Interval Scheduling and Colorful Independent Sets

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Abstract. The NP-hard Independent Set problem is to determine for a given graph G and an integer k whether G contains a set of k pairwise non-adjacent vertices. The problem has numerous applications in scheduling, including resource allocation and steel manufacturing. There, one encounters restricted graph classes such as 2-union graphs, which are edge-wise unions of two interval graphs on the same vertex set, or strip graphs, where additionally one of the two interval graphs is a disjoint union of cliques.

We prove NP-hardness of Independent Set on a very restricted subclass of 2-union graphs and identify natural parameterizations to chart the possibilities and limitations of effective polynomial-time preprocessing (kernelization) and fixed-parameter algorithms. Our algorithms benefit from novel formulations of the computational problems in terms of (list-)colored interval graphs.

1 Introduction

Many scheduling problems can be formulated as finding maximum independent sets in certain generalizations of interval graphs [14]. Intuitively, finding a maximum number of pairwise non-adjacent vertices in a graph (this is the Independent Set problem) corresponds to scheduling a maximum number of jobs (represented by intervals) without conflicts. In this context, we consider two popular and practically motivated graph models, namely 2-union interval graphs [2] (also called 2-union graphs) and strip graphs [8].

A graph $G = (V, E)$ is a *2-union graph* if it can be represented as the union of two interval graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same vertex set V, that is, $G = (V, E_1 \cup E_2)$. There are numerous applications of solving (weighted) Independent Set on 2-union graphs, including scheduling problems such as resource allocation [2] or coil coating in steel manufacturing [9].

2-Union Independent Set:
Input: Two interval graphs $G_1 = (V, E_1), G_2 = (V, E_2)$ and an integer k.
Question: Does $G = (V, E_1 \cup E_2)$ have an independent set of size k?

*Supported by the DFG, projects DAPA, NI 369/12, and AREG, NI 369/9.
†Supported by the DFG, project DARE, NI 369/11.

We found a helpful natural embedding of 2-Union Independent Set into a more general problem by replacing 2-union graphs with list-colored interval graphs and searching for *colorful* independent sets:[1]

Colorful Independent Set:

Input: An interval graph $G = (V, E)$, a multicoloring $\mathrm{col} : V \to 2^{\{1,\dots,\gamma\}}$, and an integer k.

Question: Does G have a colorful independent set of size k?

An advantage of this model is that we only have to deal with one interval graph instead of two merged ones. Indeed, the modeling proved very useful when studying Independent Set on *strip graphs*, an important subclass of 2-union graphs. We believe that introducing our colorful view on finding independent sets and scheduling is of independent interest and might be useful in further studies. This "colored view on scheduling" leads to a useful reformulation of the classic Job Interval Selection problem [8, 15]. The task is to find a maximum set of jobs that can be executed, where each job has multiple possible execution intervals, each job is executed at most once, and a machine can only execute one job at a time. We state this problem using its classical name, but formulate it in terms of colored interval graphs, where the colors correspond to jobs and intervals of the same color correspond to multiple possible execution times of this job:

Job Interval Selection:

Input: An interval graph $G = (V, E)$, a coloring $\mathrm{col} : V \to \{1, 2, \dots, \gamma\}$ and an integer k.

Question: Does G have a colorful independent set of size k?

Here, the definition of "colorful"[1] degenerates to "no two intervals in the independent set have the same color".

Previous results. For 2-Union Independent Set, the following results are known. The problem remains NP-hard even when the two interval graphs are proper (unit interval) [1]. When restricted to so-called 5-claw-free graphs (which comprises the case that both input interval graphs are proper), Bafna et al. [1] provided a polynomial-time ratio-3.25 approximation. Bar-Yehuda et al. [2] showed that the vertex-weighted optimization version of 2-Union Independent Set has a polynomial-time ratio-4 approximation (indeed, they showed a ratio-$2t$ approximation for the generalization to t-union graphs). Recently, Höhn et al. [9] considered so-called m-composite 2-union graphs (which has applications in coil coating) and developed a dynamic programming algorithm running in polynomial time with the polynomial degree depending on m. This generalizes a result of Jansen [11], who gave such an algorithm for a subclass of m-composite 2-union graphs. Concerning parameterized complexity, Jiang [13] answered an open question of Fellows et al. [6] by proving 2-Union Independent Set to be W[1]-hard for the parameter "solution size k". This W[1]-hardness result holds even when both input interval graphs are proper.

