{i+1})$ arcs to the following vertices, the value of the potential at position $i+1$ is $\omega(p_{i+1}) = \omega(p_i) - d^-(v_{i+1}) + d^+(v_{i+1})$. Generalizing this yields the following.

Observation 4.2. Let $\phi = v_1, v_2, \ldots, v_n$ be a topological ordering of a DAG D and let $1 \leq i < j \leq n$ be two integers. Then it holds that $\omega(p_j) = \omega(p_i) + \sum_{\ell=i+1}^{j}(d^+(v_\ell) - d^-(v_\ell))$.

Second, we remark that a potential stores more information than just the number of arcs "crossing" some position: The potential p_i stores *all* information about *how many* vertices from $\{v_1, \ldots, v_i\}$ have *how many* outgoing arcs to $\{v_{i+1}, \ldots, v_n\}$: In particular, for each $1 \leq j < \Delta$ there are $p_i[j] - p_i[j+1]$ vertices in $\{v_1, \ldots, v_i\}$ that have exactly j outneighbors in $\{v_{i+1}, \ldots, v_n\}$. Thus, for any potential $p_i \in \mathbb{N}^\Delta$ at any position $i \in \mathbb{N}$, it holds that $p_i[j] \geq p_i[j+1]$ for all $1 \leq j < \Delta$.

Third, observe that each vector $p \in \mathbb{N}^\Delta$ satisfying the above requirement can appear as potential in some DAG. More precisely the following holds.

Observation 4.3. Let $p \in \mathbb{N}^\Delta$ be a vector with $p_i[j] \geq p_i[j+1]$ for all $j \in \{1, \ldots, \Delta\}$. Then there exists a DAG D with a topological ordering $\phi = v_1, v_2, \ldots, v_n$ such that $p = p_i^\phi$ for some $i \in \{1, \ldots, n\}$.

Proof. Let $p \in \mathbb{N}^\Delta$ be a vector with $p_i[j] \geq p_i[j+1]$ for all $1 \leq j < \Delta$. We construct a DAG D with a topological ordering ϕ such that $p = p^\phi_{p[1]}$. The DAG D contains $p[1]$ sources and $\omega(p)$ sinks and no further vertices. In particular, D contains $p[\Delta]$ sources with outdegree Δ and for each $1 \leq j < \Delta$, D contains $p[j] - p[j+1]$ sources with outdegree j. The $\omega(p)$ sinks in D have all indegree one. Observe that the sum of the outdegrees is $\omega(p)$, that is, equal to the sum of the indegrees. We complete D by making each source adjacent with the appropriate number of sinks. Since D contains only sinks and sources, D is acyclic. Moreover, observe that in a topological ordering of D where all sources precede the sinks, the potential at position $p[1]$ (after the last source) is equal to p. $\square$

Algorithm Outline Our algorithm consists of two parts. First, as described in Section 4.2, the algorithm checks whether the DAG Realization instance admits a "high-potential" realization where at some position the value of the potential is at least Δ^2. If no high-potential realization is found, then, by exploiting the fact that the value of all potentials is upper-bounded, the algorithm checks whether a "low-potential" realization exists; see Section 4.3 for the description.

4.1 General Terms and Observations

In this section we provide some general notations, observations, and lemmas leading to the reordering operation we use in the algorithms to find high-potential as well as low-potential realizations.

Notation For a topological ordering $\phi = v_1, \dots, v_n$ and two indices $1 \leq i \leq j \leq n$ we set $\phi[i, j] := v_i, v_{i+1}, \dots, v_j$. The vertex set $\{v_i, \dots, v_j\}$ is denoted by $\phi\{i, j\}$. Analogously, for an ordered degree sequence $\sigma = \binom{a_1}{b_1}, \dots, \binom{a_n}{b_n}$ we set $\sigma[i, j] := \binom{a_i}{b_i}, \dots, \binom{a_j}{b_j}$ and we denote the multiset $\left\{\binom{a_i}{b_i}, \dots, \binom{a_j}{b_j}\right\}$ by $\sigma\{i, j\}$.

Definition 4.4. Two tuples $\binom{a}{b}$ and $\binom{a'}{b'}$ are of the same *type* if $a = a'$ and $b = b'$. Furthermore, $\binom{a}{b}$ is a *good type* tuple if $a \leq b$ and otherwise it is a *bad type* tuple.

Note that there are at most $(\Delta + 1)^2$ different types.

Well-Connected DAGs Berger and Müller-Hannemann [3] showed that, given an ordering of a degree sequence, one can check in polynomial time whether this ordering is a realizable degree ordering. The proof is done by *well-connecting* consecutive vertices in a topological ordering.

Definition 4.5. Let D be a DAG with a topological ordering $\phi[1, n]$. The *remaining outdegree* of vertex v_i at position j, $1 \leq i \leq j < n$, is the number of v_i's neighbors in the subsequence $\phi[j + 1, n]$. Furthermore, D is *well-connected* with respect to ϕ if for all vertices $v_i \in \phi\{1, n\}$ it holds that the $d^-(v_i)$ inneighbors of v_i are among the vertices in $\phi[1, i - 1]$ that have the highest remaining outdegree at position $i - 1$.

Observe that the potential p_i^ϕ at position i stores all remaining outdegrees of all vertices $v_1, \dots, v_i$ at position i. In the following, we omit ϕ and just write that D is well-connected when the corresponding topological ordering ϕ is clear from the context or implicitly given by a realizable degree ordering corresponding to D. Furthermore, during the construction of a DAG corresponding to a realizable degree ordering we write that we well-connect the vertex v_i as an abbreviation for making $d^-(v_i)$ vertices with the highest remaining outdegree at position $i - 1$ inneighbors of v_i. Berger and Müller-Hannemann [3] showed how to construct a well-connected DAG realizing a given realizable degree ordering; see Berger [1] for the complete proof of correctness.

Lemma 4.6. ([1, Theorem 4.1] and [3, Lemma 1]) *Let σ be a realizable degree ordering. Then, there exists a realizing well-connected DAG D that admits a topological ordering ϕ corresponding to σ.*

We use Lemma 4.6 in two (obvious) ways. First, it paves the way to a simple algorithm checking whether a given ordering of a degree sequence $\mathcal{S}$ is indeed a realizable degree ordering: The algorithm iteratively adds the vertices according to given ordering and well-connects each added vertex.

Lemma 4.7. *Given an ordered degree sequence σ, one can decide in $O(\Delta n)$ time whether σ is a realizable degree ordering.*

Proof. As mentioned above, the algorithm constructs the DAG stepwise by iterating over σ, adding a vertex v for the currently considered tuple, and well-connecting v. We remark that Berger [1, Theorem 4.1] actually proved that *all* possibilities of well-connecting v lead to a realizing DAG. Hence, if there are multiple possibilities to well-connect v, then we can use an arbitrary one.

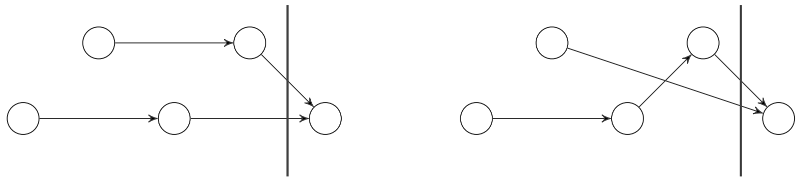

Figure 3: Two non-isomorphic well-connected DAGs realizing the realizable degree ordering $\binom{0}{1}, \binom{0}{1}, \binom{1}{1}, \binom{1}{1}, \binom{2}{0}$. Observe that for each position the two potentials at this position are the same, for example, in both graphs the potential at position four is $p_4 = (2, 0)$.

The algorithm uses Δ lists, where the i^{th} list stores all vertices having remaining outdegree i at the currently considered position. By virtually concatenating the Δ lists in $O(\Delta)$ time and then iterating over the first Δ elements one can determine in $O(\Delta)$ time up to Δ vertices with highest outdegrees. Hence, well-connecting a vertex v can be done in $O(\Delta)$ time and decreasing the remaining outdegree of the $d^-(v) \leq \Delta$ inneighbors of v by one can also be done in $O(\Delta)$ time. Furthermore, inserting v into these lists can be done in constant time. Hence, with n elements in the given realizable degree ordering σ one can decide in $O(\Delta n)$ time whether σ is a realizable degree ordering. $\square$

Second, in the following it will be important that Lemma 4.6 allows us to assume that, given a realizable degree ordering σ, the corresponding realizing DAG is well-connected if not explicitly stated otherwise. Note that there might be more than one well-connected DAG realizing σ as there might be multiple vertices with the highest outdegree at some position, see Figure 3 for an example. However, for each $x \in \mathbb{N}$ the number of vertices with remaining outdegree x at position i is the same for all well-connected DAGs realizing σ. Hence, we arrive at the following.

Lemma 4.8. Let $\phi[1, \ldots, n]$ be a topological ordering of a well-connected DAG and let p_j be the potential at position $j \in \{0, \ldots, n-1\}$. Furthermore, let $\binom{a}{b}$ be the in- and outdegree of the $(j+1)$th vertex in ϕ. Then, setting $p_j[\Delta+1] = 0$, for all $\ell \in \{1, \ldots, \Delta\}$ it holds that

$$\sum_{i=1}^{\ell} p_j[i] - p_{j+1}[i] = \max\{0, a - p_j[\ell+1]\} - \min\{i, b\}. \tag{2}$$

Proof. We first prove Lemma 4.8 in case of $b = 0$ and thus we prove that

$$\sum_{i=1}^{\ell} p_j[i] - p_{j+1}[i] = \max\{0, a - p_j[\ell+1]\}. \tag{3}$$

Observe that this is sufficient as the last additive term in Equation (2) only counts the contribution of the b outgoing arcs of the $(j+1)$th vertex in ϕ to p_{j+1}.

Denoting the $(j+1)$th vertex in ϕ by v, we next analyze its effect of "absorbing" a arcs from potential p_j. Vertex v has a vertices with highest remaining outdegree at position j as inneighbors because the underlying DAG is well-connected. Thus, for $\ell = \Delta$ the difference on the left-hand side in Equation (3) is exactly a. By definition, $p_j[\ell + 1]$ denotes the number of vertices that have at position j a remaining outdegree of at least $\ell + 1$. If $a \leq p_j[\ell + 1]$, then clearly the vertices with remaining outdegree of at most ℓ at position j are not adjacent to v and, therefore, the difference on the left-hand side is zero. Conversely, if $a > p_j[\ell+1]$, then $a - p_j[\ell+1]$ of the vertices with remaining outdegree at most ℓ at position j are adjacent to v. Thus, the difference on the left-hand side in Equation (3) is in this case exactly $a - p_j[\ell + 1]$. $\square$

Lemma 4.8 shows that in a topological ordering of a well-connected DAG, the potential at position $i + 1$ is uniquely determined by the potential at position i and the in- and outdegree of the $(i + 1)$th vertex. This allows us to define the potential p_i^σ of σ at position i independent from the DAG and its topological ordering.

Definition 4.9. Let $\sigma[1, n]$ be a realizable degree ordering. For all $0 \leq i \leq n$, the potential p_i^σ of σ at position i is the potential at position i in a corresponding topological ordering of a well-connected DAG realizing σ.

