Stable Roommate with Narcissistic, Single-Peaked, and Single-Crossing Preferences

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Abstract. The classical Stable Roommate problem asks whether it is possible to pair up an even number of agents such that no two non-paired agents prefer to be with each other rather than with their assigned partners. We investigate Stable Roommate with complete (i.e. every agent can be matched with every other agent) or incomplete preferences, with ties (i.e. two agents are considered of equal value to some agent) or without ties. It is known that in general allowing ties makes the problem NP-complete. We provide algorithms for Stable Roommate that are, compared to those in the literature, more efficient when the input preferences are complete and have some structural property, such as being narcissistic, single-peaked, and single-crossing. However, when the preferences are incomplete and have ties, we show that being single-peaked and single-crossing does not reduce the computational complexity—Stable Roommate remains NP-complete.

1 Introduction

Given $2 \cdot n$ agents, each having a preference with regard to how suitable the other agents are as potential partners, the Stable Roommate problem asks whether it is possible to pair up the agents such that no two non-paired agents prefer to be with each other rather than with their assigned partners. We call such a pairing a *stable matching*. Stable Roommate was introduced by Gale and Shapley [17] in the 1960's and has been studied extensively since then [21–23, 31, 32]. While it is quite straightforward to see that stable matchings may not always exist, it is not trivial to see whether an existent stable matching can be found in polynomial time, even when the input preference orders are *complete*

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orders without ties (i.e. each agent can be a potential partner to each other agent, and no two agents are considered to be equally suitable as a partner). For the case without ties, Irving [21] and Gusfield and Irving [19] provided $O(n^2)$-time algorithms to decide the existence of stable matchings and to find one if it exists for complete preferences and for incomplete preferences, respectively. For the case where the given preferences may have ties, deciding whether a given instance admits a stable matching is NP-complete [31].

Solving Stable Roommate has many applications, such as pairing up students to accomplish a homework project or users in a P2P file sharing network, assigning co-workers to two-person offices, partitioning players in two-player games, or finding receiver-donor pairs for organ transplants [16, 24, 26, 28, 33, 34]. In such situations, the students, the people, or the players, which we jointly refer to as *agents*, typically have certain *structurally restricted* preferences on which other agents might be their best partners. For instance, when assigning roommates, each agent may have an ideal room temperature and may prefer to be with another agent with the same penchant. Such preferences are called *narcissistic*. Moreover, if we order the agents according to their ideal room temperatures, then it is natural to assume that each agent z prefers to be with an agent x rather than with another agent y if z's ideal temperature is closer to x's than to y's. This kind of preferences is called *single-peaked* [4, 7, 20]. Single-peakedness is used to model agents' preferences where there is a criterion, e.g. room temperature, that can be used to obtain a linear order of the agents such that each agent's preferences over all agents along this order are strictly increasing until they reach the peak—their ideal partner—and then strictly decreasing. Single-peakedness is a popular concept with prominent applications in voting contexts. It can be tested for in linear time [1, 3, 8, 13] if the input preferences are complete and have no ties. Another possible restriction on the preferences is the *single-crossing* property, which was originally proposed to model individuals' preferences on income taxation [29, 30]. It requires a linear order (the so-called single-crossing order) of the agents so that for each two distinct agents x and y, there exists at most one pair of consecutive agents (the crossing point) along the single-crossing order that disagrees on the relative order of x and y. Single-crossingness can be detected in polynomial time [5, 8, 9] if the input preferences are complete and have no ties. We refer to Bredereck et al. [6] and Elkind et al. [12] for numerous references on single-peakedness and single-crossingness.

Bartholdi III and Trick [3] studied Stable Roommate with narcissistic and single-peaked preferences. They showed that for the case with linear orders (i.e. complete and without ties), Stable Roommate *always* admits a unique stable matching and provided an $O(n)$ time algorithm to find this matching. This is remarkable since restricting the preference domain does not only guarantee the existence of stable matchings, but also speeds up finding it to sublinear time. In this specific case, this speed up implies that a stable matching can be found without "reading" the whole input preferences as long as the input is assumed to be narcissistic and single-peaked.