Introduced by Nakajima and Hakimi [15] (using a different notion), Job Interval Selection was shown APX-hard by Spieksma [17], who also provided

[1] We call an independent set *colorful* if no two of its vertices share a color.

a ratio-2 greedy approximation algorithm. Chuzhoy et al. [5] improved this ratio to 1.582. Halldórsson and Karlsson [8] introduced the equivalent notion of Job Interval Selection as Independent Set on strip graphs, which are 2-union graphs where one of the two input interval graphs is a cluster graph. They showed fixed-parameter tractability for a structural parameter and for the parameter "total number of jobs". In related work, Jansen [12] considered Independent Set on unions of cographs and cluster graphs (the latter being disjoint unions of cliques).

New results. The main focus of this work is on initiating a systematic parameterized complexity study (particularly featuring kernelization results) for the three NP-hard problems Colorful Independent Set, 2-Union Independent Set, and Job Interval Selection (here listed in descending degree of generality). Doing so, we also discuss the relevance and interrelationships of several parameterizations. For Colorful Independent Set, we provide an $O(2^\gamma \cdot n^3)$-time dynamic-programming algorithm for the parameter "number γ of colors". For 2-Union Independent Set, this result translates to a $O(2^{\#\mathrm{mC}_{\mathrm{min}}} \cdot n^3)$-time algorithm, where $\#\mathrm{mC}_{\mathrm{min}}$ denotes the minimum of the numbers of maximal cliques in the two input interval graphs. Moreover, we provide an NP-hardness proof for 2-Union Independent Set, even when restricted to the case that one input graph is a collection of paths on three vertices and the other is a collection of edges and triangles. In contrast, if both input graphs are cluster graphs, we show that, 2-Union Independent Set can be solved in $O(n^{1.5})$ time, improving on the $O(n^3)$ time algorithm [16] implied by the claw-freeness of unions of two cluster graphs. Next, stimulated by Jiang's [13] W[1]-hardness result for the parameter "solution size k", we discuss natural structural parameters that are lower-bounded by or closely related to k. Systematically exploring these parameters, we chart the border between tractability and intractability for 2-Union Independent Set. In particular, we initiate the study of the power of polynomial-time data reduction (known as kernelization in parameterized algorithmics) and show that 2-Union Independent Set has a cubic-vertex problem kernel with respect to the parameter "maximum number of maximal cliques in one of the two input interval graphs". This improves to a quadratic-vertex kernel if both input interval graphs are proper. We remark that parameterizing by the number(s) of maximal cliques allows for generalizing previous results of Halldórsson and Karlsson [8]. Our results for 2-Union Independent Set carry over to the vertex-weighted case.

For Job Interval Selection (or, equivalently, Independent Set restricted to strip graphs), our main result refers to polynomial-time preprocessing: while we prove the nonexistence (assuming a standard complexity-theoretic conjecture) of polynomial-size problem kernels even for Job Interval Selection with respect to the combination of the parameters "maximum clique size ω" and "number γ of colors", we also show that, while still NP-hard, Job Interval Selection restricted to proper interval graphs has a problem kernel with $O(k^2 \cdot \omega)$ intervals that can be computed in linear time. Here, notably, $k \leq \gamma$.

Due to the lack of space, most technical details are deferred to a full version.

Preliminaries. When speaking of interval graphs, we state our running times under the assumption that an *interval representation* is given in which the intervals are sorted with respect to their starting or ending points. Given a graph G that allows for such an interval representation, the representation can be computed in $\mathrm{O}(n+m)$ time [4]. A graph is a *proper interval* graph if it allows for an interval representation such that for no two intervals v and w it holds that $v \subseteq w$. Every interval graph allows for a total and linear-time computable *clique ordering* $\prec$ of its maximal cliques such that, for each vertex, the maximal cliques containing it occur consecutively [7]. Moreover, all maximal cliques of an interval graph can be listed in linear time.

A problem is *fixed-parameter tractable* (FPT) with respect to a parameter k if there is an algorithm solving any problem instance of size n in $f(k) \cdot n^{\mathrm{O}(1)}$ time for some computable function f. A *problem kernelization* is a polynomial-time transformation of a problem instance x with a parameter k into a new instance x' with parameter k' such that $|x'|$ is bounded by a function in k (ideally, a polynomial in k), $k' \leq k$, and (x, k) is a yes-instance if and only if (x', k') is a yes-instance. We call (x', k') the *problem kernel* and $|x'|$ its *size*.