Reordering Realizable Degree Orderings Given a realizable degree ordering, cutting out subsequences and reinserting them appropriately is the main operation that we perform to reorder the degree sequence such that we can exploit the resulting regular structure in our algorithm. Basically, if a certain potential p occurs twice in a realizable degree ordering, then removing the subsequence between them results also in a realizable degree ordering. Furthermore, this subsequence can be reinserted at any position where the potential p occurs. In the following we give a formal description of this operation. To this end, we link subsequences to the two potentials that appear at the cut-positions in the realizable degree ordering as these potentials obviously do not depend solely on the subsequences but on the whole realizable degree ordering. In this way, the following definition formalizes the potentials that may fit to a subsequence if the rest of the realizable degree ordering is chosen accordingly. For notational convenience, we write $\sigma_1\sigma_2$ for the concatenation of two ordered degree sequences σ_1 and σ_2.

Definition 4.10. An ordered degree sequence $\sigma[1, n]$ is a *partial realizable degree ordering* with *input potential* p_0^σ and *output potential* p_n^σ, if there are two ordered degree sequences $\sigma_1[1, n_1]$ and σ_2 such that $\sigma' = \sigma_1\sigma\sigma_2$ is a realizable degree sequence with potential p_0^σ at position n_1 and potential p_n^σ at position $n_1 + n$. The potential of σ at position $0 \leq i \leq n$ is defined to be $p_{n_1+i}^{\sigma'}$.

By Lemma 4.8 the potential of a partial realizable degree ordering with a certain input and output potential is well-defined.

Observe that a partial realizable degree ordering with input and output potential 0^Δ is also a realizable degree ordering. Furthermore, for a realizable degree ordering $\sigma[1, n]$ it holds that $\sigma[i, j]$, for all $1 \leq i \leq j \leq n$, is a partial realizable degree ordering with input potential p_{i-1}^σ and output potential p_j^σ.

In the remainder of this subsection we show that we can concatenate two partial realizable degree orderings σ_1 and σ_2 to $\sigma_1\sigma_2$ when the output potential of σ_1 is "better" than the input potential of σ_2. To formalize what it means to be better we introduce a partial order for potentials.

Definition 4.11. For $p, p' \in \mathbb{N}^\Delta$, $p \succeq p'$ if $\forall 1 \leq j \leq \Delta\colon \sum_{i=1}^{j} p[i] \geq \sum_{i=1}^{j} p'[i]$.

Intuitively, a bad potential value represents "few" vertices with "high" outdegree. These vertices can only be connected to many vertices with low indegree. On the contrary, a good potential represents many vertices with low outdegree. These vertices can be connected to many vertices with low indegree or to few vertices with high indegree. So a good potential at some position indicates a high freedom for connecting the following vertices. Indeed, a potential p that is better than a potential p' guarantees that all subsequent vertices that can be connected with potential p' can also be connected with potential p, as shown in the next proposition.

Proposition 4.12. Let $\sigma_1[1, n_1]$ be a partial realizable degree ordering with input potential $p_0^{\sigma_1}$ and output potential $p_{n_1}^{\sigma_1} \in \mathbb{N}^\Delta$ and let $\sigma_2[1, n_2]$ be a partial realizable degree ordering with input potential $p_0^{\sigma_2}$ and output potential $p_{n_2}^{\sigma_2}$ such that $p_{n_1}^{\sigma_1} \succeq p_0^{\sigma_2}$. Then, $\sigma = \sigma_1\sigma_2$ is a partial realizable degree ordering with input potential $p_0^{\sigma} = p_0^{\sigma_1}$ and output potential $p_{n_1+n_2}^{\sigma} \succeq p_{n_2}^{\sigma_2}$ such that $p_i^{\sigma} = p_i^{\sigma_1}$ for all $1 \leq i \leq n_1$ and $p_i^{\sigma} \succeq p_{i-n_1}^{\sigma_2}$ for all $n_1 \leq i \leq n_2 + n_1$. If additionally $\omega(p_{n_1}^{\sigma_1}) = \omega(p_0^{\sigma_2})$ and $p_0^{\sigma_1} = p_{n_2}^{\sigma_2} = 0^\Delta$, then σ is also a realizable degree ordering.

The proof of Proposition 4.12 is based on the following lemma dealing with the case $n_2 = 1$.

Lemma 4.13. Let $\sigma_1[1, n_1]$ be a partial realizable degree ordering with input potential $p_0^{\sigma_1}$ and output potential $p_{n_1}^{\sigma_1} \in \mathbb{N}^\Delta$ and let $\sigma_2 = \binom{a}{b}$, $a, b \in \mathbb{N}$, be a partial realizable degree ordering with input potential $p_0^{\sigma_2}$ and output potential $p_1^{\sigma_2}$ such that $p_{n_1}^{\sigma_1} \succeq p_0^{\sigma_2}$. Then, $\sigma = \sigma_1\sigma_2$ is a partial realizable degree ordering with input potential $p_0^{\sigma_1}$ and output potential $p_{n_1+1}^{\sigma} \succeq p_1^{\sigma_2}$ such that $p_i^{\sigma} = p_i^{\sigma_1}$ for all $1 \leq i \leq n_1$.

Proof. Let $\widetilde{\sigma}_1[1, \widetilde{n}_1]$ be a realizable degree ordering corresponding to σ_1, that is, there are integers $1 \leq i \leq j \leq \widetilde{n}_1$ such that $\sigma_1 = \widetilde{\sigma}_1[i, j]$, $p_0^{\sigma_1} = p_{i-1}^{\widetilde{\sigma}_1}$, and $p_{n_1}^{\sigma_1} = p_j^{\widetilde{\sigma}_1}$. Note that, by Definition 4.10, $\widetilde{\sigma}_1$, i, and j exist. Furthermore, let $\hat{\sigma}$ be a multiset containing $\omega(p_{n_1}^{\sigma_1}) - a + b$ times the sink $\binom{1}{0}$. In order to prove the lemma, we will show that $\widetilde{\sigma} = \widetilde{\sigma}_1[1, j]\sigma_2\hat{\sigma}$ is a realizable degree ordering such that $p_{i-1}^{\widetilde{\sigma}} = p_0^{\sigma_1}$ and $p_{j+1}^{\widetilde{\sigma}} \succeq p_1^{\sigma_2}$.

First, we show that $\widetilde{\sigma} = \widetilde{\sigma}_1[1, j]\sigma_2\hat{\sigma}$ is indeed a realizable degree ordering: Let D be a well-connected realizing DAG for $\widetilde{\sigma}_1[1, \widetilde{n}_1]$. We next describe how to transform D into a realizing DAG for $\widetilde{\sigma}$. Remove all vertices in D that correspond to tuples in $\widetilde{\sigma}[j+1, \widetilde{n}_1]$. Next, add to D the vertex v corresponding to $\sigma_2 = \binom{a}{b}$ and well-connect v. This is possible since $a \leq p_0^{\sigma_2}[1] \leq p_{n_1}^{\sigma_1}[1]$. Finally, repeatedly add sinks corresponding to the tuples in $\hat{\sigma}$ that all are of type $\binom{1}{0}$. Note that, since $|\hat{\sigma}| = \omega(p_{n_1}^{\sigma_1}) - a + b$ this leads to a well-connected DAG for $\widetilde{\sigma}$. Furthermore, since we did not change the arcs between the vertices corresponding to $\widetilde{\sigma}[1, j]$, it follows that $p_t^{\widetilde{\sigma}} = p_{t-i+1}^{\sigma_1}$ for all $i - 1 \leq t \leq j$. Hence, it remains to show that $p_{j+1}^{\widetilde{\sigma}} \succeq p_1^{\sigma_2}$.

We now complete the proof by showing that $p_{j+1}^{\widetilde{\sigma}} \succeq p_1^{\sigma_2}$. To this end, we first assume that $b = 0$ and deal afterwards with the case $b > 0$. For every $\ell \in \{1, \ldots, \Delta\}$ by Equation (2) from Lemma 4.8 we have

$$\sum_{i=1}^{\ell} p_{j+1}^{\widetilde{\sigma}}[i] \overset{(2)}{=} \left(\sum_{i=1}^{\ell} p_j^{\widetilde{\sigma}}[i]\right) - \max\{0, a - p_j^{\widetilde{\sigma}}[\ell+1]\}. \tag{4}$$

If $p_j^{\widetilde{\sigma}}[\ell+1] \geq a$, then we obtain

$$\sum_{i=1}^{\ell} p_{j+1}^{\widetilde{\sigma}}[i] \overset{(4)}{=} \sum_{i=1}^{\ell} p_j^{\widetilde{\sigma}}[i] \geq \sum_{i=1}^{\ell} p_0^{\sigma_2}[i] \geq \sum_{i=1}^{\ell} p_1^{\sigma_2}[i], \tag{5}$$

as by assumption we have $p_j^{\widetilde{\sigma}} \succeq p_0^{\sigma_2}$ and $p_0^{\sigma_2} \succeq p_1^{\sigma_2}$ since we have $b = 0$. In the remaining case of $p_j^{\widetilde{\sigma}}[\ell+1] < a$, we obtain

$$\begin{aligned}\sum_{i=1}^{\ell} p_{j+1}^{\widetilde{\sigma}}[i] &\overset{(4)}{=} \left(\sum_{i=1}^{\ell} p_j^{\widetilde{\sigma}}[i]\right) - (a - p_j^{\widetilde{\sigma}}[\ell+1]) = \left(\sum_{i=1}^{\ell+1} p_j^{\widetilde{\sigma}}[i]\right) - a \geq \left(\sum_{i=1}^{\ell+1} p_0^{\sigma_2}[i]\right) - a \\ &= \left(\sum_{i=1}^{\ell} p_0^{\sigma_2}[i]\right) - (a - p_0^{\sigma_2}[\ell+1]) \geq \left(\sum_{i=1}^{\ell} p_0^{\sigma_2}[i]\right) - \max\{0, a - p_0^{\sigma_2}[\ell+1]\} \\ &\overset{(2)}{=} \sum_{i=1}^{\ell} p_1^{\sigma_2}[i]. \end{aligned} \tag{6}$$

Since Inequality (5) and Inequality (6) hold for every $\ell \in \{1, \ldots, \Delta\}$, it follows that $p_{j+1}^{\widetilde{\sigma}} \succeq p_1^{\sigma_2}$.

It remains to consider the case $b > 0$. To this end, we denote by $c \in \mathbb{N}^{\Delta}$ the vector having ones in the first b entries and zeros in the remaining entries. Intuitively, c is the "emission" the vertex v realizing $\binom{a}{b}$ adds to both potentials $p_{j+1}^{\widetilde{\sigma}}$ and $p_1^{\sigma_2}$. As this emission is the same for both potentials, we can subtract c from $p_{j+1}^{\widetilde{\sigma}}$ and from $p_1^{\sigma_2}$ to obtain the potentials only containing the "absorption" of v. It is thus sufficient to show that $p_{j+1}^{\widetilde{\sigma}} - c \succeq p_1^{\sigma_2} - c$. In this way, the case $b > 0$ reduces to the case $b = 0$. □

Now, we prove Proposition 4.12 by repeatedly invoking Lemma 4.13.

Proof. (of Proposition 4.12) We first prove, by induction on j, that for each $0 \leq j \leq n_2$ the sequence $\sigma[1, n_1 + j] = \sigma_1[1, n_1]\sigma_2[1, j]$ ($\sigma[1, n_1 + j] = \sigma[1, n_1]$ for $j = 0$) is a partial realizable degree ordering with input potential $p_0^{\sigma_1}$ and output potential $p_{n_1+j}^{\sigma} \succeq p_j^{\sigma_2}$ such that $p_i^{\sigma} = p_i^{\sigma_1}$ for all $1 \leq i \leq n_1$ and $p_i^{\sigma} \succeq p_{i-n_1}^{\sigma_2}$ for all $n_1 \leq i \leq n_2 + n_1$.

