Multi-Player Diffusion Games on Graph Classes*

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Abstract. We study competitive diffusion games on graphs introduced by Alon et al. [1] to model the spread of influence in social networks. Extending results of Roshanbin [7] for two players, we investigate the existence of pure Nash equilibria for at least three players on different classes of graphs including paths, cycles, and grid graphs. As a main result, we answer an open question proving that there is no Nash equilibrium for three players on $m \times n$ grids with $\min\{m, n\} \geq 5$.

1 Introduction

Social networks, and the diffusion of information within them, yields an interesting and well-researched field of study. Among other models, competitive diffusion games have been introduced by Alon et al. [1] as a game-theoretic approach towards modelling the process of diffusion (or propagation) of influence (or information in general) in social networks. Such models have applications in "viral marketing" where several companies (or brands) compete in influencing as many customers (of products) or users (of technologies) as possible by initially selecting only a "small" subset of target users that will "infect" a large number of other users. Herein, the network is modeled as an undirected graph where the vertices correspond to the users, with edges modeling influence relations between them. The companies, being the players of the corresponding diffusion game, choose an initial subset of target vertices which then influence other neighboring vertices via a certain propagation process. More concretely, a vertex adopts a company's product at some specific time during the process if he is influenced by (that is, connected by an edge to) another vertex that already adopted this product. After adopting a product of one company, a vertex will never adopt any other product in the future. However, if a vertex gets influenced by several companies at the same time, then he will not adopt any of them and he is removed from the game (the reason being that the effects of these influencing companies on the customer cancel out each other such that the customer is "too confused" to adopt any of the products). See Section 1.3 for the formal definitions of the game.

In their initial work, Alon et al. [1] studied how the existence of pure Nash equilibria is influenced by the diameter of the underlying graph. Following this

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line of research, Roshanbin [7] investigated the existence of Nash equilibria for competitive diffusion games with two players on several classes of graphs such as paths, cycles, and grid graphs. Notably, she proved that on sufficiently large grids, there always exists a Nash equilibrium for two players, further conjecturing that there is no Nash equilibrium for three players on grids. We extend the results of Roshanbin [7] for two players to three or more players on paths, cycles, and grid graphs, proving the conjectured non-existence of a pure Nash equilibrium for three players on grids as a main result. An overview of our results is given in Section 1.2. After introducing the preliminaries in Section 1.3, we discuss our results for paths and cycles in Section 2, followed by the proof of our main theorem on grids in Section 3. We finish with some statements considering general graphs in Section 4.

1.1 Related Work

The study of influence maximization in social networks was initiated by Kempe et al. [5]. Several game-theoretic models have been suggested, including our model of reference, introduced by Alon et al. [1]. Some interesting generalizations of this model are the model by Tzoumas et al. [11], who considered a more complex underlying diffusion process (there, depending on its neighborhood, a general scheme is used to determine whether a vertex adopts a product), and the model studied by Etesami and Basar [3], allowing each player to choose multiple vertices. Dürr and Thang [2] and Mavronicolas et al. [6] studied so-called Voronoi games, which are closely related to our model (but not identical; there, instead of an underlying diffusion process, each vertex is assigned to its closest player and vertices can be shared). Concerning our model, Alon et al. [1] claimed the existence of pure Nash equilibria for any number of players on graphs of diameter at most two, however, Takehara et al. [10] gave a counterexample consisting of a graph with nine vertices and diameter two with no Nash equilibrium for two players.

Our main point of reference is the work of Roshanbin [7], who studied the existence (and non-existence) of pure Nash equilibria mainly for two players on special graph classes (paths, cycles, trees, unicycles, and grids); indeed, our work can be seen as an extension of that work to more than two players. Small [8] already showed that there is a Nash equilibrium for any number of players on any star or clique. Small and Mason [9] proved that there is always a pure Nash equilibrium for two players on a tree, but not always for more than two players. Janssen and Vautour [4] considered safe strategies on trees and spider graphs, where a safe strategy is a strategy which maximizes the minimum pay-off of a certain player, when the minimum is taken over the possible unknown actions of the other players.