2 Independent Set and 2-Union Graphs

This section mainly investigates the standard and parameterized complexity of 2-Union Independent Set. We start by discussing a complexity dichotomy and thereafter consider various parameterizations of the problem. Finally, we provide parameterized tractability results with respect to number of maximal cliques in the input graphs.

A complexity dichotomy. 2-Union Independent Set is known to be NP-hard [1] and APX-hard for 2-union graphs of maximum degree three [2] and for graphs that are the union of an interval graph with pairwise disjoint edges [17]. Using a reduction from 3-Sat, we can impose further restrictions on NP-hard instances, which are important for showing kernelization lower bounds in Section 3.

Theorem 1. 2-Union Independent Set *is* NP*-hard, even if one of the input graphs is restricted to be a disjoint union of altogether k edges and triangles and the other is restricted to contain only paths of length two.*

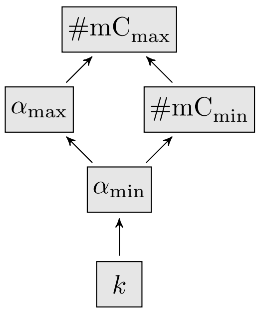
In the context of Theorem 1, note that paths of length two are the simplest graphs that are not cluster graphs. If, in contrast, G would be the union of two cluster graphs, then G is claw-free. Independent Set on claw-free graphs is solvable in $\mathrm{O}(n^3)$ time [16]. However, for the union of two cluster graphs, we can provide a $\mathrm{O}(n^{1.5})$ time algorithm based on computing a matching of the cliques in the two input cluster graphs.

Proposition 1. 2-Union Independent Set *is solvable in* $\mathrm{O}(n^{1.5})$ *time if both input interval graphs are cluster graphs.*

Theorem 1 and Proposition 1 give rise to a complexity dichotomy, stating that 2-Union Independent Set is polynomial-time solvable if both inputs are restricted to be cluster graphs, and NP-complete otherwise, even in the simplest case of non-cluster graphs. A more detailed investigation of our proof of Theorem 1 together with a result of Impagliazzo et al. [10] yields that, even in the restricted case covered by Theorem 1, there is no algorithm with running time $2^{o(k)} \cdot \text{poly}(n)$ for 2-Union Independent Set unless the Exponential Time Hypothesis[2] fails.

Corollary 1. *Under the prerequisites of Theorem 1, there is no algorithm with running time* $2^{o(k)} \cdot \text{poly}(n)$ *for* 2-Union Independent Set *unless the Exponential Time Hypothesis*[2] *fails.*

Parameter identification. We now consider suitable parameters for 2-Union Independent Set. Since we have two input graphs, we often consider the maximum or minimum value of parameters taken over the two input graphs. For example, considering the maximum degrees Δ_1 and Δ_2 of G_1 and G_2, respectively, natural parameters are $\Delta_{\min} := \min\{\Delta_1, \Delta_2\}$ and $\Delta_{\max} := \max\{\Delta_1, \Delta_2\}$. However, Theorem 1 implies that 2-Union Independent Set is NP-hard even if $\Delta_{\max} \leq 2$.



A second view on the parameterized landscape is centered around the fact that 2-Union Independent Set is W[1]-hard with respect to the parameter "solution size k" [13]. We therefore consider parameters that are lower-bounded by k. Unfortunately, the W[1]-hardness proof for parameter k due to Jiang [13] also shows that 2-Union Independent Set is W[1]-hard for the the maximum $\alpha_{\max}$ of the respective independence numbers α_1 and α_2 of G_1 and G_2. In interval graphs, a parameter that is lower bounded by the independence number α is the number of maximal cliques #mC. Indeed, we can show fixed-parameter tractability with respect to $\#\text{mC}_{\min}$ and $\#\text{mC}_{\max}$, denoting the minimum, respspectively the maximum, of the numbers of maximal cliques in the two input interval graphs. For the parameter $\#\text{mC}_{\min}$, we exploit an alternative problem formulation, additionally allowing us to obtain results for the well-known Job Interval Selection problem [15]. An overview of the parameters that are lower-bounded by k is shown above.

Parameterized tractability. In the quest for polynomial-time preprocessing for 2-Union Independent Set, we considered simple-to-implement reduction rules. Surprisingly, a single twin-type reduction rule is sufficient to provide a polynomial-size problem kernel with respect to the parameter $\#\text{mC}_{\max}$. We reduce the number of vertices having a given "signature" and then bound the number of signatures in a 2-union graph.