Table 1: Complexity of Stable Roommate for restricted domains: narcissistic, single-peaked, and single-crossing preferences. Entries marked with $^\diamondsuit$ are from Irving [21]. Entries marked with $^\spadesuit$ are from Gusfield and Irving [19]. Entries marked with $^\triangle$ are from Ronn [31]. Entries marked with $^\heartsuit$ are from Bartholdi III and Trick [3]. Entries marked with * and boldfaced are new results shown in this paper. Note that our polynomial-time results also include the existence of a stable matching and that our hardness result even holds for the more restricted tie-sensitive single-crossing property.

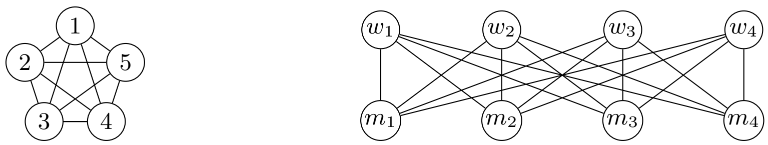
	Complete preferences		Incomplete preferences	
	w/o ties	w/ ties	w/o ties	w/ ties
no restriction	$O(n^2)^\diamondsuit$	NP-c$^\triangle$	$O(n^2)^\spadesuit$	NP-c$^\triangle$
single-peaked & single-crossing	$O(n^2)^\diamondsuit$	?	$O(n^2)^\spadesuit$	**NP-c***
narcissistic & single-peaked	$O(n)^\heartsuit$	$\boldsymbol{O(n^2)}^\star$	$O(n^2)^\spadesuit$	?
narcissistic & single-crossing	$\boldsymbol{O(n)}^\star$	$\boldsymbol{O(n^2)}^\star$	$O(n^2)^\spadesuit$	?

In this paper, we first discuss natural generalizations of the well-known single-peaked and single-crossing preferences (that were originally introduced for linear orders) for incomplete preferences with ties. Then, we investigate how some structural preference restrictions can help in guaranteeing the existence of stable matchings and in designing more efficient algorithms for finding one, including the case when the input preferences are not linear orders. We found that for complete preference orders, structurally restricted preferences such as being narcissistic and single-crossing or being narcissistic and single-peaked guarantee the existence of stable matchings. Moreover, we showed that when the preferences are complete, even with ties, narcissistic and single-crossing or narcissistic and single-peaked, then the algorithm of Bartholdi III and Trick [3] always finds a stable matching. The running time for 2· agents increases to $O(n^2)$. However, when the preferences are incomplete and ties are allowed, Stable Roommate becomes NP-complete, even if the given preferences are single-peaked as well as single-crossing. Our results on Stable Roommate, together with those from related work, are summarized in Table 1. Due to space constraints, some proofs are omitted.

2 Fundamental Concepts and Basic Observations

Let $V = \{1, 2, \ldots, 2\cdot n\}$ be a set of $2\cdot n$ agents. Each agent $i \in V$ has a *preference order* $\succeq_i$ *over a subset* $V_i \subseteq V$ *of agents* that i finds *acceptable* as a partner[3]. We note that although in our stable roommate problem, an agent cannot be matched to itself, it may still make sense to include an agent x in its preference orders, for instance when x represents someone which is very close to its ideal. The set V_i is called the *acceptable set* of i and a *preference order* $\succeq_i$ *over* V_i

[3] For technical reasons, an agent may find itself acceptable, which means that $\{i\} \subseteq V_i$.



(a) The underlying acceptability graph of a STABLE ROOMMATE instance with complete preferences, where *each* two distinct agents can be matched to each other.

(b) The underlying acceptability graph of a classical STABLE MARRIAGE instance, which is bipartite. In such an instance, each woman from the top row can only be matched with a man from the bottom row, and the converse.

Fig. 1: Acceptability graphs of two special cases of STABLE ROOMMATE.

is a *weak order on* V_i, i.e. a transitive and complete binary relation on V_i. For instance, $x \succeq_i y$ means that i weakly prefers x over y (i.e. x is better than or as good as y). We will use $\succ_i$ to denote the asymmetric part of $\succeq_i$ (i.e. $x \succeq_i y$ and $\neg(y \succeq_i x)$, meaning that i strictly prefers x to y) and $\sim_i$ to denote the symmetric part of $\succ_i$ (i.e. $x \succeq_i y$ and $y \succeq_i x$, meaning that i values x and y equally). We call an agent x a *most acceptable* agent of another agent y if for all $z \in V_y \setminus \{x, y\}$ it holds that $x \succeq_y z$. Note that an agent can have more than one most acceptable agent.