For the base case $\sigma[1, n_1 + 0] = \sigma[1, n_1]$, the statement is fulfilled due to the assumptions of the proposition. For the induction step, assume that the statement is true for some $j \in \{0, \ldots, n_2 - 1\}$. Then, by Lemma 4.13, $\sigma[1, n_1 + j + 1] = \sigma[1, n_1 + j]\sigma_2[j + 1, j + 1]$ is a partial realizable degree ordering with input potential p_0^{σ} and output potential $p_{n_1+j+1}^{\sigma} \succeq p_{j+1}^{\sigma_2}$ such that $p_i^{\sigma[1,n_1+j+1]} = p_i^{\sigma[1,n_1+j]}$ for all $1 \leq i \leq n_1 + j$. Hence, by induction hypothesis, it holds that $p_i^{\sigma[1,n_1+j+1]} = p_i^{\sigma_1}$ for all $1 \leq i \leq n_1$ and $p_i^{\sigma[1,n_1+j+1]} \succeq p_{i-n_1}^{\sigma_2}$ for all $n_1 \leq i \leq n_1 + j + 1$. This proves the first statement of the proposition.

For the second statement of the proposition observe that if $\omega(p_{n_1}^{\sigma_1}) = \omega(p_0^{\sigma_2})$, then $\omega(p_{n_1+n_2}^{\sigma}) = \omega(p_{n_2}^{\sigma_2})$. Thus, if $p_0^{\sigma_1} = p_{n_2}^{\sigma_2} = 0^{\Delta}$, then σ is a partial realizable degree ordering with input and output potential 0^{Δ}. Hence, σ is in this case a realizable degree ordering. □

Proposition 4.12 shows that we can "merge" two partial realizable degree orderings σ_1 and σ_2 to $\sigma_1\sigma_2$, if for the output potential $p_{n_1}^{\sigma_1}$ of σ_1 and the input potential $p_0^{\sigma_2}$ it holds that $p_{n_1}^{\sigma_1} \succeq p_0^{\sigma_2}$ and $\omega(p_{n_1}^{\sigma_1}) = \omega(p_0^{\sigma_2})$. This provides the basis for cutting out a subsequence in a realizable degree ordering and reinserting it at another position. First, consider the cut out of subsequences.

Lemma 4.14. Let $\sigma[1,n]$ be a realizable degree ordering. If there are two indices $1 \le i < j \le n$ such that $p_i^\sigma = p_j^\sigma$, then $\sigma' = \sigma[1,i]\sigma[j+1,n]$ is a realizable degree ordering with $p_{i+\ell}^{\sigma'} = p_{j+\ell}^{\sigma}$ for all $0 \le \ell \le n-j$.

Proof. Let $\sigma[1,n]$ be a realizable degree ordering and let $1 \le i < j \le n$ be two indices such that $p_i^\sigma = p_j^\sigma$. First, by definition, $\sigma[1,i]$ is a partial realizable degree ordering with input potential 0^Δ and output potential p_i^σ. Furthermore, $\sigma[j+1,n]$ is a partial realizable degree ordering with input potential p_j^σ and output potential 0^Δ. Hence, since $\omega(p_i^\sigma) = \omega(p_j^\sigma)$ and $p_i^\sigma \succeq p_j^\sigma$, it follows from Proposition 4.12 that $\sigma' = \sigma[1,i]\sigma[j+1,n]$ is a realizable degree ordering.

It remains to show that $p_{i+\ell}^{\sigma'} = p_{j+\ell}^{\sigma}$ for all $0 \le \ell \le n-j$. We show this by induction on ℓ. Since $p_i^\sigma = p_j^\sigma$, the statement is true for the base case $\ell = 0$. For $\ell \ge 1$, the induction hypothesis states that $p_{i+\ell-1}^{\sigma'} = p_{j+\ell-1}^{\sigma}$. From Lemma 4.8 follows that $p_{i+\ell}^{\sigma'} = p_{j+\ell}^{\sigma}$, which proves the induction step. □

Lemma 4.14 shows that from a realizable degree ordering σ we can cut out a subsequence $\sigma[i+1,j]$ whenever $p_i^\sigma = p_j^\sigma$. The next observation shows that we can reinsert this subsequence in the remaining realizable degree ordering σ' at any position ℓ with $p_\ell^{\sigma'} = p_i^\sigma = p_j^\sigma$.

Lemma 4.15. Let $\sigma_1[1,n_1]$ be a realizable degree ordering. Furthermore, let $\sigma_2[1,n_2]$ be a partial realizable degree ordering with input and output potential p. Then, for all indices $1 \le i \le n_1$ where $p_i^{\sigma_1} = p$, the ordering $\sigma = \sigma_1[1,i]\sigma_2[1,n_2]\sigma[i+1,n_1]$ is a realizable degree ordering with $p_j^{\sigma_1} = p_{j+n_2}^{\sigma}$ for all $i < j \le n_1$.

Proof. Let $\sigma_1[1,n_1]$ be a realizable degree ordering and let $\sigma_2[1,n_2]$ be a partial realizable degree ordering with input and output potential $p \in \mathbb{N}^\Delta$. Furthermore, let i be a position in σ_1 such that $p_i^{\sigma_1} = p$. Then, by Proposition 4.12, $\sigma_1[1,i]\sigma_2[1,n_2]$ is a partial realizable degree ordering with input potential 0^Δ and output potential $p' \succeq p$. Since the input and output potential of σ_2 are equal, it follows that $\omega(p') = \omega(p_i^{\sigma_1}) = \omega(p)$. Hence, again applying Proposition 4.12, it follows that $\sigma = \sigma_1[1,i]\sigma_2[1,n_2]\sigma_1[i+1,n_1]$ is a realizable degree ordering with $p_j^{\sigma_1} \succeq p_{j+n_2}^{\sigma}$ for all $i < j \le n_1$.

As in the proof of Lemma 4.14, one can show by induction on j that $p_j^{\sigma_2} = p_{j+i}^{\sigma}$ for all $0 \le j \le n_2$. Thus, $p_{i+n_2}^{\sigma} = p$. Then, the statement $p_j^{\sigma_1} = p_{j+n_2}^{\sigma}$ for all $i < j \le n_1$ follows from a similar induction over j which invokes Lemma 4.8 in the induction step. □

Given a realizable degree ordering σ where at the three positions i, j, k the same potential occurs, we can now cut out (Lemma 4.14) the part between positions i and j and then insert (Lemma 4.15) it at position k. The following proposition formalizes this "reordering operation".

Proposition 4.16. Let $\sigma[1,n]$ be a realizable degree ordering and let $1 \le i < j < k \le n$ be three positions with $p_i = p_j = p_k$. Then $\sigma[1,i]\sigma[j+1,k]\sigma[i+1,j]\sigma[k+1,n]$ is a realizable degree ordering.

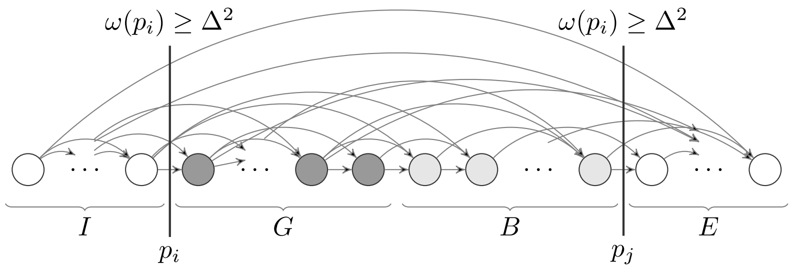

Figure 4: A schematic illustration of a realizing high-potential DAG that corresponds to the pattern $I\,G\,B\,E$. Thereby, I is a subsequence of length at most $\Delta^{2\Delta}$ such that the first high potential occurs at position i. Correspondingly, j is the last position with high potential and E is a sequence of length at most $\Delta^{2\Delta}$. The sequence G (resp. B) consists of only good (bad) type vertices but is of arbitrary length. All high-potential realizations can be reordered to fit into this pattern.

4.2 High-Potential Sequences

In this subsection we show that if a realizable degree sequence admits a realizable degree ordering where at some position the value of the potential is at least Δ^2, a so-called *high-potential realizable degree ordering*, then there is also a realizable degree ordering σ that is of the following "pattern" (see Figure 4 for an illustration): The ordering σ can be partitioned into four subsequences I, G, B, E. It starts with sequence I "establishing" a potential of value at least Δ^2, a so-called *high potential*. Correspondingly, at the end there is a sequence E that reduces the value of the potential from a value that is at least Δ^2 to zero. Furthermore, I and E are of length at most $\Delta^{2\Delta}$. The subsequence G, which is of arbitrary length, only consists of good type tuples (indegree at most outdegree) in arbitrary order and, correspondingly, B is of arbitrary length but only consists of bad type tuples (indegree larger than outdegree) in arbitrary order.

This characterization allows us to check whether there is a high-potential realizable degree ordering as follows: First, branch into all possibilities to choose I and E. Second, insert in each branch the remaining vertices sorted by good and bad types between I and E and, third, check whether this ordering is a realizable degree ordering. There are at most $((\Delta+1)^2)^{2\Delta^{2\Delta}}) = \Delta^{\Delta^{O(\Delta)}}$ possibilities for choosing I and E. Furthermore, the insertion and checking can be done in polynomial time, see Lemma 4.7. Hence, this branching algorithm yields fixed-parameter tractability with respect to Δ for the high-potential case.

Our strategy to prove that there is indeed a high-potential realizable degree ordering with the pattern $I\,G\,B\,E$ is as follows. Let $\sigma[1, n]$ be an arbitrary high-potential realizable degree ordering and let $1 \leq i \leq n$ be the first position with a high potential and, symmetrically, let j be the last position with a high potential. In the first part of this subsection (see Proposition 4.21), we show that σ can be restructured such that $i \leq \Delta^{2\Delta}$ and $j \geq n - \Delta^{2\Delta}$. To prove this the main argument is that if $i > \Delta^{2\Delta}$, then, since there are at most $\Delta^{2\Delta}$ different potentials with value less than Δ^2, there have to be two positions $1 \leq \ell_1 < \ell_2 < i$

with $p_{\ell_1} = p_{\ell_2}$. Then, by Lemma 4.14, we can cut out $\sigma[\ell_1+1, \ell_2]$ from σ and we will show (see Lemma 4.20) that we can reinsert it right behind i, resulting in a realizable degree ordering $\sigma[1, \ell_1]\sigma[\ell_2+1, i]\sigma[\ell_1+1, \ell_2]\sigma[i+1, n]$. By iteratively applying this operation, we end up with a realizable degree ordering where the first position with high potential is at most $\Delta^{2\Delta}$. A symmetric argument holds for the last position j with high potential.

In the second part we show that we can arbitrarily sort the vertices in $\sigma[i+1, j]$ under the constraint that at first vertices of good type occur in any order, and then they are followed by the bad type vertices (see Proposition 4.22). The basic idea herein is that if the value of a potential at some position ℓ is at least Δ^2, then there are at least Δ vertices with remaining outdegree at least one at position ℓ. Hence, one can always connect the next vertex to the preceding vertices. Here, the sorting such that in $\sigma[i+1, j]$ first the good type vertices occur ensures that at each position $\ell \in \{i+1, \ldots, j-1\}$ the value of the potential is at least Δ^2.