1.2 Our Results

We begin by characterizing the existence of Nash equilibria for paths and cycles, showing that, except for three players on paths of length at least six, a Nash equilibrium exists for any number of players playing on any such graph (Theorem 1 and 2). We then prove Conjecture 1 of Roshanbin [7], showing that there is no Nash equilibrium for three players on $G_{m\times n}$, as long as both m and n are at least 5 (Theorem 3). Finally, we investigate the minimum number of vertices such that there is an arbitrary graph with no Nash equilibrium for k players. We prove an upper bound showing that there always exists a tree on $\lfloor \frac{3}{2}k \rfloor + 2$ vertices with no Nash equilibrium for k players (Theorem 4). Due to space constraints, some of the proofs are omitted. Please refer to the full version (available at http://arxiv.org/abs/1412.2544).

1.3 Preliminaries

Notation. For $i, j \in \mathbb{N}$ with $i < j$, we define $[i, j] := \{i, \dots, j\}$ and $[i] := \{1, \dots, i\}$. We consider simple, finite, undirected graphs $G = (V, E)$ with vertex set V and edge set $E \subseteq \{\{u, v\} \mid u, v \in V\}$. A path $P_n = (V, E)$ on n vertices is the graph with $V = [n]$ and $E = \{\{i, i+1\} \mid i \in [n-1]\}$. A cycle $C_n = (V, E)$ on n vertices is the graph with $V = [n]$ and $E = \{\{i, i+1\} \mid i \in [n-1]\} \cup \{\{n, 1\}\}$.

For $m, n \in \mathbb{N}$, the $m \times n$ grid $G_{m\times n} = (V, E)$ is a graph with vertices $V = [m] \times [n]$ and edges $E = \{\{(x, y), (x', y')\} \mid |x - x'| + |y - y'| = 1\}$. We use the term *position* for a vertex $x \in V$. We define the *distance* of two positions $x = (x_1, y_1)$, $y = (x_2, y_2) \in V$ as $\|x - y\|_1 := |x_1 - x_2| + |y_1 - y_2|$ (note that this corresponds to the length of a shortest path from x to y in the grid). We denote the number of players by k and enumerate the players as Player $1, \dots,$ Player k.

Diffusion Game on Graphs. A *game* $\Gamma = (G, k)$ is defined by an undirected graph $G = (V, E)$ and a number k of players, each having its distinct color in $[k]$. The *strategy space* of each player is V, such that each Player i selects a single vertex $v_i \in V$ at time 0, which is then colored by her color i. If two players choose the same vertex v, then this vertex is removed from the graph. For Player i, we use the terms strategy and position interchangeably, referring to its chosen vertex. A *strategy profile* is a tuple $(v_1, \dots, v_k) \in V^k$ containing the initially chosen vertex for each player. The *pay-off* $U_i(v_1, \dots, v_k)$ of Player i is the number of vertices with color i after the following propagation process. At time $t + 1$, any so far uncolored vertex that has only uncolored neighbors and neighbors colored in i (and no neighbors with other colors $j \in [k] \setminus \{i\}$) is colored in i. Any uncolored vertex with more than two different colors among its neighbors is removed from the graph. The process terminates when the coloring of the vertices does not change between consecutive steps. A strategy profile $(v_1, \dots, v_k)$ is a (pure) *Nash equilibrium* if, for each Player $i \in [k]$ and each vertex $v' \in V$, it holds that $U_i(v_1, \dots, v_{i-1}, v', v_{i+1}, \dots, v_k) \leq U_i(v_1, \dots, v_k)$.

2 Paths and Cycles

In this section, we fully characterize the existence of Nash equilibria on paths and cycles, for any number k of players.

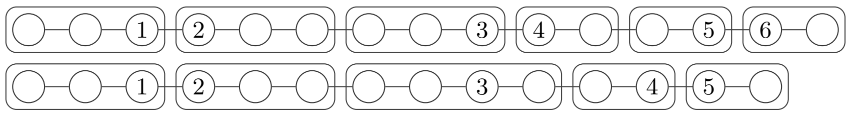


Fig. 1: Illustrations for Theorem 1, showing a Nash equilibrium for 6 players on P_{15} (top) and a Nash equilibrium for 5 players on P_{14} (bottom). The boxes show the colored regions of each player.

Theorem 1. *For any $k \in \mathbb{N}$ and any $n \in \mathbb{N}$, there is a Nash equilibrium for k players on P_n, except for $k = 3$ and $n \geq 6$.*

The general idea of the proof is to pair the players and distribute these pairs evenly. In the rest of this section, we prove three lemmas whose straightforward combination proves Theorem 1.