Let $X \subseteq V$ and $Y \subseteq V$ be two disjoint sets of agents and $\succeq$ be a binary relation over V. To simplify notation, we write $X \succeq Y$ to denote that for each two agents x and y with $x \in X$ and $y \in Y$ it holds that $x \succeq y$. (We use $X \succeq y$ as shortcut for $X \succeq \{y\}$ and $X \succ Y$ as well as $X \sim Y$ in an analogous way.)

To model which agent is considered as acceptable in a preference order we introduce the notion of an *acceptability graph* G for V. It is an undirected graph without loops. An edge signifies whether two distinct agents find each other acceptable. We use V to also denote the vertex set of G. There is an edge $\{i, j\}$ in G if $i \in V_j \setminus \{j\}$ and $j \in V_i \setminus \{i\}$. We assume without loss of generality that G does not contain isolated vertices, meaning that each agent could be matched to at least one other agent. We illustrate two prominent special cases of acceptability graphs in Figure 1.

Blocking pairs and stable matchings. Given a preference profile $\mathcal{P}$ for a set V of agents, a *matching* $M \subseteq E(G)$ is a subset of disjoint pairs of agents $\{x, y\}$ with $x \neq y$ (or edges in $E(G)$), where $E(G)$ is the set of edges in the corresponding acceptability graph G). For a pair $\{x, y\}$ of agents, if $\{x, y\} \in M$, then we denote $M(x)$ as the corresponding partner y; otherwise we call this pair *unmatched*. We write $M(x) = \bot$ if agent x has *no partner*, that is, if agent x is not involved in any pair in M. An unmatched pair $\{x, y\} \in E(G) \setminus M$ *is blocking* M if the pair "prefers" to be matched to each other, i.e. it holds that

$$(M(x) = \bot \vee y \succ_x M(x)) \wedge (M(y) = \bot \vee x \succ_y M(y)).$$

A matching M is *stable* if no unmatched pair is blocking M. Note that this stability concept is called *weak stability* when we allow ties in the preferences.

We refer to the textbook by Gusfield and Irving [19], Manlove [27] for two other popular stability concepts for preferences with ties.

We focus on the following stable matching problem.

Stable Roommate
Input: A preference profile $\mathcal{P}$ for a set $V = \{1, 2, \ldots, 2 \cdot n\}$ of $2 \cdot n$ agents.
Question: Does $\mathcal{P}$ admit a stable matching?

The profile given in Figure 3 admits a (unique) stable matching: $\{\{1, 4\}, \{2, 3\}\}$. In fact, as we will see in Section 3, a narcissistic and single-peaked preference profile always admits a stable matching. However, if agent 3 changes its preference order to $3 \succ 1 \succ 2 \succ 4$, then the resulting profile is not single-peaked anymore, nor does it admit any stable matching: One can check that any agent i, $1 \leq i \leq 3$ that is matched to agent 4 will form a blocking pair together with the agent that is at the third position of the preference order of i.

Preference profiles and their properties. A preference profile $\mathcal{P}$ for V is a collection $(\succeq_i)_{i \in V}$ of preference orders for each agent $i \in V$. A profile $\mathcal{P}$ may have the following three simple properties:

1. Profile $\mathcal{P}$ is *complete* if for each agent $i \in V$ it holds that $V_i \cup \{i\} = V$; otherwise it is *incomplete*.
2. Profile $\mathcal{P}$ has a *tie* if there is an agent $i \in V$ and there are two distinct agents $x, y \in V_i$ with $x \sim_i y$. Note that linear orders are exactly those orders that are complete and have no ties.
3. Profile $\mathcal{P}$ is *narcissistic* if each agent i *strictly* prefers itself to every other acceptable agent, i.e. for each $j \in V_i$ it holds that $i \succ_i j$.