Bounding the Length of I and E With the next lemmas and observations we show that the subsequences I and E of the above pattern can be assumed to be of length at most $\Delta^{2\Delta}$. As already mentioned above, if $i > \Delta^{2\Delta}$, then there have to be two positions $1 \leq \ell_1 < \ell_2 < i$ such that $p^\sigma_{\ell_1} = p^\sigma_{\ell_2}$. Hence, by Lemma 4.14, $\sigma' = \sigma[1, \ell_1]\sigma[\ell_2+1, n]$ is a realizable degree ordering with $p^{\sigma'}_{\ell_1+\ell} \succeq p^\sigma_{\ell_2+\ell}$ for all $1 \leq \ell \leq n - \ell_2$. Next, we show that $\sigma[\ell_1+1, \ell_2]$ can be reinserted behind $\sigma[i, i]$, meaning that $\sigma[1, \ell_1]\sigma[\ell_2+1, i]\sigma[\ell_1+1, \ell_2]\sigma[i+1, n]$ is a realizable degree ordering with a high potential at position $i - (\ell_2 - \ell_1 + 1)$. However, observe that to prove this, Lemma 4.15 cannot be used since $\omega(p^{\sigma'}_{i-(\ell_2-\ell_1+1)}) \geq \Delta^2 > \omega(p^\sigma_{\ell_1})$. Thus, in the following we prove that we can reinsert the cut out subsequence in the high-potential part (Lemma 4.20). Before that, we formalize the observation that among potentials with the same value there is one that is minimum concerning the ordering introduced in Definition 4.11.

Lemma 4.17. For a fixed positive integer x let $p(x) \in \mathbb{N}^\Delta$ be the potential with

$$p(x)[j] := \begin{cases} \left\lceil \frac{x}{\Delta} \right\rceil, & \text{if } j \leq x \text{ modulo } \Delta \\ \left\lfloor \frac{x}{\Delta} \right\rfloor, & \text{otherwise} \end{cases}$$

for all $1 \leq j \leq \Delta$. Then, for all potentials $p' \in \mathbb{N}^\Delta$ with $x = \omega(p') = \omega(p(x))$ it holds that $p' \succeq p(x)$.

Proof. Let $p(x) \in \mathbb{N}^\Delta$ be the potential as defined in Lemma 4.17 and let $p' \in \mathbb{N}^\Delta$ be a potential with $\omega(p') = \omega(p(x))$. Clearly, by definition it holds that $\omega(p(x)) = x$. Towards a contradiction assume that $p' \succeq p(x)$ does *not* hold. Then, there is a position $1 \leq j \leq \Delta$ with $\sum_{\ell=1}^{j} p(x)[\ell] > \sum_{\ell=1}^{j} p'[\ell]$. From this it follows that there is a position $1 \leq t \leq j$ such that $p(x)[t] > p'[t]$ and since $p(x)[t] \leq \lceil x/\Delta \rceil$ it follows that $p'[t] \leq \lfloor x/\Delta \rfloor$. Recall that for any potential p it holds that $p[\ell_1] \geq p[\ell_2]$ for all $1 \leq \ell_1 \leq \ell_2 \leq \Delta$ (see remark after Definition 4.1). Thus, from $p'[t] \leq \lfloor x/\Delta \rfloor$ it follows that $\sum_{\ell=j+1}^{\Delta} p'[\ell] \leq (\Delta - j)\lfloor x/\Delta \rfloor \leq \sum_{\ell=j+1}^{\Delta} p(x)[\ell]$. Together with $\sum_{\ell=1}^{j} p(x)[\ell] > \sum_{\ell=1}^{j} p'[\ell]$ this yields a contradiction to $\omega(p(x)) = \omega(p')$. $\square$

The reason for considering the "worst" potential $p(x)$ is demonstrated in the next lemma. It shows that, if the input potential of any partial realizable degree ordering has value $x \geq \Delta^2$, then even the "worst" potential of value x, namely $p(x)$ as defined in Lemma 4.17, suffices as input potential.

Lemma 4.18. Let σ be a partial realizable degree ordering with input potential p_0^σ, $x = \omega(p_0^\sigma) \geq \Delta^2$, and output potential 0^Δ. Then, σ is a partial realizable degree ordering with input potential $p(x)$ and output potential 0^Δ.

Proof. Let $\sigma[1, n]$ be a partial realizable degree ordering with input potential p_0^σ, $x = \omega(p_0^\sigma) \geq \Delta^2$, and output potential 0^Δ. We first show that we may assume without loss of generality that σ consists only of sink-tuples, meaning that all tuples have outdegree zero: By Definition 4.10, there is a degree ordering $\tilde{\sigma}$ such that $\tilde{\sigma}\sigma$ is a realizable degree ordering with potential p_0^σ at position $|\tilde{\sigma}|$. Let D be a fixed realizing DAG corresponding to $\tilde{\sigma}\sigma$. Then delete from D each arc between two vertices corresponding to tuples from σ. We denote by $S(\sigma)$ be the degree ordering that is obtained from σ where for each deleted arc the in- and outdegree of the corresponding tuples is decreased by one. Clearly, in any topological ordering for D that corresponds to $\tilde{\sigma}\sigma$, the potential at position $|\tilde{\sigma}|$ does not change by removing the arcs between vertices corresponding to the tuples in σ. In the reverse direction, having shown the correctness of Lemma 4.18 for $S(\sigma)$, "reinserting" these arcs does not change the input potential of $S(\sigma)$. Hence, in the following we may assume that σ only consists of sink-tuples.

Setting $p(x)[\Delta + 1] = 0$, let σ' be the ordered degree sequence consisting of $p(x)[\ell] - p(x)[\ell + 1]$ many (source-)tuples $\binom{0}{\ell}$, for all $1 \leq \ell \leq \Delta$, that are arbitrarily ordered. Observe that σ' is a partial realizable degree ordering with input potential 0^Δ and output potential $p(x)$. To prove Theorem 4.18 we show that $\sigma'\sigma$ is a realizable degree ordering. To this end, by induction on i we show for all $0 \leq i \leq |\sigma|$

(i) that $\sigma'\sigma[1, i]$ (σ' for $i = 0$) is a partial realizable degree ordering with input potential 0^Δ and output potential p_i such that $\omega(p_i) = \omega(p_i^\sigma)$, and

(ii) that the potential is used in a "balanced way", that is, the following holds:

$$\forall \ell \in \{1, \ldots, \Delta - 1\} \colon p_i[\ell] < p(x)[\ell] \Rightarrow p_i[\ell + 1] = 0. \tag{7}$$

The induction base $i = 0$ follows from the assumption that $\omega(p_0^\sigma) = x$ and that the output potential of σ' is $p(x)$.

For the induction step, assume that $\sigma'\sigma[1, i]$ is a partial realizable degree ordering with input potential 0^Δ and output potential p_i with $\omega(p_i) = \omega(p_i^\sigma)$. Furthermore, assume that p_i satisfies Property (7). Observe that Property (7) implies Part (i), that is, $\sigma'\sigma[1, i + 1]$ is partial realizable degree ordering with input potential 0^Δ and output potential p_{i+1} such that $\omega(p_{i+1}) = \omega(p_{i+1}^\sigma)$: By Lemma 4.13, it suffices to argue that $p_i[1] \geq a_{i+1}$, for $\sigma[i + 1, i + 1] = \binom{a_{i+1}}{0}$. Since $p(x)[1] \geq \Delta \geq a_{i+1}$, we only need to consider the case where $p_i[1] < p(x)[1]$. In this case, from Property (7) it follows that $p_i[2] = 0$ and thus $\omega(p_i) = p_i[1]$. Since $\omega(p_i) = \omega(p_i^\sigma)$, this implies that $p_i[1] \geq p_i^\sigma[1]$ and thus $p_i[1] \geq p_i^\sigma[1] \geq a_{i+1}$, implying the claim for $\sigma'\sigma[1, i + 1]$.

It remains to prove the correctness of Property (7) for $i + 1$. The intuition behind the proof is as follows: Recall that by Definitions 4.9 and 4.10 potentials in a partial realizable degree ordering are defined to correspond to the potentials in a well-connected DAG. Furthermore, we assumed that we start with a high potential with value at least Δ^2. Hence, in each step there will be enough vertices with high remaining outdegree such that no vertex with low remaining outdegree is used when well-connecting some vertex v. Next, we formalize this.

Similar to the base case, let $\binom{a_{i+1}}{0} = \sigma[i+1, i+1]$. Thus, for all $\ell \in \{1, \ldots, \Delta\}$, Equation (2) (see Lemma 4.8) implies that

$$\left(\sum_{j=1}^{\ell} p_i[j]\right) - \max\{0, a_{i+1} - p_i[\ell+1]\} = \sum_{j=1}^{\ell} p_{i+1}[j]. \tag{8}$$

For all indices $1 \le \ell < \Delta$ with $p_i[\ell+1] = p(x)[\ell+1]$, Equation (8) and $p(x)[\ell+1] \ge \Delta \ge a_{i+1}$ immediately imply Property (7). It remains to consider an index $1 \le \ell < \Delta$ with $p_i[\ell+1] < p(x)[\ell+1]$. By induction hypothesis, it follows that $p_i[\ell+2] = 0$ (potentially, $\ell+2 = \Delta+1$). If $p_i[\ell+1] \ge a_{i+1}$, then Equation (8) implies that $p_{i+1}[\ell] = p_i[\ell]$. Otherwise, in case $p_i[\ell+1] < a_{i+1}$, Equation (8) implies $p_{i+1}[\ell+1] = 0$ (since $p_i[\ell+2] = 0$)). This completes the proof of Part (ii). ☐

Combining Proposition 4.12 with Lemmas 4.17 and 4.18 yields the following.

Corollary 4.19. Let $\sigma_1[1, n_1]$ be a partial realizable degree ordering with input potential 0^Δ and output potential $p_{n_1}^{\sigma_1}$. Furthermore, let σ_2 be a partial realizable degree ordering with input potential $p_0^{\sigma_2}$ and output potential 0^Δ such that $\omega(p_{n_1}^{\sigma_1}) = \omega(p_0^{\sigma_2}) \ge \Delta^2$. Then, $\sigma = \sigma_1\sigma_2$ is a realizable degree ordering.

Lemma 4.14 shows that we can cut out a partial realizable degree ordering with equal input and output potential. Based on Corollary 4.19, the following lemma shows that we can reinsert it right behind a high potential in any realizable degree ordering.

Lemma 4.20. Let $\sigma_1[1, n_1]$ be a realizable degree ordering and let $\sigma_2[1, n_2]$ be a partial realizable degree ordering with input and output potential p. Then, for any position $1 \le i \le n_1$ with $\omega(p_i^{\sigma_1}) \ge \max\{\Delta^2, \omega(p)\}$ it holds that $\sigma = \sigma_1[1, i]\sigma_2[1, n_2]\sigma[i+1, n_1]$ is a realizable degree ordering.

Proof. Let $\sigma_1[1, n_1]$ be a realizable degree ordering and let $\sigma_2[1, n_2]$ be a partial realizable degree ordering with input and output potential p. Furthermore, let $1 \le i \le n_1$ be a position with $\omega(p_i^{\sigma_1}) \ge \max\{\Delta^2, \omega(p)\}$. We prove that $\sigma = \sigma_1[1, i]\sigma_2[1, n_2]\sigma_1[i+1, n_1]$ is a realizable degree ordering. To this end, we show that $\sigma_1[1, i]\sigma_2[1, n_2]$ is a partial realizable degree ordering with input potential 0^Δ and output potential p' where $\omega(p') = \omega(p_i^{\sigma_1})$. Then, from Corollary 4.19 it follows that σ is a realizable degree ordering.