Lemma 1. *For any even $k \in \mathbb{N}$ and any $n \in \mathbb{N}$, there is a Nash equilibrium for k players on P_n.*

Proof. If $n \leq k$, then any strategy profile where each vertex of the path is chosen by at least one player is clearly a Nash equilibrium.

Otherwise, if $n > k$, then the idea is to build pairs of players, which are then placed such that two paired players are neighboring and the distance of any two consecutive pairs is roughly equal (specifically, differs by at most two). See Figure 1 for an example. Intuitively, this yields a Nash equilibrium since each player obtains roughly the same pay-off (specifically, differing by at most one), therefore no player can improve. Since we have n vertices, we want each player's pay-off to be at least $z := \lfloor \frac{n}{k} \rfloor$. This leaves $r := n(\mod k)$ other vertices, which we distribute between the first r players such that the pay-off of any player is at most $z + 1$. This can be achieved as follows. Let $p_i \in [n]$ denote the position of Player i, that is, the index of the chosen vertex on the path. We define

$$p_i := \begin{cases} z \cdot i + \min\{i, r\} & \text{if } i \text{ is odd,} \\ p_{i-1} + 1 & \text{if } i \text{ is even.} \end{cases}$$

Note that, by construction, it holds that $p_1 \in \{z, z+1\}$ and $p_k = n - z + 1$. Moreover, for each odd indexed player $i \geq 3$, we have that $2z - 1 \leq p_i - p_{i-1} \leq 2z + 1$. We claim that $u_i := U_i(p_1, \ldots, p_k) \in \{z, z+1\}$ holds for each $i \in [k]$. Clearly, $u_1 = p_1 \in \{z, z+1\}$ and $u_k = n - p_k + 1 = z$. For all odd $i \geq 3$, it is not hard to see that $u_i = u_{i-1} = 1 + \lfloor (p_i - p_{i-1} - 1)/2 \rfloor \in \{z, z+1\}$, proving the claim.

To see that the strategy profile $(p_1, \ldots, p_k)$ is a Nash equilibrium, consider an arbitrary player i and any other strategy $(p_i \neq)\ p'_i \in [n]$ that she picks. Clearly, we can assume that $p'_i \neq p_j$ holds for all $j \neq i$ since otherwise Player i's pay-off is zero. If $p'_i < p_1$ or $p'_i > p_k$, then Player i gets a pay-off of at most z. If $p_j < p'_i < p_{j+1}$ for some even $j \in [2, k-2]$, then her pay-off is at most $1 + \lfloor (p_{j+1} - p_j - 2)/2 \rfloor \leq z$. ◻

We can modify the construction given in the proof of Lemma 1 to also work for odd numbers k greater than three.

Lemma 2. *For any odd $k > 3 \in \mathbb{N}$ and for any $n \in \mathbb{N}$, there is a Nash equilibrium for k players on P_n.*

Proof. We give a strategy profile based on the construction for an even number of players (proof of Lemma 1). The idea is to pair the players, placing the remaining lonely player between two consecutive pairs.

This is best explained using a reduction to the even case. Specifically, given the strategy profile $(p'_1, \ldots, p'_{k+1})$ for an even number $k+1$ of players on P_{n+1} as constructed in the proof of Lemma 1, we define the strategy profile $(p_1, \ldots, p_k) := (p'_1, \ldots, p'_{k-2}, p'_k - 1, p'_{k+1} - 1)$. To see why this results in a Nash equilibrium, let $z := \lfloor (n+1)/(k+1) \rfloor$ and note that by construction it holds that $p_1 \in \{z, z+1\}$, $p_k = n - z + 1$, and $2z - 1 \le p_{i+1} - p_i \le 2z + 1$, for all $i \in [2, k-1]$. Moreover, each player receives a pay-off of at least z, therefore all players (except for Player $(k-2)$) cannot improve by the same arguments as in the proof of Lemma 1. Regarding Player $(k-2)$, note that her pay-off is

$$1 + \lfloor (p_{k-1} - p_{k-2} - 1)/2 \rfloor + \lfloor (p_{k-2} - p_{k-3} - 1)/2 \rfloor \ge 2z - 1.$$