Definition 1. Let (G_1, G_2, k) denote an instance of 2-Union Independent Set and let v be a vertex of G_1 and G_2. The *signature* $\text{sig}(v)$ of v is the set of all vertex sets C that contain v and form a maximal clique in either G_1 or G_2.

[2] The *Exponential-Time Hypohesis* basically states that there is no $2^{o(n)}$-time algorithm for n-variable 3SAT.

Reduction Rule 1. Let (G_1, G_2, k) denote an instance of 2-Union Independent Set. For each pair of vertices u, v of G_1 and G_2 such that $\mathrm{sig}(v) \subseteq \mathrm{sig}(u)$, delete u from G_1 and G_2.

Theorem 2. 2-Union Independent Set *admits a cubic-vertex problem kernel with respect to the parameter "larger number of maximal cliques* $\#\mathrm{mC}_{\max}$*". A quadratic-vertex problem kernel can be shown if one of the input graphs is a proper interval graph. Both kernels can be computed in* $\mathrm{O}(n \log^2 n)$ *time.*

We can generalize Theorem 2 for the problem of finding an independent set of *weight* at least k: we keep the vertex with highest weight for each signature in the graph. Since each signature is uniquely determined by its first and last maximal cliques in G_1 and G_2 with respect to a clique ordering, there are at most $\#\mathrm{mC}_{\min}^2 \cdot \#\mathrm{mC}_{\max}^2$ different signatures and we obtain a problem kernel with $\mathrm{O}(\#\mathrm{mC}_{\min}^2 \cdot \#\mathrm{mC}_{\max}^2)$ vertices for the weighted variant.

In the following, we describe a dynamic programming algorithm that solves 2-Union Independent Set in $\mathrm{O}(2^{\#\mathrm{mC}_{\min}} \cdot \#\mathrm{mC}_{\min} \cdot \#\mathrm{mC}_{\max} \cdot n)$ time. To this end, we reformulate the problem in terms of interval graphs in which each vertex has a list out of at most $\#\mathrm{mC}_{\min}$ colors. We call a subset of vertices *colorful* if their color sets are pairwise disjoint. Recall the definition of Colorful Independent Set in Section 1. We reduce 2-Union Independent Set to Colorful Independent Set by assigning a color to each maximal clique in G_2 and giving G_1 as input to Colorful Independent Set such that each vertex has the colors of the maximal cliques of G_2 containing it. Since the color lists generated in this reduction form intervals with respect to a clique ordering of G_2, Colorful Independent Set can be considered a more general problem than 2-Union Independent Set. Regarding parameters, the numbers $\#\mathrm{mC}_{\min}$ and $\#\mathrm{mC}_{\max}$ of maximal cliques in the input interval graphs translate to the number γ of colors and the number $|\mathcal{C}|$ of maximal cliques in G, respectively.

Given a list-colored interval graph G, the algorithm computes a table T indexed by pairs in $\{0, \ldots, |\mathcal{C}|\} \times 2^{\{1,\ldots,\gamma\}}$ using the clique ordering $\prec$ of G. Let $\mathcal{C}[j]$ denote the j'th element in the ordering $\prec$, and let $G^i = G - \bigcup_{1 \le \ell \le i} \mathcal{C}[\ell]$. We define $T[i, C]$ so that it contains the maximum cardinality of a colorful independent set of G minus the first i maximal cliques (with respect to $\prec$) using only colors in C. For the base case, we set $T[|\mathcal{C}|, C] = 0$ for all $C \subseteq \{1, \ldots, \gamma\}$. Next, observe that for each interval v, there is a unique maximal clique with largest index i_v (according to the ordering $\prec$ of $\mathcal{C}$) containing v. The dynamic programming table can now be filled according to the following recursion:

$$T[i-1, C] = \max\Big\{T[i, C], \max_{\substack{v \in \mathcal{C}[i] \\ \mathrm{col}(v) \subseteq C}} \{1 + T[i_v, C \setminus \mathrm{col}(v)]\}\Big\}. \tag{1}$$

The cardinality of a maximum colorful independent set of G can be read from $T[0, \{1, \ldots \gamma\}]$. This approach is easily-modifiable to also compute a maximum *weighted* independent set if the input graph is vertex-weighted.

Theorem 3. Colorful Independent Set *can be solved in* $O(2^\gamma \cdot \gamma \cdot |\mathcal{C}| \cdot n)$ *time*[3], *even if all vertices are integer-weighted.*

In terms of 2-Union Independent Set, Theorem 3 can be stated as follows.