By Definition 4.10 there exists a realizable degree ordering $\widetilde{\sigma}_2[1, \widetilde{n}_2]$ such that $\sigma_2 = \widetilde{\sigma}_2[\ell, j]$ for some $1 \le \ell < j \le \widetilde{n}_2$ and $p = p_{\ell-1}^{\widetilde{\sigma}_2} = p_j^{\widetilde{\sigma}_2}$. Now, in case of $\omega(p_i^{\sigma_1}) > \omega(p)$ we add $\omega(p_i^{\sigma_1}) - \omega(p)$ tuples of type $\binom{1}{0}$ at the beginning of $\widetilde{\sigma}_2$ and the same number of tuples of type $\binom{0}{1}$ at the end of $\widetilde{\sigma}_2$. This shows that σ_2 is a partial realizable degree ordering with input potential p' and output potential p' with $\omega(p') = \omega(p_i^{\sigma_1}) \ge \Delta^2$. Hence, by Corollary 4.19, $\sigma_2[1, n_2]\widetilde{\sigma}_2[j+1, \widetilde{n}_2]$ is a partial realizable degree ordering with input potential p' and output potential 0^Δ.

From this together with Corollary 4.19 it follows that $\sigma_1[1, i]\sigma_2[1, n_2]\widetilde{\sigma}_2[j+1, \widetilde{n}_2]$ is a realizable degree ordering. Since, by our assumption σ_2 is a partial realizable degree ordering with input and output potential p', it follows that

$\sum_{v\in\sigma_2} d^-(v) = \sum_{v\in\sigma_2} d^+(v)$. Hence, from Observation 4.2 and the fact that the potential at position i in $\sigma_1[1,i]\sigma_2[1,n_2]\widetilde{\sigma}_2[j+1,\widetilde{n}_2]$ is $p_i^{\sigma_1}$, it follows that $\sigma_1[1,i]\sigma_2[1,n_2]$ is a partial realizable degree ordering with input potential 0^Δ and output potential p'' with $\omega(p'') = \omega(p_i^{\sigma_1}) \geq \Delta^2$. Thus, by Corollary 4.19 it follows that σ is a realizable degree ordering. □

With Lemma 4.20 we are able to bound the length of the parts I and E, that is, the first position of a high potential is "near" the start and the last position of a high potential is "near" the end.

Proposition 4.21. If a DAG Realization instance consisting of n tuples admits a high-potential realization, then there is also a corresponding high-potential realizable degree ordering such that the first position with high potential is at most $\Delta^{2\Delta}$ and the last position with high potential is at least $n - \Delta^{2\Delta}$.

Proof. Let $\sigma[1,n]$ be a high-potential realizable degree ordering and let $1 \leq i \leq n$ be the first position where $\omega(p_i) \geq \Delta^2$. Consider the case where $i > \Delta^{2\Delta}$. Thus, for all $1 \leq \ell < i$ it holds that $\omega(p_\ell) < \Delta^2$. There are $\Delta^{2\Delta}$ integer Δ-tuples with elements between 0 and $\Delta^2 - 1$ and not every Δ-tuple is a potential. Hence, there are less than $\Delta^{2\Delta}$ potentials with value less than Δ^2. Thus, there are two indices $1 \leq \ell_1 < \ell_2 < i$ with $p_{\ell_1} = p_{\ell_2}$. By Lemma 4.14, the ordering $\sigma[1,\ell_1]\sigma[\ell_2+1,n]$ is a realizable degree ordering where the value of the potential at position $i-(\ell_2-\ell_1)$ is $\omega(p_i)$. Moreover, by definition $\sigma[\ell_1+1,\ell_2]$ is a partial realizable degree ordering with input and output potential p_{ℓ_1} where $\omega(p_{\ell_1}) < \Delta^2 \leq \omega(p_i)$. Thus, by Lemma 4.20, it holds that $\sigma[1,\ell_1]\sigma[\ell_2+1,i]\sigma[\ell_1+1,\ell_2]\sigma[i+1,n]$ is a realizable degree ordering. Moreover, since $\sum_{v\in\sigma\{\ell_1+1,\ell_2\}} d^-(v) - d^+(v) = 0$ it follows from Observation 4.2 that in this realizable degree ordering the first position with high potential is $i-(\ell_2-\ell_1)$. Applying the same operation iteratively as long as there are two positions with equal potential before the first high potential results in a realizable degree ordering where the first position with high potential is at most $\Delta^{2\Delta}$.

Basically, the same argumentation can be applied for the last position j where a high potential occurs. In case of $j < n - \Delta^{2\Delta}$, there have to be two indices $j < \ell_1 < \ell_2 \leq n$ where $p_{\ell_1} = p_{\ell_2}$. Then, by Lemma 4.14 the ordering $\sigma[1,\ell_1]\sigma[\ell_2+1,n]$ is a realizable degree ordering and $\sigma[\ell_1+1,\ell_2]$ is a partial realizable degree ordering with input and output potential p_{ℓ_1} with $\omega(\ell_1) < \omega(p_j)$. Thus, by Lemma 4.20 the ordering $\sigma[1,j]\sigma[\ell_1+1,\ell_2]\sigma[j+1,\ell_1]\sigma[\ell_2+1,n]$ is a realizable degree ordering. Since $\sum_{v\in\sigma\{\ell_1+1,\ell_2\}} d^-(v) - d^+(v) = 0$ it follows from Observation 4.2 that the last position with high potential is $j+(\ell_2-\ell_1)$. Again, by applying this operation iteratively we get an ordering where the last position with high potential is at least $n - \Delta^{2\Delta}$. □

Sorting the Remaining Vertices Having shown that we can assume that for the first position i and the last position j with high potential it holds that $i \leq \Delta^{2\Delta}$ and $j \geq n - \Delta^{2\Delta}$, we next prove that one can sort all vertices between i and j arbitrarily by good (indegree at most outdegree) and bad (indegree larger than outdegree) types.

Proposition 4.22. Let $\sigma[1,n]$ be a high-potential realizable degree ordering and let $1 \leq i < j \leq n$ be two arbitrary positions such that $\omega(p_i) \geq \Delta^2$ and $\omega(p_j) \geq \Delta^2$. Furthermore, let $\sigma'[i+1,j]$ be a permutation of the tuples in $\sigma[i+1,j]$ such that there is a position $0 \le \ell \le j-i$ with the property that the first ℓ tuples in $\sigma'[i+1,j]$ are of good type and all subsequent tuples are of bad type. Then, the ordering $\sigma[1,i]\sigma'[i+1,j]\sigma[j+1,n]$ is a realizable degree ordering.

Proof. Assume that there is a high-potential realizable degree ordering with two indices $1 \le i \le j \le n$ such that $\omega(p_i) \ge \Delta^2$ and $\omega(p_j) \ge \Delta^2$. We prove that $\sigma[1,i]\sigma'[i+1,j]\sigma[j+1,n]$ is a realizable degree ordering for any reordering $\sigma'[i+1,j]$ of $\sigma[i+1,j]$ where the first ℓ tuples are of good types and the remaining ones of bad types.

To this end, by induction on h with $0 \le h \le j-i$ we show that the sequence $\sigma[1,i]\sigma'[i+1,i+h]$ (the sequence $\sigma[1,i]$ if $h=0$) is a partial realizable degree ordering with input potential 0^Δ and output potential p_h with $\omega(p_h) \ge \Delta^2$. The base case $h=0$ is given by the assumptions of the lemma. By induction hypothesis the output potential p_{h-1} of $\sigma[1,i]\sigma'[i+1,i+h-1]$ is a high potential and hence $p_{h-1}[1] \ge \Delta$. Thus, by Lemma 4.13 $\sigma[1,i]\sigma'[i+1,i+h]$ is a partial realizable degree ordering. It remains to show that the value of the output potential p_h of $\sigma[1,i]\sigma'[i+1,i+h]$ is at least Δ^2. Towards a contradiction suppose that it is not. This implies

$$\sum_{\binom{a}{b} \in \sigma'\{i+1,i+h\}} a-b > \omega(p_i) - \Delta^2. \tag{9}$$

Clearly, $\sigma'[i+h,i+h]$ has to be a bad type, otherwise Inequality (9) cannot be true. However, it holds that

$$\omega(p_i) - \sum_{\binom{a}{b} \in \sigma\{i+1,j\}} a-b = \omega(p_j) \ge \Delta^2$$

and thus

$$\sum_{\binom{a}{b} \in \sigma\{i+1,j\}} a-b \le \omega(p_i) - \Delta^2. \tag{10}$$

Since $\sigma'[i+1,j]$ is sorted by good and bad types and $\sigma'[i+h,i+h]$ is of bad type, all tuples in $\sigma'[i+h,j]$ are bad type tuples with $a-b<0$. Thus, Inequality (10) yields a contradiction to Inequality (9). Hence, $\sigma[1,i]\sigma'[i+1,j]$ is a partial realizable degree ordering with input potential 0^Δ and output potential p_{j-i} with $\omega(p_{j-i}) \ge \Delta^2$. This completes the proof of the induction.

By Observation 4.2, it holds that $\omega(p_{j-i}) = \omega(p_j^\sigma)$. Thus, by Corollary 4.19, $\sigma[1,i]\sigma'[i+1,j]\sigma[j+1,n]$ is indeed a realizable degree ordering. ☐

Propositions 4.21 and 4.22 lead to the central result of this subsection:

Theorem 4.23. If a DAG Realization instance admits a high-potential realizable degree ordering, then it can be solved in $\Delta^{\Delta^{O(\Delta)}} \cdot n$ time.

Proof. If an instance of DAG Realization admits a high-potential realizable degree ordering, then by Proposition 4.21 there is also a high-potential realizable degree ordering in which the occurrence of the first high potential is at most at position i with $i \le \Delta^{2\Delta}$ and the last occurrence of a high potential is at

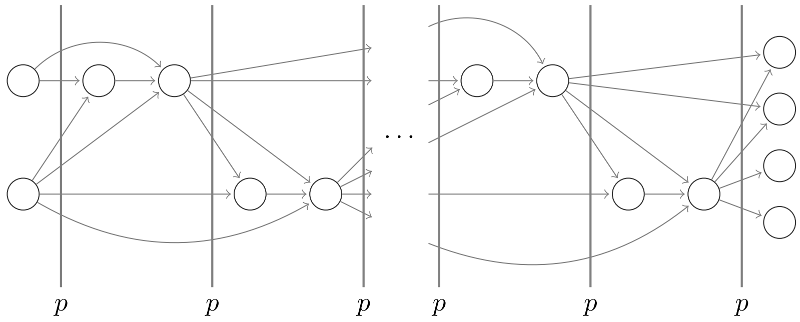

Figure 5: Realization for the degree sequence $\binom{0}{2}, \binom{0}{4}, \binom{2}{1}, \binom{3}{4}, \binom{2}{1}, \binom{3}{4}, \ldots, \binom{2}{1}, \binom{3}{4}, \binom{2}{1}, \binom{3}{4}, \binom{2}{0}, \binom{2}{0}, \binom{1}{0}, \binom{1}{0}$. Since this sequence basically consists of only two different types (not regarding types with indegree or outdegree equal to zero), it is easy to check that the pictured low-potential realization (the highest occurring value of a potential is $\omega(p) = 6$) is the only one. The displayed potentials are all identical and they furthermore indicate the repetitions of the *neutral block* $\binom{2}{1}, \binom{3}{4}$.

least at position j with $j \geq n - \Delta^{2\Delta}$. Recall that there are at most $(\Delta + 1)^2$ types of tuples in the given degree sequence, and thus the subsequences $\sigma[1, i]$ and $\sigma[j, n]$ of a realizable degree ordering $\sigma[1, n]$ can be found by exhaustive search in $\Delta^{\Delta^{O(\Delta)}}$ time. Proposition 4.22 shows that the remaining tuples can be arbitrarily inserted between them, as long as they are sorted by good and bad types. This can be done in $O(n)$ time. Finally, we check whether the produced ordering is indeed a realizable degree ordering, which can be done by Lemma 4.7 in $O(\Delta^2 n)$ time. Altogether, we arrive at a running time of $\Delta^{\Delta^{O(\Delta)}} \cdot n$. $\square$

4.3 Low-Potential Sequences

In this section, we will provide an algorithm that finds a *low-potential realization* (if one exists) for a DAG Realization instance, that is, a realization such that the value of all potentials is strictly less than Δ^2. See Figure 5 for an example of such a realization.