Hence, she clearly cannot improve by choosing any position outside of $[p_{k-3}, p_{k-1}]$. Moreover, she cannot improve by choosing any other position in $[p_{k-3}, p_{k-1}]$. To see this, note that her maximum pay-off from any position in $[p_{k-3}, p_{k-1}]$ is

$$1 + \lfloor (p_{k-1} - p_{k-3} - 2)/2 \rfloor = 1 + \lfloor (p_{k-1} - p_{k-2} - 1 + p_{k-2} - p_{k-3} - 1)/2 \rfloor,$$

which is equal to the above pay-off since $p_{k-1} - p_{k-2}$ and $p_{k-2} - p_{k-3}$ cannot both be even, by construction. □

It remains to discuss the fairly simple (non)-existence of Nash equilibria for three players. Note that Roshanbin [7] already stated without proof that there is no Nash equilibrium for three players on $G_{2\times n}$ and $G_{3\times n}$ and that Small and Mason [9] showed that there is no Nash equilibrium for three players on P_7. For the sake of completeness, we prove the following lemma.

Lemma 3. *For three players, there is a Nash equilibrium on P_n if and only if $n \le 5$.*

Proof. If $n \le 3$, then a strategy profile where each vertex of the path is chosen by at least one player is clearly a Nash equilibrium. For $n \in \{4, 5\}$, the strategy profile $(2, 3, 4)$ is a Nash equilibrium.

To see that there is no Nash equilibrium for $n \ge 6$, consider an arbitrary strategy profile (p_1, p_2, p_3). Without loss of generality, we can assume that $p_1 < p_2 < p_3$ and consider the following two cases. First, we assume that $p_2 = p_1 + 1$ and $p_3 = p_2 + 1$. If $p_1 > 2$, then Player 2 increases her pay-off by choosing $p_1 - 1$. Otherwise, it holds that $p_3 < n - 1$ and Player 2 increases her pay-off by moving to $p_3 + 1$. Therefore, this case does not yield a Nash equilibrium. For the remaining case, it holds that $p_1 < p_2 - 1$ or $p_3 > p_2 + 1$. If $p_1 < p_2 - 1$, then Player 1 increases her pay-off by moving to $p_2 - 1$, while if $p_3 > p_2 + 1$, then Player 3 increases her pay-off by moving to $p_2 + 1$. Thus, this case does not yield a Nash equilibrium as well, and we are done. □

We close this section with the following result considering cycles. Interestingly, for cycles there exists a Nash equilibrium also for three players.

Theorem 2. *For any $k, n \in \mathbb{N}$, there is a Nash equilibrium for k players on C_n.*

Proof. It is an easy observation that the constructions given in the proofs of Lemma 1 and 2 also yield Nash equilibria for cycles, that is, when the two endpoints of the path are connected by an edge. Thus, it remains to show a Nash equilibrium for $k = 3$ players for any C_n. We set $p_1 := 1$, $p_2 := n$ and

$$p_3 := \begin{cases} \lfloor n/2 \rfloor & \text{if } n \mod 4 = 1, \\ \lceil n/2 \rceil & \text{otherwise.} \end{cases}$$

It is not hard to check that (p_1, p_2, p_3) is a Nash equilibrium. □

3 Grid Graphs

In this section we consider three players on the $m \times n$ grid $G_{m\times n}$ and prove the following main theorem.

Theorem 3. *If $n \geq 5$ and $m \geq 5$, then there is no Nash equilibrium for three players on $G_{m\times n}$.*

Before proving the theorem, let us first introduce some general definitions and observations. Throughout this section, we denote the strategy of Player i, that is, the initially chosen vertex of Player i, by $p_i := (x_i, y_i) \in [m] \times [n]$. Note that any strategy profile where more than one player chooses the same position is never a Nash equilibrium since in this case each of these players gets a pay-off of zero, and can improve its pay-off by choosing any free vertex (to obtain a pay-off of at least one). Therefore, we will assume without loss of generality that $p_1 \neq p_2$, $p_2 \neq p_3$, and $p_1 \neq p_3$. Further, note that the game is symmetric with respect to the axes. Specifically, reflecting coordinates along a dimension or rotating the grid by 90 degrees yields the same outcome for the game. Thus, in what follows, we only consider possible cases up to these symmetries.