We note that the completeness concept basically means that each two distinct agents can be matched together. Thus, it does not matter whether $V_i = V$ or $V_i \cup \{i\} = V$ because i cannot be matched to itself anyway. By the same reasoning, the narcissistic property alone, which reflects the fact that each agent prefers to be with someone like itself among all alternatives, does not really restrict the input of our stable roommate problem. However, one can further restrict single-peaked preferences or single-crossing preferences by additionally requiring them to be narcissistic and we show that this affects the existence of stable matchings.

As already discussed in Section 1, the single-peaked and the single-crossing properties were originally introduced and studied mainly for linear preference orders (i.e. orders without ties). For preferences with ties, a natural generalization is to think of a possible linear extension of the preferences for which the single-peaked or single-crossing property holds. We consider this variant in our paper. Profile $\mathcal{P}$ is *single-peaked* if there is a linear order $\rhd$ over V such that the preference order of each agent i is *single-peaked with respect to* $\rhd$:

$$\forall x, y, z \in V_i \text{ with } x \rhd y \rhd z \text{ it holds that } (x \succ_i y \text{ implies } y \succeq_i z).$$

Just as for the single-peaked property, the single-crossing property also requires a natural linear order of the agents, the so-called single-crossing order. However, unlike the single-peaked property which assumes that the preferences of an agent i over two agents are measured by their "distance" to the peak along the single-peaked order, the single-crossing property assumes that the agents' preferences over each two distinct agents change (cross) at most once. In fact, for preferences with ties, two natural single-crossing notions are of interest. To define them, we first introduce a notion which denotes a subset of voters that have the same preferences over two distinct agents x and y: Let $V[x \succ y] \coloneqq \{i \in V \mid x \succ_i y\}$ be the subset of voters i that strictly prefer x to y, and let $V[x \sim y] \coloneqq \{i \in V \mid x \sim_i y\}$ be the subset of voters i that find x and y to be of equal value. We say that profile $\mathcal{P}$ is *single-crossing* if there is a linear extension of $\mathcal{P}$ to a profile $\mathcal{P}' = (\succ'_1, \succ'_2, \ldots, \succ'_{2 \cdot n})$ without ties and there is a linear order $\rhd$ over V such that for each two distinct agents x and y, $\mathcal{P}'$ is *single-crossing with respect to* $\rhd$, i.e.

$$V[x \sim' y] = \emptyset \text{ and either } V[x \succ' y] \rhd V[y \succ' x] \text{ or } V[y \succ' x] \rhd V[x \succ' y].$$

We also consider a more restricted single-crossing concept which compared the single-crossing property introduced above requires that the agents that have ties are ordered in the middle. A profile $\mathcal{P}$ is called *tie-sensitive single-crossing* if there is a linear order $\rhd$ over V such that each pair $\{x, y\}$ of two distinct agents is *tie-sensitive single-crossing with respect to* $\rhd$, i.e.

$$\text{either } V[x \succ y] \rhd V[x \sim y] \rhd V[y \succ x] \text{ or } V[y \succ x] \rhd V[x \sim y] \rhd V[x \succ y].$$

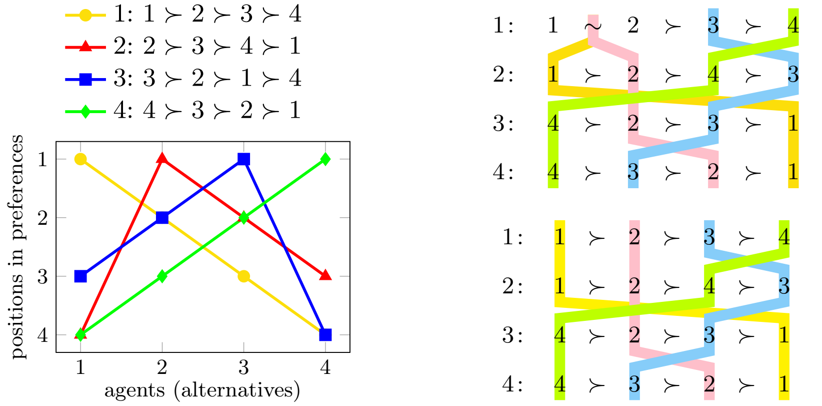
See Figure 2 for an illustration of the different types of restricted preferences for the case where the preferences are linear orders.