As in the high-potential case, the main idea is to restrict the length of the parts in a realizable degree ordering that have to be guessed by brute force. To this end, fix a realizable degree ordering σ_{sol} that our algorithm is supposed to find. In the low-potential case, we can exploit that there are at most $\Delta^{2\Delta}$ potentials with value less than Δ^2 and, thus, if the length of a realizable degree ordering is greater than $\Delta^{2\Delta}$, then there are two positions with equal potential. By Lemma 4.14, we can remove in σ_{sol} the part between two positions with the same potential and obtain another realizable degree ordering. We call such parts with equal potential at the beginning and end *neutral blocks*. By repeatedly cutting out neutral blocks in σ_{sol}, we end up with a realizable degree ordering σ_{short} of length at most $\Delta^{2\Delta}$. In the first step, our algorithm branches into all possibilities to choose σ_{short}. Since σ_{short} does not contain all tuples of the input sequence, the next step of the algorithm is to try to extend σ_{short}. Here, Lemma 4.15 allows us to reinsert a neutral block we cut out; we only need to find a position in σ_{short} with the correct potential. Thus, the remaining problem is "just" to find a combination of neutral blocks such that after inserting them the resulting sequence contains the very same tuples as our input degree sequence. This problem is solved in our algorithm by utilizing an ILP (integer linear program) formulation. For this second step, we need two conditions to hold: First, we need a bound on the number of different neutral blocks we can possibly reinsert. Second, in order to insert the removed neutral blocks, the requirement is that the corresponding potentials are contained in σ_{short}. To this end, we will remove in σ_{sol} only neutral blocks of length at most $\Delta^{2\Delta}$. Furthermore, we restrict the removal of neutral blocks such that all potentials that occur in σ_{sol} do also occur at least once in σ_{short}. These two requirements will increase the length bound for σ_{short}, but we can provide a bound that only depends on a function in Δ.

In the following we introduce some notation to formalize the above concepts.

Definition 4.24. A *neutral block* of potential $p \in \mathbb{N}^{\Delta}$ is a partial realizable degree ordering $\sigma[1, n]$ with input and output potential p. If $n \le \Delta^{2\Delta}$, then a neutral block $\sigma[1, n]$ is called *short.*

The *set of potentials* $\operatorname{pot}(\sigma')$ is defined to contain all potentials of a realizable degree ordering $\sigma' = \sigma'[1, n']$, that is,

$$\operatorname{pot}(\sigma') := \{p \in \mathbb{N}^{\Delta} \mid \exists i \in \{0, \dots, n'\} \colon p = p_i^{\sigma'}\}.$$

A realizable degree ordering $\sigma'[1, n']$ *contains* the neutral block σ of potential p if there exist two integers $1 \le i \le j \le n'$ such that $\sigma'[i, j] = \sigma$ and $p_{i-1}^{\sigma'} = p_j^{\sigma'} = p$. If σ is short and $\operatorname{pot}(\sigma') = \operatorname{pot}(\sigma'[1, i]\sigma'[j + 1, n'])$, then σ is called *removable* from σ'.

We now prove the claim that deleting all removable neutral blocks in a realizable degree ordering results in another realizable degree ordering of bounded length. To this end, our two core tools are the repetition removals and the reordering operation provided in Proposition 4.16.

Lemma 4.25. Let σ be a low-potential realizable degree ordering, that is, $\omega(p) < \Delta^2$ for all $p \in \operatorname{pot}(\sigma)$. Then there is a realizable degree ordering σ' of length at most $\Delta^{4\Delta}$ such that

(i) $\operatorname{pot}(\sigma) = \operatorname{pot}(\sigma')$ and

(ii) σ' can be obtained by repeatedly deleting removable neutral blocks in σ.

Proof. Let σ' be a shortest realizable degree ordering satisfying Properties (i) and (ii). Such a realizable degree ordering has to exists as $\sigma' = \sigma$ satisfies these properties. Assume towards a contradiction that σ' has length more than $\Delta^{4\Delta}$. Since there are less than $\Delta^{2\Delta}$ potentials of value less than Δ^2, it follows that there is a potential $p \in \operatorname{pot}(\sigma')$ that occurs more than $\Delta^{2\Delta}$ times in σ'. These occurrences of p define $\Delta^{2\Delta}$ (pairwise disjoint) neutral blocks of potential p that are contained in σ'. Since there are less than $\Delta^{2\Delta}$ potentials of value less than Δ^2, it follows that σ' contains a neutral block σ_1 of potential p such that all potentials occurring within σ_1 also occur outside of it. Denote by σ_2 a shortest neutral block for some $p' \in \operatorname{pot}(\sigma')$ that is contained within σ_1 (allowing $\sigma_1 = \sigma_2$). By the minimality of the length of σ_2, it follows that the potentials occurring within σ_2 are all different. Thus, σ_2 is short, that is, has length at most $\Delta^{2\Delta}$. By construction of σ_2 it follows that all potentials within σ_2 occur also outside σ_2. Thus, σ_2 is removable from σ'. Deleting σ_2 from σ' yields by Lemma 4.14 a shorter realizable degree ordering $\widetilde{\sigma}$. Since σ_2 is removable, it follows that $\widetilde{\sigma}$ satisfy Properties (i) and (ii); a contradiction to the assumption that σ' is the shortest realizable degree ordering satisfying Properties (i) and (ii). $\square$

Lemma 4.25 shows that deleting all removable neutral blocks yields indeed a short realizable degree ordering. Our algorithm branches into all possibilities to choose the short realizable degree ordering σ_{short} of length at most $\Delta^{4\Delta}$. This gives at most $(\Delta+1)^{2\Delta^{4\Delta}}$ cases. For each produced ordered degree sequence σ_{short} the algorithm checks whether σ_{short} is indeed a realizable degree ordering. By Lemma 4.7, this can be done in $O(\Delta n)$ time. In the cases that pass the test, the algorithm computes the short neutral blocks that can be (re-)inserted. To this end, we need some further notation. For a multiset S and an element $e \in S$, the number of occurrences of e in S is denoted by $o(e, S)$.

Definition 4.26. Let $\mathcal{S}$ be a DAG Realization-instance with maximum degree Δ. A realizable degree ordering $\sigma[1, n]$ is called *short realizable degree ordering for* $\mathcal{S}$ if σ has length at most $\Delta^{4\Delta}$ and for all $a, b \in \{0, \ldots, \Delta\}$ it holds that

$$o(\binom{a}{b}, \sigma\{1, n\}) \leq o(\binom{a}{b}, \mathcal{S}).$$

Furthermore, $\mathcal{B}(\Delta, \sigma_{\text{short}})$ is the set that contains for each $p \in \text{pot}(\sigma_{\text{short}})$ all possible short neutral blocks of potential p containing only tuples with degree at most Δ.

Given the realizable degree ordering σ_{short} and Δ, the algorithm computes $\mathcal{B}(\Delta, \sigma_{\text{short}})$ by trying for all possible $(\Delta + 1)^{2\Delta^{2\Delta}}$ ordered degrees sequences of length at most $\Delta^{2\Delta}$ whether they are indeed neutral blocks of some potential $p \in \text{pot}(\sigma_{\text{short}})$. Checking whether a partial realizable degree ordering is a neutral block of potential p is done as follows: Using the construction in the proof of Observation 4.3, our algorithm constructs a realizable degree ordering with $p[1]$ sources and $\omega(p)$ sinks. The potential at position $p[1]$ is p. Next, our algorithm inserts the partial realizable degree ordering at position $p[1]$ and constructs the corresponding well-connected DAG. The length of the DAG is at most $\Delta^{2\Delta}+2\Delta^2$, because $\omega(p) < \Delta^2$. Hence, by Lemma 4.7, the DAG can be constructed in $O(\Delta^{2\Delta+1})$ time. Then, the algorithm checks (in the well-connected DAG) whether the potential at the end of the neutral block is p. Recall that by Lemma 4.8 the potential at position $p[1] + 1$ depends only on p and the first tuple in the neutral block, and not on the individual tuples before position $p[1]$. Thus, the above construction yields for every neutral block of potential p a realizable realizable degree ordering. Overall, constructing $\mathcal{B}(\Delta, \sigma_{\text{short}})$ can be done in $O((\Delta + 1)^{2\Delta^{2\Delta}} \cdot \Delta^{2\Delta} \cdot \Delta^{2\Delta}) = \Delta^{\Delta^{O(\Delta)}}$ time since there are at most $\Delta^{2\Delta}$ potentials of value less than Δ^2.

Now, given the set $\mathcal{B}(\Delta, \sigma_{\text{short}})$ of neutral blocks, the algorithm determines whether it is possible to insert these a certain number of times such that a realizable degree ordering for $\mathcal{S}$ is obtained. To this end, we formalize the problem and call it Exact Multiset Multicover.

Exact Multiset Multicover (EMM)

Input: A collection $\mathcal{C} = \{S_1, \ldots, S_\ell\}$ of multisets over a universe $U = \{e_1, \ldots, e_m\}$ and a demand function dem: $U \to \mathbb{N}$.

Question: Is there an exact multicover, that is, a function c: $\mathcal{C} \to \mathbb{N}$ such that for each element $e \in U$ it holds that

$$\sum_{i=1}^{\ell} \mathrm{c}(S_i) \cdot o(e, S_i) = \mathrm{dem}(e)?$$

Before providing an ILP-formulation for Exact Multiset Multicover, we show that it indeed captures our remaining problem.

Lemma 4.27. Let $\mathcal{S}$ be a DAG Realization instance with maximum degree Δ and let σ_{short} be a short realizable degree ordering for $\mathcal{S}$. Set $U := \{\binom{a}{b} \mid a, b \in \{0, \ldots, \Delta\}\}$ and $\mathrm{dem}(t) := o(t, \mathcal{S}) - o(t, \sigma_{\text{short}})$ for each tuple $t \in U$. Then, σ_{short} can be extended to a realizable degree ordering for $\mathcal{S}$ by inserting neutral blocks of $\mathcal{B}(\Delta, \sigma_{\text{short}})$ if and only if the EMM instance $(\mathcal{B}(\Delta, \sigma_{\text{short}}), U, \mathrm{dem})$ is a yes-instance.

Proof. "$\Rightarrow$:" Let σ be the realizable degree ordering for $\mathcal{S}$ that is obtained by inserting neutral blocks of $\mathcal{B}(\Delta, \sigma_{\text{short}})$ in σ_{short}. For each neutral block $\sigma^B \in \mathcal{B}(\Delta, \sigma_{\text{short}})$, let $\#\mathrm{ins}(\sigma^B) \in \mathbb{N}$ denote the number how often the neutral block σ^B is inserted in σ_{short}. Hence, for each tuple $t \in U$ we have

$$\sum_{\sigma^B \in \mathcal{B}(\Delta, \sigma_{\text{short}})} \#\mathrm{ins}(\sigma^B) o(t, \sigma^B) = o(t, \mathcal{S}) - o(t, \sigma_{\text{short}}) = \mathrm{dem}(t).$$

Thus setting $\mathrm{c}(\sigma^B) = \#\mathrm{ins}(\sigma^B)$ for all $\sigma^B \in \mathcal{B}(\Delta, \sigma_{\text{short}})$ is an exact multicover.