We define $\Delta_x := \max_{i,j\in[k]} |x_i - x_j|$ and $\Delta_y := \max_{i,j\in[k]} |y_i - y_j|$ to be the maximum coordinate-wise differences among the positions of the players. We say that a player *strictly controls* the other two players, if both reside on the same side of the player, in both dimensions.

Definition 1. Player i *strictly controls* the other players, if either

$$\begin{aligned} &\forall j \neq i : x_i < x_j \wedge y_i < y_j, \\ \text{or } &\forall j \neq i : x_i < x_j \wedge y_i > y_j, \\ \text{or } &\forall j \neq i : x_i > x_j \wedge y_i < y_j, \\ \text{or } &\forall j \neq i : x_i > x_j \wedge y_i > y_j \text{ holds.} \end{aligned}$$

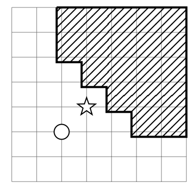


Fig. 2: Example of a strategy profile where Player 1 (white circle) has both other players to her top right with distance at least three (the shaded region denotes the possible positions for Player 2 and 3). Player 1 can increase her pay-off by moving closer to the others (star).

The proof of Theorem 3 proceeds as follows.

Proof (Theorem 3). Let $m \geq 5$ and $n \geq 5$. We perform a case distinction based on the relative positions of the three players. As a first case, we consider strategy profiles where the players are playing "far" from each other, that is, there are two players whose positions differ by at least four in some coordinate (formally, $\max\{\Delta_x, \Delta_y\} \geq 3$). For these profiles, we distinguish two subcases, namely, whether there exists a player who strictly controls the others (Lemma 4) or not (Lemma 5). We prove that none of these cases yields a Nash equilibrium by showing that there always exists a player who can improve her pay-off. Notably, the improving player always moves closer to the other two players. We are left with the case where the players are playing "close" to each other, specifically, their positions all lie inside a 3×3 subgrid (that is, $\max\{\Delta_x, \Delta_y\} \leq 2$). For these strategy profiles, we show that there always exists a player who can improve her pay-off (Lemma 6), however the improving position depends not only on the relative positions between the players, but also on the global positioning of this subgrid on the main grid. This leads to a somewhat erratic behaviour, which we overcome by considering all possible close positions (up to symmetries) in the proof of Lemma 6. Altogether, Lemmas 4, 5, and 6, cover all possible strategy profiles (ruling them out as Nash equilibria), thus implying the theorem. □

In order to conclude Theorem 3, it remains to prove the lemmas mentioned in the case distinction discussed above. To this end, we start with two easy preliminary results. First, we observe (as can be easily proven by induction) that a vertex for which the player with the shortest distance to it is unique is colored in that player's color.

Observation 1 *Let $x \in [m] \times [n]$ and $i \in [k]$. If $\|p_i - x\|_1 < \|p_j - x\|_1$ holds for all $j \neq i$, then x will be colored in color i at the end of the propagation process.*

Based on Observation 1, we show that if a player has distance at least three to the other players and both of them are positioned on the same side of that player (with respect to both dimensions), then she can improve her pay-off by moving closer to the others (see Figure 2 for an illustration).

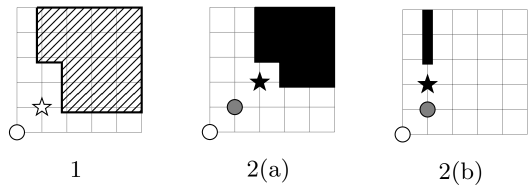


Fig. 3: Possible cases (up to symmetry) for Player 1 (white) strictly controlling Player 2 (gray) and Player 3 (black). Circles denote the player's strategies. The shaded region contains the possible positions of both Player 2 and 3, whereas the black regions denote possible positions for Player 3 only. A star marks the position improving the pay-off of the respective player.

Proposition 1. *If* $x_1 \leq x_j$, $y_1 \leq y_j$, *and* $\|p_1 - p_j\|_1 \geq 3$ *holds for* $j \in \{2, 3\}$, *then Player 1 can increase her pay-off by moving to* $(x_1 + 1, y_1 + 1)$.