For partial orders, our two single-crossing concepts are *incomparable*. In particular, there are incomplete preferences with ties which are single-crossing but *not* tie-sensitive single-crossing, and the converse also holds. For weak orders and for preferences without ties, however, the following holds. (Notably, a large part of the observation can be found in a long version of Elkind et al. [11].)

Observation 1. *Let $\mathcal{P}$ be an arbitrary preference profile: (i) If $\mathcal{P}$ is complete, then $\mathcal{P}$ is single-crossing if it is tie-sensitive single-crossing. (ii) If $\mathcal{P}$ is without ties, then $\mathcal{P}$ is single-crossing if and only if it is tie-sensitive single-crossing.*

Figure 2a demonstrates that the converse of the first statement in Observation 1 does not hold.

There are many slightly different concepts of single-peakedness and single-crossingness for partial orders (a generalization of incomplete preferences with ties) [11, 15, 25]. It is known that detecting single-peakedness or single-crossingness is NP-hard for partial orders under most of the concepts studied in the literature. For linear orders, all these concepts (including ours) are equivalent to those introduced by Black [4] and Mirrlees [29] and can be detected in polynomial time [1, 3, 5, 8, 9, 13]. For incomplete preferences with ties, Lackner [25] showed that detecting single-peakedness is NP-complete. For complete preferences with ties, while Elkind et al. [11] showed that detecting single-crossingness is NP-complete, Fitzsimmons [14] and Elkind et al. [11] provided polynomial-time algorithms for detecting single-peakedness and ties-sensitive single-crossingness. All these known hardness results seem to hold only when the preferences have ties. However, we observe that the hardness proof for Corollary 6 by Elkind et al.



(a) A Stable Roommate instance with narcissistic and single-peaked preferences. They are *not* single-crossing since $\{1, 4\}$ forces the two agents 1 and 3 (resp. 2 and 4) to be ordered next to each other in a single-crossing order whereas $\{2, 3\}$ forces the two agents 1 and 2 (resp. 3 and 4) to be ordered next to each other. All these four conditions, however, cannot be satisfied by a linear order.

(b) Top: A Stable Roommate instance with single-crossing preferences. They are *not* tie-sensitive single-crossing since $\{2, 3, 4\}$ implies that $1 \rhd 2 \rhd 3 \rhd 4$ and its reverse are the only possible single-crossing orders. But, $\{1, 2\}$ is not tie-sensitive single-crossing wrt. either $\rhd$ or its reverse. Bottom: A possible linear extension, showing single-crossingness.

Fig. 2: Visualization of different restricted profiles.

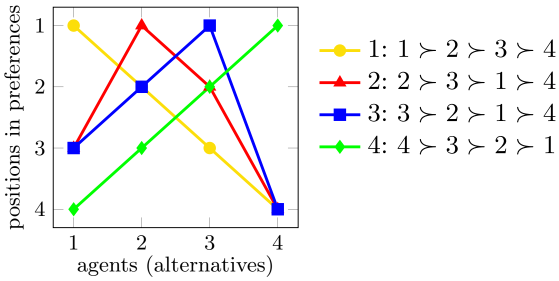
[11] indeed can be adapted to show NP-completeness for deciding whether an incomplete preference profile without ties is single-peaked or single-crossing.

Observation 2. *Deciding whether an incomplete preference profile without ties is single-crossing (or equivalently tie-sensitive single-crossing) or single-peaked is NP-complete.*

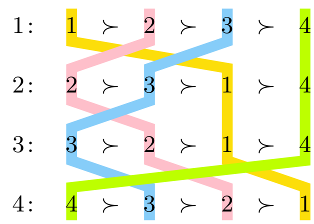
Barberà and Moreno [2] as well as Elkind et al. [10] noted that for complete preferences without ties, narcissistic and single-crossing preferences are also single-peaked. We show that the relation also holds when ties are allowed. We note that Barberà and Moreno [2] also considered complete preferences with ties. However, their single-crossingness for the case with ties only resembles our tie-sensitive single-crossing definition, which is a strict subset of our single-crossingness (Observation 1).