"$\Leftarrow$:" Conversely, let c: $\mathcal{B}(\Delta, \sigma_{\text{short}}) \to \mathbb{N}$ denote an exact multicover, that is, for every tuple $t \in U$ we have

$$\sum_{\sigma^B \in \mathcal{B}(\Delta, \sigma_{\text{short}})} \mathrm{c}(\sigma^B) \cdot o(t, \sigma^B) = \mathrm{dem}(t) = o(t, \mathcal{S}) - o(t, \sigma_{\text{short}}).$$

Observe that inserting a neutral block into σ, by Lemma 4.15, does not change the potentials that occur after the insert position. Hence, we can insert any subset of neutral blocks of $\mathcal{B}(\Delta, \sigma_{\text{short}})$ in σ_{short}. Thus, if we insert each neutral block $\sigma^B \in \mathcal{B}(\Delta, \sigma_{\text{short}})$ exactly $\mathrm{c}(\sigma^B)$ times in σ_{short}, then we obtain a realizable degree ordering for $\mathcal{S}$. □

Next, we show fixed-parameter tractability of EMM with respect to the parameter $|\mathcal{B}(\Delta, \sigma_{\text{short}})| = \ell$. Since the number ℓ of neutral blocks is bounded by a function only depending on Δ, this completes our algorithm for the case that an input of DAG Realization admits a low-potential realization.

Lemma 4.28. EMM is fixed-parameter tractable with respect to the parameter $|\mathcal{C}| = \ell$.

Proof. We show the fixed-parameter tractability result by giving an ILP-formulation of the problem with ℓ variables. It has been shown that an ILP with p variables can be solved in $O(p^{2.5p+o(p)} \cdot L)$ time and space polynomial in L where L is the input size [9, 16, 20]. To solve the EMM instance, we use the following ILP-formulation:

$$\forall 1 \le i \le \ell : \qquad x_i \in \mathbb{N} \tag{11}$$

$$\forall e \in U : \qquad \sum_{i=1}^{\ell} x_i \cdot o(e, S_i) = \mathrm{dem}(e) \tag{12}$$

Each of the ℓ integer variables $x_1, \ldots, x_\ell$ denotes how often the multiset $S_i \in \mathcal{C}$ is used in the solution. The function $o(e, s_i)$ denotes the number of occurrences of the tuple e the multiset S_i. It follows from the definition of EMM that a solution to the ILP directly corresponds to an exact multicover ($\mathrm{c}(S_i) = x_i$) for the EMM instance.

The ILP consists of $\ell + |U|$ equations that contain together $O(\ell \cdot |U|)$ integers, each upper-bounded by $\max_{e \in U, S_i \in \mathcal{C}}\{\mathrm{dem}(e), o(e, S_i)\}$. Hence, the ILP can be solved in $O(\ell^{2.5\ell + o(\ell)} \cdot \ell \cdot |U| \cdot \log(\max_{e \in U, S_i \in \mathcal{C}}\{\mathrm{dem}(e), o(e, S_i)\}))$ time. □

Combining Lemma 4.25 and Lemma 4.28 shows fixed-parameter tractability for the low-potential case: The algorithm first tries all possibilities for the short realizable degree ordering and in each branch it tries to solve the corresponding EMM-instance. As to the running time observe that there are $\Delta^{\Delta^{O(\Delta)}}$ possibilities for the short realizable degree ordering σ_{short}. Then the algorithm constructs in $\Delta^{\Delta^{O(\Delta)}}$ time $\mathcal{B}(\Delta, \sigma_{\text{short}})$. To check whether σ_{short} is realizable requires $O(\Delta n)$ time (Lemma 4.7), the construction of the EMM instance $O(\Delta^{\Delta^{O(\Delta)}} \cdot \log n)$ time, and solving the ILP requires $\Delta^{\Delta^{\Delta^{O(\Delta)}}} \cdot \log n$ time. Since the algorithm has to read the input of size $O(n \log \Delta)$ we arrive at the following theorem.

Theorem 4.29. If a degree sequence admits a low-potential realization, then it can be found in $\Delta^{\Delta^{\Delta^{O(\Delta)}}} \cdot n$ time.

Theorems 4.23 and 4.29 together lead to the main result of this section.

Theorem 4.30. DAG Realization is fixed-parameter tractable with respect to the parameter maximum degree Δ.

Note that Theorem 4.30 is a mere classification result: The corresponding running time is $\Delta^{\Delta^{\Delta^{O(\Delta)}}} \cdot n$. It is dominated by the low-potential case.

5 Fixed-Parameter Tractability with Respect to $m - n + 1$ and $\binom{n}{2} - m$

In the end of Section 3 we showed that DAG Realization remains NP-hard even on sparse and on dense instances. More specifically, for every constant $\ell > 1$, DAG Realization remains NP-hard when restricted to instances with $m < \ell n$ or $m > \binom{n}{2} \cdot \ell^{-1}$ (see Theorem 3.8). In contrast, in this section we prove that if we measure the sparseness or denseness by the number of arcs that a realizing DAG is away from a tree or a tournament instead of using a fraction of m and n, then the problem becomes tractable. In particular, we show that DAG Realization is fixed-parameter tractable with respect to each of the parameters $\binom{n}{2} - m$ and $m - n + 1$. To this end, we first consider the (very) dense setting and then the (very) sparse setting.

Dense Setting Our algorithm for the dense setting relies on two simple observations: First, for each tuple $\binom{a}{b}$ with degree $a+b=n-1$ the position in a realizable degree ordering σ is precisely defined: The a inneighbors of the vertex v realizing the tuple are ahead in the topological ordering ϕ corresponding to σ and the b outneighbors of v are behind in ϕ. As $|\phi|=|\sigma|=n$ it follows that v is the $(a+1)^{\text{th}}$ vertex in ϕ, that is, $\binom{a}{b}$ occurs in any realizable degree ordering at the $(a+1)^{\text{th}}$ positions. Second, any realizing DAG can be obtained from a transitive tournament by exactly $k := \binom{n}{2} - m$ arc removals. Thus, all but $2k$ vertices in the realizing DAG have a degree (indegree plus outdegree) of $n-1$. Putting this together, there are at most $2k$ positions "left" in a realizable degree ordering. Hence, a simple search-tree algorithm tries all possibilities to insert the tuples $\binom{a}{b}$ with $a+b<n-1$ in these "free positions" and checks whether the resulting ordering is indeed a realizable degree ordering. As there are $(2k)!$ possibilities to insert the tuples and, by Lemma 4.7, the checking can be done in $O(\Delta n)$ time, we arrive at the following.

Theorem 5.1. DAG Realization is fixed-parameter tractable with respect to the parameter $k := \binom{n}{2} - m$. The corresponding running time is $O((2k)!\Delta n)$.

Sparse Setting The sparse setting requires more effort. Let $k := m - n + 1$. We develop a polynomial-time executable data reduction rule whose exhaustive application to a degree sequence $\mathcal{S}$ results in an instance which is equivalent to $\mathcal{S}$ and whose maximum degree is at most $2k$. Then, the claimed fixed-parameter tractability follows from Theorem 4.30.

We exploit the following observations: For an instance $\mathcal{S}$ with $m \le n-1$, Berger and Müller-Hannemann [3] have shown that DAG Realization is polynomial-time solvable (recall that we assume $\binom{0}{0} \notin \mathcal{S}$). Moreover, they proved that if $m \ge n-1$ and $\mathcal{S}$ is realizable, then there is a realizing DAG for $\mathcal{S}$ that consists of only one connected component. (Recall that by "connectedness" in a directed graph we always refer to the connectivity in the underlying undirected graph.) Thus, if a degree sequence is realizable, then there is a connected realizing DAG D such that the parameter k denotes the size of a feedback edge set of the underlying undirected graph of D. A feedback edge set F of an undirected graph G is a subset of the edges whose removal makes the graph acyclic, that is, G can be seen as a tree with the additional edges in F. This implies that deleting in D a vertex with in- or outdegree at least $k+2$ results in a disconnected DAG: Deleting a vertex of degree $k+2$ in a tree results in $k+2$ connected components. At most $k+1$ of these $k+2$ components can be pairwise connected by the edges in the feedback edge set.

If there are two connected components in a realizing DAG D, then we can restructure D by copying parts from one component into the other component without creating cycles. This restructuring of a realizing DAG is our core tool to develop the data reduction rule and is formally stated in the next lemma. See Figure 6 for a schematic view on the requirements and the statement of the lemma.

Lemma 5.2. Let $D=(V,A)$ be a realizing DAG for a degree sequence $\mathcal{S}$, and let $v^p, v^t \in V$ be two vertices with $(v^p, v^t) \in A$. Furthermore, let K be the vertices of the connected component in $D[V \setminus \{v^t\}]$ containing the vertex v^p and let $v^s, u, w, w' \in V \setminus \{v^t\}$ be vertices such that:

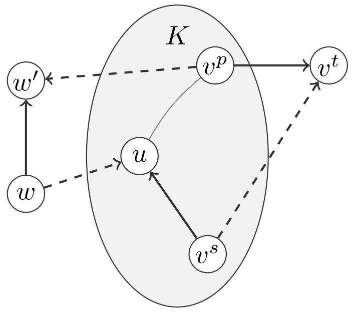

Figure 6: The situation when Lemma 5.2 is applicable: The underlying undirected graph of the induced subgraph K is acyclic and there is only one arc connecting a vertex inside K with a vertex outside K: the arc (v^p, v^t). The vertex v^s is inside of K with the outneighbor u lying on the uniquely defined undirected path (indicated by the thin edge) between v^s and v^p (with the possibility that $u = v^p$). Furthermore (w, w') is some arc with w and w' lying outside of K (with $w \neq v^t$ and $w' \neq v^t$). Then deleting the solid arcs in the picture and adding the dashed arcs results again in a DAG where the degrees of the vertices remain unchanged.

(i) $(v^s, u), (w, w') \in A$,

(ii) $v^s, v^p \in K$, but $w, w' \notin K$,

(iii) the underlying undirected graph of $D[K]$ is acyclic,

(iv) u lies on the uniquely defined undirected path between v^p and v^s or $u = v^p$, and

(v) v^p is the only neighbor of v^t in K.

Then, the digraph $D' = (V, (A \setminus \{(v^s, u), (v^p, v^t), (w, w')\}) \cup \{(v^s, v^t), (w, u), (v^p, w')\})$ is a realizing DAG for $\mathcal{S}$. Furthermore, the underlying undirected graph of the connected component in $D'[V \setminus \{v^t\}]$ which contains v^s is acyclic and contains no further neighbor of v^t.

Proof. Let K' be the connected component in $D'[V \setminus \{v^t\}]$ which contains the vertex v^s and let K'_u be the underlying undirected graph of K'. We first prove that K'_u is acyclic. Observe that by Assumption (iii) the underlying undirected graph K_u of K is acyclic. Hence, K_u is a tree. Next, root the tree K_u in the vertex v^p. Since by Assumption (iv) the vertex u lies on the (uniquely defined) path between v^s and v^p, the graph K'_u accords in K_u to the subtree with root v^s. Hence, K'_u is acyclic and contains no further neighbor of v^t.