Proof. Let $p_1' := (x_1 + 1, y_1 + 1)$ and $x \in [x_1] \times [y_1]$. Note that $\|p_1' - x\|_1 = \|p_1 - x\|_1 + 2 < \|p_j - x\|_1 = \|p_1 - p_j\|_1 + \|p_1 - x\|_1 \geq \|p_1 - x\|_1 + 3$ holds for $j \in \{2, 3\}$. Hence, Player 1 still has the unique shortest distance to x. By Observation 1, x gets color 1. Moreover, for any other position $x \notin [x_1] \times [y_1]$, there is a shortest path from p_1 to x going through at least one of the positions (x_1+1, y_1), (x_1, y_1+1), or p_1'. Clearly, there is also a shortest path from p_1' to x of at most the same length going through one of these positions. Thus, if x was colored in color 1 before, then x is still colored in color 1.

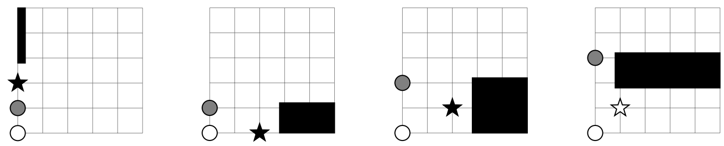
To see that Player 1 strictly increases her pay-off, note that $\|p_1' - x\|_1 = \|p_1 - x\|_1 - 2$ holds for all $x \in [x_1 + 1, n] \times [y_1 + 1, m]$. Hence, Player 1 now has the unique shortest distance to all those positions where the distance from p_1 was at most one larger than the shortest distance from any other player (clearly, there exists at least one such position with color $j \neq 1$). By Observation 1, these positions now get color 1, thus Player 1 strictly increases her pay-off. □

We go on to prove the lemmas needed for Theorem 3, starting with the case that the players play far from each other. The following lemma handles the first subcase, that is, where one of the players strictly controls the others.

Lemma 4. *A strategy profile with* $\max\{\Delta_x, \Delta_y\} \geq 3$ *where one of the players strictly controls the others is not a Nash equilibrium.*

Proof. We assume without loss of generality that Player 1 strictly controls Player 2 and Player 3, specifically, we assume that $x_1 < x_2$ and $y_1 < y_2$ and $x_1 < x_3$ and $y_1 < y_3$ holds. Figure 3 depicts the three possible cases for the positions of Player 2 and Player 3. For each case, we show that a player which can improve her pay-off exists.

Case 1: We assume that $(x_2, y_2) \neq (x_1+1, y_1+1)$ and $(x_3, y_3) \neq (x_1+1, y_1+1)$. By Proposition 1, Player 1 gets a higher pay-off from $(x_1 + 1, y_1 + 1)$.



1 2(a) 2(b) 2(c)

Fig. 4: Possible cases (up to symmetry) when no player strictly controls the others. Circles denote the positions of Player 1 (white) and Player 2 (gray). The black regions contain the possible positions for Player 3. A star marks the position improving the pay-off of the respective player.

Case 2: We assume without loss of generality that $(x_2, y_2) = (x_1 + 1, y_1 + 1)$.

- **(a)** We assume $x_2 < x_3$ and $y_2 < y_3$. Then, $x_3 > x_2 + 1$ or $y_3 > y_2 + 1$ holds since $\max\{\Delta_x, \Delta_y\} \geq 3$. Note that Player 3 strictly controls Player 1 and Player 2 and that this case is symmetric to Case 1.
- **(b)** We assume $x_2 \geq x_3$ or $y_2 \geq y_3$. Then, it holds that $x_3 = x_2$ or $y_3 = y_2$. We assume $x_3 = x_2$ (the argument for $y_3 = y_2$ being analogous). Since $\max\{\Delta_x, \Delta_y\} \geq 3$, we have $y_3 > y_2 + 1$, thus Player 3 can improve by moving to $(x_2, y_2 + 1)$ because then all positions in $[m] \times [y_2 + 1, n]$ are colored in color 3, and before only a strict subset of these positions were colored in her color. □

The other subcase, where no player strictly controls the others, is handled by the following lemma.

Lemma 5. *A strategy profile with* $\max\{\Delta_x, \Delta_y\} \geq 3$ *where no player strictly controls the others is not a Nash equilibrium.*

Proof. If no player strictly controls the others, then it follows that at least two players have the same coordinate in at least one dimension. We perform a case distinction on the cases as depicted in Figure 4.