Proposition 1. *If a complete, even with ties, and narcissistic preference profile $\mathcal{P}$ has a single-crossing order $\rhd$, then this order $\rhd$ is also a single-peaked order.*

Proof. Suppose for the sake of contradiction that $\rhd$ with $a_1 \rhd a_2 \rhd \cdots \rhd a_{2 \cdot n}$ is not single-peaked. This means that there exists an agent a_i that is not single-peaked



(a) A single-peaked visualization



(b) A single-crossing visualization.

Fig. 3: A narcissistic, single-peaked, and single-crossing profile.

wrt. $\rhd$, and there are three agents a_j, a_k, a_ℓ with $j < k < \ell$ such that $a_j \succ_{a_i} a_k$ and $a_\ell \succ_{a_i} a_k$. Together with the narcissistic property, the following holds:

agent a_i: $a_i \succ_{a_i} a_j \succ_{a_i} a_k$ and $a_i \succ_{a_i} a_\ell \succ_{a_i} a_k$, agent a_j: $a_j \succ_{a_j} a_k$,
agent a_k: $a_k \succ_{a_k} a_j$ and $a_k \succ_{a_k} a_\ell$, agent a_ℓ: $a_\ell \succ_{a_\ell} a_k$.

On the one hand, the agents' preferences over the pair $\{a_j, a_k\}$ implies that $i < k$. On the other hand, the pair $\{a_k, a_\ell\}$ implies that $i > k$—a contradiction. □

The profile shown in Figure 3 is narcissistic and single-crossing wrt. the order $1 \rhd 2 \rhd 3 \rhd 4$ and it is also single-peaked with respect to the same order $\rhd$.

3 Complete Preferences

In this section, we consider profiles with complete preferences. It is known that if ties do not exist, then Stable Roommate can be solved in $O(n^2)$ time [21], while the existence of ties makes the problem NP-hard [31]. For the case of complete, narcissistic, and single-peaked preferences without ties, Bartholdi III and Trick [3] showed that Stable Roommate is even solvable in $O(n)$ time. Their algorithm is based on the following two facts (referred to as Propositions 2 and 3) that are related to the concept of most acceptable agents. We show that the facts transfer to the case with ties.

Proposition 2. *If the given preference profile $\mathcal{P}$ is complete (even with ties), narcissistic, and single-peaked, then there are two distinct agents i, j that are each other's most acceptable agents.*

Proof. The statement for complete, narcissistic, and single-peaked preferences without ties was shown by Bartholdi III and Trick [3]. It turns out that this also holds for the case when ties are allowed. Let V be the set of all $2 \cdot n$ agents and consider a single-peaked order $\rhd$ of the agents V with $x_1 \rhd x_2 \rhd \cdots \rhd x_n$. For each agent $x \in V$, let M_x be the set of all most acceptable agents of x. Towards a contradiction, suppose that each two distinct agents x and y have $x \notin M_y$

Algorithm 1: The algorithm of Bartholdi III and Trick [3] for computing a stable matching with input $\mathcal{P}$ being complete, narcissistic, and single-peaked.

```
M ← ∅;
while P ≠ ∅ do
    Find two agents x, y in P that consider each other as most acceptable;
    Delete x and y from profile P;
    M ← M ∪ {x, y};
return M;
```

or $y \notin M_x$. By the narcissistic property and single-peakedness, each $M_x \cup \{x\}$ forms an interval in $\rhd$. This implies that the first agent x_1 and the last agent x_n in the order $\rhd$ have $x_2 \in M_{x_1}$ and $x_{n-1} \in M_{x_n}$. By our assumption, however $x_2 \in M_{x_1}$ implies that for each $i \in \{2, \ldots, n\}$ the following holds: $x_{i-1} \notin M_x$—a contradiction to $x_{n-1} \in M_{x_n}$. □

By the stability definition, we have the following for complete preferences.