Corollary 2. 2-Union Independent Set *can be solved in* $O(2^{\#\mathrm{mC}_{\min}} \cdot \#\mathrm{mC}_{\max} \cdot \#\mathrm{mC}_{\min} \cdot n)$ *time.*

3 Colorful Independent Sets and Strip Graphs

In Section 2 we reformulated 2-Union Independent Set in terms of finding a maximum colorful independent set in an interval graph and gave a fixed-parameter algorithm for the more general problem Colorful Independent Set. We now consider the variant of Colorful Independent Set where each vertex (resp. interval) has only one color instead of a list of colors. This restriction is equivalent to 2-Union Independent Set for input graphs that are the edge-wise union of an interval graph and a cluster graph,[4] a class of graphs called *strip graphs* by Halldórsson and Karlsson [8]. They interpreted each clique in the input cluster graph as an equivalence class of vertices of the input interval graph; we reinterpret these equivalence classes in a natural way: as colors. In the literature, this problem is known as Job Interval Selection [15] (see the definition in Section 1). In our model, colors represent jobs and intervals of the same color in the input graph are possible execution intervals for one job. A solution then shows how to execute at least k jobs.

Jansen [11] showed a polynomial-time algorithm for Job Interval Selection for a constant number γ of colors. The dynamic programming algorithm given by Höhn et al. [9] can be seen as a generalization of this algorithm, since strip graphs are a special case of m-composite graphs. In both cases, the degree of the polynomial depends on γ. Halldórsson and Karlsson [8] gave a fixed-parameter algorithm running in $O(2^Q \cdot n)$ time with Q denoting the "maximum number of live intervals". Omitting the detailed description of the parameter Q, we note that, in instances of the underlying scheduling problem in which there is more than one machine, Q equals the number γ of colors in our interpretation.

Fixed-parameter algorithms for combinations with k. As 2-Union Independent Set is W[1]-hard for the single parameter "solution size k" [13], we combine k with the maximum clique size ω in the input interval graph G, the maximum number ϕ of cliques in G that have a vertex in common, and the number γ of colors in G. These combinations allow for fixed-parameter tractability and kernelization results. In the following, let $C_1, C_2, \dots$ denote the maximal cliques of G in order of the clique ordering of G. We will reuse the notion of "signatures" (see Definition 1). In the context of colored interval graphs, the *signature* $\mathrm{sig}(v)$ of an interval v is the pair of its color and the set of maximal cliques it is contained in.

[3]Assuming that adding, subtracting, and comparing of integers work in $O(1)$ time.

[4]Recall that 2-Union Independent Set is solvable in $O(n^{1.5})$ time if both input graphs are cluster graphs (see Proposition 1).

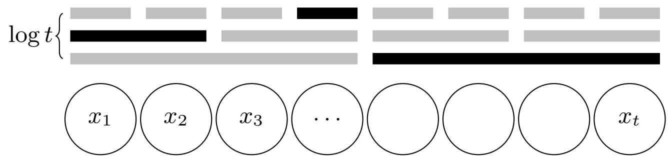


Fig. 1: Schematic view of the construction of the cross-composition. Circles at the bottom represent the t input instances. Bars at the top represent the newly added intervals spanning over the input instances. Here, each of the $\log t$ rows stands for a new color. A solution (black intervals) for the instance must select one interval in each row, thereby selecting one of the t input instances (x_3 in this example).

The algorithms presented in this section rely on the observation that an optimal solution can be assumed to contain an interval v of the first maximal clique C_1. In the following, assume that there is an interval $v \in C_1$ that is in the sought colorful independent set. Our fixed-parameter algorithms branch on properties of v that allow us to either identify v or remove intervals from G so that an isolated clique containing v is created. These properties are (a) the size of C_1, (b) the last clique containing v, and (c) the color of v. After at most k branchings, we end up with a cluster graph, on which the problem can be solved in polynomial time using Proposition 1. Depending on what property of v we branched on, the exponential components of the running times can be bounded in ω^k, ϕ^k, or γ^k.