Case 1: All three players have the same coordinate in one dimension. We assume that $x_1 = x_2 = x_3$ (the case $y_1 = y_2 = y_3$ is analogous). Without loss of generality also $y_1 < y_2 < y_3$ holds. Since $\max\{\Delta_x, \Delta_y\} \geq 3$, it follows that $y_{i+1} - y_i \geq 2$ holds for some $i \in \{1, 2\}$, say for $i = 2$. Clearly, Player 3 can improve her pay-off by choosing $(x_3, y_2 + 1)$ (analogous to Case 2b in the proof of Lemma 4).

Case 2: There is a dimension where two players have the same coordinate but not all three players have the same coordinate in any dimension. We assume $x_1 = x_2 < x_3$ and $y_1 < y_2$ (all other cases are analogous). We also assume that $y_1 \leq y_3 \leq y_2$, since otherwise Player 3 strictly controls the others, and this case is handled by Lemma 4.

- **(a)** We assume that $y_2 = y_1 + 1$. Then $x_3 \geq x_1 + 3$ holds since $\max\{\Delta_x, \Delta_y\} \geq 3$. Player 3 increases her pay-off by moving to $(x_1 + 2, y_1)$ (analogous to Case 2b in the proof of Lemma 4).

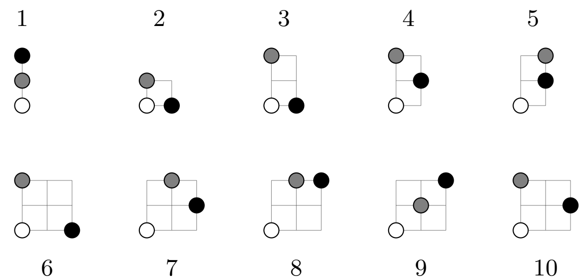


Fig. 5: Possible positions (up to symmetry) of three players playing inside a subgrid of size at most 3×3.

(b) We assume that $y_2 = y_1 + 2$. Then $x_3 \geq x_1 + 3$ holds since $\max\{\Delta_x, \Delta_y\} \geq 3$. Player 3 increases her pay-off by moving to $(x_1 + 2, y_1 + 1)$ (analogous to Case 2b in the proof of Lemma 4).

(c) We assume that $y_2 > y_1 + 2$ and $|y_2 - y_3| \leq |y_1 - y_3|$. That is, without loss of generality, Player 3 is closer to Player 2. Then, by Proposition 1, Player 1 increases her pay-off by moving to $(x_1 + 1, y_1 + 1)$. □

It remains to consider the cases where the players play close to each other.

Lemma 6. *A strategy profile with* $\max\{\Delta_x, \Delta_y\} \leq 2$ *is not a Nash equilibrium.*

Proof. First, we assume that $\Delta_x + \Delta_y \geq 2$, as otherwise there would be at least two players on the same position (so each one of them can improve by moving to any free vertex). Without loss of generality, we also assume that $\Delta_x \leq \Delta_y$, leaving the cases depicted in Figure 5 for consideration. Due to space constrains, we omit this case analysis. Please refer to the full version. □

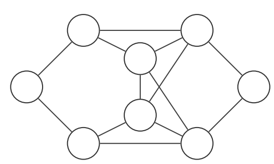


Fig. 6: A graph on 8 vertices with no Nash equilibrium for two players.

4 General Graphs

In this section, we study the existence of Nash equilibria on arbitrary graphs. Using computer simulations, we found that for two players, a Nash equilibrium

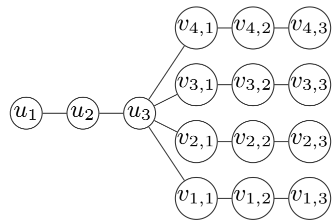


Fig. 7: A tree with no Nash equilibrium for 9 players.

exists on any graph with at most $n = 7$ vertices. For $n = 8$, we obtained the graph depicted in Figure 6, for which there is no Nash equilibrium for two players. As it is clear that adding isolated vertices to the graph in Figure 6 does not allow for a Nash equilibrium, we conclude the following.

Corollary 1. *For two players, there is a Nash equilibrium on each n-vertex graph if and only if $n \leq 7$.*

For more than two players, we can show the following.

Theorem 4. *For any $k > 2$ and any $n \geq \lfloor \frac{3}{2}k \rfloor + 2$, there exists a tree with n vertices such that there is no Nash equilibrium for k players.*

Proof. We describe a construction only for $n = \lfloor \frac{3}{2}k \rfloor + 2$, as we can add arbitrarily many isolated vertices without introducing a Nash equilibrium.