Next, we prove that D' is a realizing DAG for $\mathcal{S}$. Clearly, the vertices v^s, u, v^t, v^p, w, and w' have the same indegree and outdegree in D' as in D. To show that D' does not contain any cycle, assume towards a contradiction that there is a directed cycle C' in D'. Recall that K'_u is acyclic and, hence, also K' is acyclic. From this and from Assumption (v) it follows that C' cannot contain any vertex of K'. Since D is acyclic, this implies that at least one of the arcs $(w, u), (v^p, w')$ is contained in C'. Denote with $\widetilde{K}$ the subgraph containing all vertices that are in K but not in K', that is, the graph induced by the vertices that are cut-out of K. By Assumption (iii) also $\widetilde{K}$ is acyclic and, thus, by Assumption (ii) C' has to contain both arcs $(w, u), (v^p, w')$. This implies that in D' and so in D there is a directed path from w' to w, implying that, since $(w, w') \in A$, there is also a cycle in D, a contradiction. □

Observe that the version of Lemma 5.2, where all arcs appear in the reversed direction, is also true. To see this, first swap in every tuple $\binom{a}{b}$ in $\mathcal{S}$ the values of a and b and, correspondingly, swap in a realizing DAG the direction of each arc. Then, apply Lemma 5.2 and finally swap again the values in the tuples and the arcs in the restructured DAG.

Using the restructuring operation provided by Lemma 5.2 and its reversed-arc version, we can show the following.

Lemma 5.3. Let $\mathcal{S}$ be a realizable degree sequence with $k = m - n + 1 > 0$ and let $t = \binom{a}{b} \in S$ be a tuple with $a > 2k$. Furthermore, let $s_{\min} \in \mathcal{S}$ be a source with minimum outdegree. Then, there is a realizing DAG for $\mathcal{S}$ such that the vertex that corresponds to t is an outneighbor of the vertex $v^s_{\min}$ which corresponds to $s_{\min}$. Furthermore, all other outneighbors of $v^s_{\min}$ are degree-one sinks.

Proof. Let $\mathcal{S}$ be a realizable degree sequence and let $t = \binom{a}{b} \in \mathcal{S}$ be a tuple with $a > 2k$. Furthermore, let $D = (V, A)$ be a connected realizing DAG for $\mathcal{S}$ and denote by v^t the vertex that corresponds to t. We prove the statement of the lemma in three steps. First, we show in Step 1 that we can assume that v^t has some source v^s as inneighbor such that the underlying undirected graph of the connected component K_s of $D[V \setminus \{v^t\}]$ containing v^s is acyclic, v^s is the only vertex in K_s that is adjacent to v^t, and v_s is a minimum-outdegree source in K_s. In Step 2, we prove that we can replace this source by $v^s_{\min}$ such that the connected component of the underlying undirected graph of $D[V \setminus \{v^t\}]$ containing $v^s_{\min}$ is acyclic and $v^s_{\min}$ is the only vertex in this connected component that is adjacent to v^t. Finally, in Step 3 we show that we can replace all outneighbors of $v^s_{\min}$ except v^t by degree-one sinks. The reason for these restrictive requirements for the outcome in Steps 1 and 2 is that we apply Lemma 5.2 in Steps 2 and 3.

Step 1: Assume that v^t has no source as inneighbor satisfying the outcome we want after Step 1; otherwise go to Step 2. As k is the size of a feedback edge set in the underlying undirected graph of D and $a > 2k$, there are at least $k + 1$ connected components in $D[V \setminus \{v^t\}]$ and at most k of them can contain a cycle or more than one neighbor of v^t. Thus, there is at least one connected component K in $D[V \setminus \{v^t\}]$ that contains only one neighbor (an inneighbor), say v^p, of v^t and does not contain any cycle in the underlying undirected graph of $D[V \setminus \{v^t\}]$. Let v^s be a source in K that has a minimum outdegree (within K). The underlying undirected graph of K is a tree and, hence, contains a unique path from v^s to v^p. Let u be the outneighbor of v^s lying on this path (with u not necessarily being distinct from v^p). Furthermore, let $(w, w') \in A$ be an arc where both endpoints w and w' are in $D[V \setminus \{v^t\}]$ in a connected component different from that of v^s. Observe that such an arc must exist, as otherwise, by the choice of v^s, the underlying undirected graph of D would be acyclic, contradicting the assumption $k > 0$. Then, by Lemma 5.2,

$$D' = (V, (A \setminus \{(v^s, u), (v^p, v^t), (w, w')\}) \cup \{(v^s, v^t), (w, u), (v^p, w')\})$$

is also a realizing DAG for $\mathcal{S}$. Furthermore, denoting the connected component in $D'[V \setminus \{v^t\}]$ containing v^s by K_s, the underlying undirected graph of K_s is acyclic, only v^s in K_s is a neighbor of v^t, and no source in K_s has a smaller outdegree than v^s.

Step 2: Towards proving that v^t has $v^s_{\min}$ as inneighbor, assume that $d^+(v^s) > d^+(v^s_{\min})$; otherwise go to Step 3. We show that we can replace v^s as inneighbor of v^t by $v^s_{\min}$. Observe that since v^s is within K_s a source with minimum outdegree, it follows that $v^s_{\min} \notin K_s$. Let $s_1, \ldots, s_x$ be the outneighbors of v^s with $s_1 = v^t$ and let $o^s_1, \ldots, o^s_y$ be the outneighbors of $v^s_{\min}$. Observe that $x > y$ and that $v^s_{\min}$ and v^s can have at most one common outneighbor: v^t. Thus, the vertices $s_2, \ldots, s_x$ are no outneighbors of $v^s_{\min}$. Now, we obtain a directed graph D'' from D' by deleting the arcs from v^s to all $s_{y+1}, \ldots, s_x$ and by adding arcs from $v^s_{\min}$ to all $s_{y+1}, \ldots, s_x$. Clearly, since in D'' in- and outdegrees of $v^s_{\min}$ and v^s have been exchanged, v^s is a minimum-degree source in D''. Since $v^s_{\min}$ and v^s are sources and all arcs that have been modified to get D'' from D' have either $v^s_{\min}$ or v^s as an endpoint and a source can never be contained in a directed cycle, D'' is a realizing DAG for $\mathcal{S}$. Furthermore, since K_s does not contain any cycle in $D[V \setminus \{v^t\}]$ and all modifications involving v^s did only delete arcs, it follows that also the connected component of v^s in $D''[V \setminus \{v^t\}]$ does not contain any undirected cycle. For convenience, we also exchange the names of v^s and $v^s_{\min}$ so that in the following $v^s_{\min}$ is the source with minimum outdegree.

Step 3: By the argumentation above there is a realizing DAG D'' for $\mathcal{S}$ such that there is an inneighbor of v^t that is a minimum degree source $v^s_{\min}$. Furthermore, the connect component K'' that contains $v^s_{\min}$ in $D''[V \setminus \{v^t\}]$ in $v^s_{\min}$ contains no further in- or outneighbor of v^t, and the underlying undirected graph of K'' is acyclic.

By restructuring the arcs in D'', we show that there is also a realizing DAG such that except for v^t all outneighbors of $v^s_{\min}$ are degree-one sinks. Towards this, assume that $v^s_{\min}$ does have an outneighbor u that is not a degree-one sink and that is different from v^t; otherwise we are done. Root the connected component K'' of $v^s_{\min}$ in $D''[V \setminus \{v^t\}]$ in $v^s_{\min}$. Then there is at least one leaf v^ℓ in the subtree with root u. Since $v^s_{\min}$ is the source with the minimal outdegree and $v^s_{\min}$ has at least two outneighbors (v^t and u), it follows that v^ℓ is a degree-one sink. Let $v^{\ell'}$ be the inneighbor of this sink. Furthermore, let $(w, w') \in A''$ be an arc that is not contained in K''. Since the underlying undirected graph of K'' is a tree, by the version of Lemma 5.2 with all arcs being reversed (see the discussion before Lemma 5.3), the digraph

$$D''' = (V, A'' \setminus \{(w, w'), (v^s_{\min}, u), (v^{\ell'}, v^\ell)\} \cup \{(v^s_{\min}, v^\ell), (w, u), (v^{\ell'}, w')\})$$

is also a realizing DAG for $\mathcal{S}$. Let K''' the connected component of $v^s_{\min}$ in $D'''[V \setminus \{v^t\}]$. Observe that the underlying undirected graph of K''' is, again, acyclic: the underlying undirected graph of K'' is acyclic and K''' was obtained by replacing a subtree in K'' by a leaf. By the same argument, K''' contains only one neighbor of v^t, namely $v^s_{\min}$. Thus, D''' fulfills all conditions required in Step 3, but compared to D'', $v^s_{\min}$ has one more degree-one sink as outneighbor. Hence, by induction on the number of degree-one sinks $v^s_{\min}$ has as outneighbor, it follows that there is a realizing DAG such that all outneighbors of $v^s_{\min}$ except v^t are degree-one sinks. □

Lemma 5.3 shows how we can restructure a realizing DAG if there is a vertex with indegree greater than $2k$. By applying the same procedure as for Lemma 5.2, one obtains similar results in case of a high outdegree: First, swap in every tuple $\binom{a}{b}$ in the corresponding degree sequence the values of a and b and, correspondingly, swap in a realizing DAG the direction of each arc. Then, apply the Lemmas 5.2 and 5.3 and finally swap again the values in the tuples and the arcs in the restructured DAG. This proves the correctness of an analogous version of Lemma 5.3 for a vertex with outdegree greater than $2k$, which leads to the following data reduction rule.

Reduction Rule 5.4. Let $\mathcal{S}$ be a degree sequence containing a tuple $\binom{a_t}{b_t}$ with $a_t > 2k$ ($b_t > 2k$). Furthermore, let $s = \binom{a_s}{b_s} \in \mathcal{S}$ be a tuple with $a_s = 0$ ($b_s = 0$) and b_s (a_s) be minimal among all tuples with $a_s = 0$ ($b_s = 0$). Then, replace $\binom{a_t}{b_t}$ by $\binom{a_t-1}{b_t}$ ($\binom{a_t}{b_t-1}$), delete s, and delete $b_s - 1$ ($a_s - 1$) tuples of the form $\binom{1}{0}$ ($\binom{0}{1}$).

Clearly, Reduction Rule 5.4 can be applied in polynomial time. The correctness is also not hard to see: Lemma 5.3 shows that if $\mathcal{S}$ is a yes-instance, then so is the reduced instance. Conversely, if the reduced instance is a yes-instance, then adding a star realizing the removed tuples and making the center of the star adjacent to the vertex realizing $\binom{a_t-1}{b_t}$ (respectively $\binom{a_t}{b_t-1}$) shows that also $\mathcal{S}$ is a yes-instance. Furthermore, each application of Reduction Rule 5.4 decreases the number of tuples in the input by one and does not change the parameter k. Thus, exhaustively applying Reduction Rule 5.4 can be done in polynomial time. In addition, the maximum degree in an instance that is reduced with respect to Reduction Rule 5.4 is upper-bounded by $2k$. Together with Theorem 4.30 this implies the following.

Theorem 5.5. DAG Realization is fixed-parameter tractable with respect to the parameter $k = m - n + 1$.

6 Outlook

Answering an open question of Berger and Müller-Hannemann [2] we proved the NP-completeness of DAG Realization even in sparse and in dense graphs. Following the spirit of deconstructing intractability [19] we proved the necessity of large degrees in the NP-hardness proof by showing fixed-parameter tractability of DAG Realization with respect to the maximum degree Δ. Furthermore, we showed fixed-parameter tractability with respect to the feedback edge set size of the underlying undirected graph of a realizing DAG. It is open whether DAG Realization is solvable in single-exponential FPT time and whether it admits polynomial-size problem kernels with respect to these two parameters. In our NP-hardness reduction other parameters occur with unbounded values, for instance, the number of types. Note that this is a "stronger parameterization" [18] than the parameter maximum degree Δ as the number of types is at most $(\Delta + 1)^2$. Hence, investigating this parameter is an interesting task for future work.

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