Proposition 2. Job Interval Selection *can be solved in* $\mathrm{O}(\omega^k \cdot n)$*,* $\mathrm{O}(\phi^k \cdot n^{1.5})$*, and* $\mathrm{O}(\gamma^k \cdot n^2 \log^2 n)$ *time.*

Non-existence of polynomial-size kernels for Job Interval Selection. We show that Job Interval Selection is unlikely to admit polynomial-size problem kernels with respect to various parameters. To this end, we employ the technique of "cross-composition" introduced by Bodlaender et al. [3] using a bitmasking approach as standard in previous publications that exclude polynomial-size kernels for other problems. A *cross-composition* is a polynomial-time algorithm that, given t instances x_i with $0 \leq i < t$ of an NP-hard starting problem A, outputs an instance (y, k) of a parameterized problem B such that $k \in \mathrm{poly}(\max_i\{|x_i|\}+\log t)$ and $(y, k) \in B$ if and only if there is some $0 \leq i < t$ with $x_i \in A$. A theorem by Bodlaender et al. [3] states that if a problem B admits such a cross-composition, then there is no polynomial-size kernel for B unless coNP $\subseteq$ NP/poly.

We use an operation on binary-encoded numbers: *shifting* a number i by j bits to the right, denoted by $\mathrm{shift}(i, j) := \lfloor i/2^j \rfloor$. In the following, we present a cross-composition for Job Interval Selection with respect to (ω, γ). For the NP-hard starting problem we use the unparameterized version of Job Interval Selection with the restriction that $k = c$. The NP-hardness of this problem is a direct consequence of Theorem 1. We assume, without loss of generality, that $\log t$ is an integer. The framework of Bodlaender et al. [3] allows us to force the input instances to all have the same value for k and, thus, each instance uses the same color set $\{1, 2, \ldots, k\}$. The steps of the composition are as follows (see Figure 1):

Step 1. Place the t input instances in order of their index on the real line such that no interval of one instance overlaps an interval of another instance.

Step 2. Introduce $\log t$ more colors $k+1, k+2, \ldots, k+\log t$ (the resulting instance then asks for an independent set of size $k + \log t$).

Step 3. For each $1 \leq i \leq \log t$, introduce 2^i new intervals $v_0^i, v_1^i, \ldots, v_{2^i-1}^i$, each with color $k+i$, such that the new interval v_j^i spans over all instances x_ℓ with $\text{shift}(\ell, \log t - i) = j$.

It is easy to see that both the number of colors γ and the maximum clique size ω of the constructed instance are at most $\max_i |x_i| + \log t$. In order to show that the presented algorithm constitutes a cross-composition, it remains to prove that the resulting instance has a colorful independent set of size $k + \log t$ if and only if there is an input instance x_i that has a colorful independent set of size k. The presented cross-composition implies the following theorem [3].

Theorem 4. Job Interval Selection *does not admit a polynomial-size problem kernel with respect to the combination of the parameters "number of colors γ" and "maximum clique size ω" unless* $\text{coNP} \subseteq \text{NP/poly}$.

Polynomial-size kernel for proper interval graphs. We further restrict Job Interval Selection to proper interval graphs, on which it is still NP-hard, as evident from Section 2. Surprisingly, simple data reduction rules enable us to construct a problem kernel comprising $2\omega k(k-1)$ intervals in this case, sharply contrasting Theorem 4, which excludes a polynomial-size problem kernel with respect to the combined parameter (k, ω) (since $\gamma \geq k$).

Reduction Rule 2. Delete from G every interval that has a color that appears more than $2\omega(k-1)$ times and decrease k by the number of removed colors.

Reduction Rule 3. If G contains more than $2\omega k(k-1)$ intervals, then return a trivial yes-instance.

Reduction Rule 2 can be applied exhaustively in $O(n)$ time. Thereafter executing Reduction Rule 3 immediately yields the following theorem.

Theorem 5. Job Interval Selection *on proper interval graphs admits a problem kernel with at most $2\omega k(k-1)$ intervals that can be computed in $O(n)$ time.*

4 Outlook

Besides hardness results, we also developed encouraging algorithmic results which might find use in practical applications, so future empirical studies seem worthwhile (also see the strong practical results of Höhn et al. [9] with respect to steel manufacturing). As a future challenge, it is interesting to know whether 2-Union Independent Set admits a polynomial-size problem kernel with respect to the parameter $\#\text{mC}_{\text{min}}$, denoting the smaller number of maximal cliques in one of the input interval graphs. Furthermore, we conjecture that Job Interval Selection with respect to the parameter "solution size k" is fixed-parameter tractable, whereas 2-Union Independent Set is known to be W[1]-hard for this parameter [13].

Acknowledgment. We are grateful to Michael Dom and Hannes Moser for earlier discussions on coil coating.

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