We first describe the construction for k being odd. We create one P_3, whose vertices we denote by u_1, u_2, and u_3, such that u_2 is the middle vertex of this P_3. For each $i \in [2, \lceil \frac{k}{2} \rceil]$, we create a copy of P_3, denoted by P_i, whose vertices we denote by $v_{i,1}$, $v_{i,2}$, and $v_{i,3}$, such that $v_{i,2}$ is the middle vertex of P_i. For each $i \in [2, \lceil \frac{k}{2} \rceil]$, we connect $v_{i,1}$ to u_3. An example for $k = 9$ is depicted in Figure 7.

For k being even, we create one P_2, whose vertices we denote by u_1, u_2. For each $i \in [2, \frac{k}{2} + 1]$, we create a copy of P_3, denoted by P_i, whose vertices we denote by $v_{i,1}$, $v_{i,2}$, and $v_{i,3}$, such that $v_{i,2}$ is the middle vertex of P_i. For each $i \in [2, \frac{k}{2} + 1]$, we connect $v_{i,1}$ to u_2. Due to space constraints, the analysis showing that no Nash equilibrium exists for k players playing on these graphs is omitted. Please refer to the full version. □

5 Conclusion

We studied competitive diffusion games for three or more players on paths, cycles, and grid graphs, answering—as a main contribution—an open question concerning the existence of a Nash equilibrium for three players on grids [7] negatively. Moreover, we provide a first systematic study of this game for more than two players. However, there are several questions left open, of which we mention some here.

An immediate question (generalizing Theorem 3) is whether a Nash equilibrium exists for more than three players on grids. Also, giving a lower bound for the number of vertices n such that there is a graph with n vertices with no Nash equilibrium for k players is an interesting question as it is not clear that the upper bounds given in Theorem 4 are optimal. In other words, is it true that $n \leq \frac{3}{2}k + 1$ implies the existence of a Nash equilibrium for k players?

References

[1] N. Alon, M. Feldman, A. D. Procaccia, and M. Tennenholtz. A note on competitive diffusion through social networks. *Information Processing Letters*, 110(6):221–225, 2010.

[2] C. Dürr and N. K. Thang. Nash equilibria in Voronoi games on graphs. In *Proceedings of the 15th Annual European Symposium on Algorithms (ESA'07)*, volume 4698 of *LNCS*, pages 17–28. Springer, 2007.

[3] S. R. Etesami and T. Basar. Complexity of equilibrium in diffusion games on social networks. In *Proceedings of the 2014 American Control Conference*, pages 2065–2070, 2014.

[4] J. Janssen and C. Vautour. Finding safe strategies for competitive diffusion on trees. *Internet Mathematics*, 2014. Accepted for publication.

[5] D. Kempe, J. Kleinberg, and É. Tardos. Maximizing the spread of influence through a social network. In *Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 137–146. ACM, 2003.

[6] M. Mavronicolas, B. Monien, V. G. Papadopoulou, and F. Schoppmann. Voronoi games on cycle graphs. In *Proceedings of the 33rd International Symposium on Mathematical Foundations of Computer Science (MFCS'08)*, volume 5162 of *LNCS*, pages 503–514. Springer, 2008.

[7] E. Roshanbin. The competitive diffusion game in classes of graphs. In *Proceedings of the 10th International Conference on Algorithmic Aspects in Information and Management (AAIM'14)*, volume 8546 of *LNCS*, pages 275–287. Springer, 2014.

[8] L. Small. *Information Diffusion on Social Networks*. PhD thesis, National University of Ireland Maynooth, 2012.

[9] L. Small and O. Mason. Nash equilibria for competitive information diffusion on trees. *Information Processing Letters*, 113(7):217–219, 2013.

[10] R. Takehara, M. Hachimori, and M. Shigeno. A comment on pure-strategy Nash equilibria in competitive diffusion games. *Information Processing Letters*, 112(3):59–60, 2012.

[11] V. Tzoumas, C. Amanatidis, and E. Markakis. A game-theoretic analysis of a competitive diffusion process over social networks. In *Proceedings of the Eighth Workshop on Internet and Network Economics (WINE'12)*, volume 7695 of *LNCS*, pages 1–14. Springer, 2012.