Proposition 3. *Let $\mathcal{P}$ be a preference profile and let M be a stable matching for $\mathcal{P}$. Let $\mathcal{P}'$ be a preference profile resulting from $\mathcal{P}$ by adding two agents x, y who are each other's most acceptable agents (and the preferences of other agents over x, y are arbitrary but fixed). Then, matching $M \cup \{\{x, y\}\}$ is stable for $\mathcal{P}'$.*

Proof. Suppose for the sake of contradiction that $M \cup \{\{x, y\}\}$ is not stable for $\mathcal{P}'$. This means that $\mathcal{P}'$ has an unmatched blocking pair $\{u, w\} \notin M$. It is obvious that $|\{u, w\} \cap \{x, y\}| = 1$ as otherwise $\{u, w\}$ would also be an unmatched blocking pair for $\mathcal{P}$. Assume without loss of generality that $u = x$. Then, by the definition of blocking pairs, it must hold that $w \succ_x y$—a contradiction to y being one of the most acceptable agents of x. □

Utilizing Propositions 2 and 3 (in more restricted variants), Bartholdi III and Trick [3] derived an algorithm to construct a *unique* stable matching when the preferences are linear orders (i.e. complete and without ties) and are narcissistic and single-peaked (see Algorithm 1). For $2 \cdot n$ agents their algorithm runs in $O(n)$ time. We will show that Algorithm 1 also works when ties are allowed. The stable matching, however, may not be unique anymore and the running time is $O(n^2)$ since we need to update the preferences of each agent after we match one pair of two agents.

Theorem 1. *Algorithm 1 finds a stable matching for profiles with $2 \cdot n$ agents that are complete, with ties, narcissistic and single-peaked in $O(n^2)$ time.*

Proof. The correctness follows directly from Propositions 2 and 3 and the narcissistic and single-peaked property is preserved when deleting any agent. As for the running time, there are n rounds to build up M, and in each round we find two distinct agents x and y whose most acceptable agent sets M_x and M_y

include each other: $x \in M_y$ and $y \in M_x$. Note that Proposition 2 implies that such two agents exist. After each round we need to update the most acceptable agents of at most $2 \cdot n$ agents. Thus, in total the running time is $O(n^2)$. □

Now, we move on to (tie-sensitive) single-crossingness.

Corollary 1. *Algorithm 1 finds a stable matching for preference profiles with $2 \cdot n$ agents that are complete, with ties, narcissistic and single-crossing (or tie-sensitive single-crossing) in $O(n^2)$ time. The running time for the case without ties is $O(n)$.*

Proof. By Proposition 1 and Observation 1 (i), the stated profiles are single-peaked. The result of Bartholdi III and Trick [3] and Theorem 1 imply the desired statement. □

4 Incomplete Preferences

Incomplete preferences mean that some agents do not appear in the preferences of an agent, for instance, because two agents are unacceptable to each other or they are not "allowed" to be matched to each other. If in this case no two agents are considered of equal value by any agent (i.e. the preferences are without ties), then Stable Roommate still remains polynomial-time solvable [19]. However, once ties are involved, Stable Roommate becomes NP-complete [31] even for complete preferences. In this section, we consider the case where the input preferences may be narcissistic, single-peaked, or single-crossing. First of all, we note that these preference restrictions can no longer guarantee the existence of two consecutive agents that are each other's most acceptable agent. However, this guarantee is crucial for the existence of a stable matching and for why the algorithm by Bartholdi III and Trick [3] can work in time linear in the number of agents. Moreover, for incomplete preferences, even without ties, narcissistic and single-crossing preferences do not imply single-peakedness anymore.

Proposition 4. *For incomplete preferences without ties, the following holds: Narcissistic and single-crossing preferences are* not *necessarily single-peaked. Narcissistic and single-peaked (resp. single-crossing) preferences guarantee neither the uniqueness nor the existence of stable matchings.*

Proof. Consider the following profile with six agents $1, 2, \ldots, 6$:

agent 1: $1 \succ_1 5 \succ_1 6$, agent 3: $3 \succ_3 5 \succ_3 6$, agent 5: $5 \succ_5 1 \succ_5 2 \succ_5 3 \succ_5 4$,
agent 2: $2 \succ_2 5 \succ_2 6$, agent 4: $4 \succ_4 5 \succ_4 6$, agent 6: $6 \succ_6 4 \succ_6 2 \succ_6 3 \succ_6 1$.

It is single-crossing wrt. the order $1 \rhd 2 \rhd \cdots \rhd 6$, but it is not single-peaked because of the last two agents' preference orders over $1, 2, 3, 4$. It does not admit a stable matching of size three. But it admits a stable matching of size two: $\{\{1, 5\}, \{4, 6\}\}$.

The following profile with four agents $1, 2, 3, 4$ is narcissistic and single-peaked wrt. the order $1 \rhd 2 \rhd 3 \rhd 4$, and single-crossing wrt. the order $1 \rhd' 3 \rhd' 2 \rhd' 4$. It admits two different stable matchings $\{\{1, 2\}, \{3, 4\}\}$ and $\{\{1, 3\}, \{2, 4\}\}$.

agent 1: $1 \succ_1 2 \succ_1 3 \succ_1 4$, agent 2: $2 \succ_2 4 \succ_2 1$,
agent 3: $3 \succ_3 1 \succ_3 4$, agent 4: $4 \succ_4 3 \succ_4 2 \succ_4 1$.

The following profile with ten agents $1, 2, \ldots, 10$ is narcissistic and single-peaked wrt. the order $4 \rhd 2 \rhd 1 \rhd 3 \rhd 5 \rhd 9 \rhd 7 \rhd 6 \rhd 8 \rhd 10$. But, no matching M is stable for this profile: First, the agents can be partitioned into two subsets $V_1 = \{1, 2, \ldots, 5\}$ and $V_2 = \{6, 7, \ldots, 10\}$ such that only agents within the same subset can be matched together. Since $|V_1|$ is odd, at least one agent $i \in V_1$ is not matched by M. But, agent i and the agent at the third position of the preference order of i would form a blocking pair.

agent 1: $1 \succ_1 4 \succ_1 3$, agent 2: $2 \succ_2 5 \succ_2 4$, agent 3: $3 \succ_3 1 \succ_3 5$,
agent 4: $4 \succ_4 2 \succ_4 1$, agent 5: $5 \succ_5 3 \succ_5 2$, agent 6: $6 \succ_6 9 \succ_6 8$,
agent 7: $7 \succ_7 10 \succ_7 9$, agent 8: $8 \succ_8 6 \succ_8 10$, agent 9: $9 \succ_9 7 \succ_9 6$,
agent 10: $10 \succ_{10} 8 \succ_{10} 7$. □

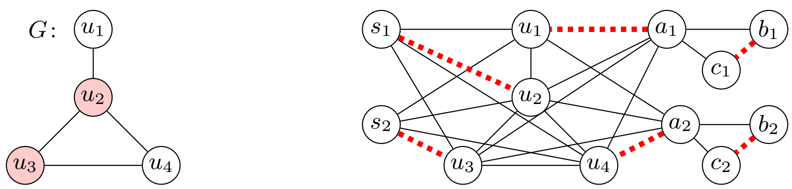
For the case with ties allowed, Ronn [31] showed that Stable Roommate becomes NP-hard even if the preferences are complete. The constructed instances in his hardness proof, however, are not always single-peaked or single-crossing. It is even not clear whether the problem remains NP-hard for this restricted case. If we abandon the completeness of the preferences, then we obtain NP-hardness, by another and simpler reduction. Before we state the corresponding theorem, we prove the following lemma which is heavily used in our preference profile construction to force two agents to be matched together.

Lemma 1. *Let $\mathcal{P}$ be a* Stable Roommate *instance for a given voter set V, and let a, b, and c be three distinct agents with the following preferences:*
agent a: $X \succ b \succ c \succ V_a \setminus (X \cup \{b, c\})$,
agent b: $c \succ a \succ V_b \setminus \{a, c\}$, *agent c:* $a \succ b \succ V_c \setminus \{a, b\}$,
where $X \subseteq (V_a \cap V_b \cap V_c) \setminus \{a, b, c\}$ is a non-empty subset. Then, every stable matching M for $\mathcal{P}$ must fulfill that (i) $M(a) \in X$ and (ii) $\{b, c\} \in M$.

Proof. Assume towards a contradiction to (i) that $\mathcal{P}$ admits a stable matching M with $M(a) \notin X$. There are three cases: (1) $M(a) = b$, implying the blocking pair $\{b, c\}$, (2) $M(a) = c$, implying the blocking pair $\{a, b\}$, and (3) $M(a) \notin \{b, c\}$, implying the blocking pair $\{a, c\}$. Thus, a must be matched with some agent from X. For (ii), statement (i) implies that c cannot be matched with a. Now, if $\{b, c\} \notin M$, then $\{b, c\}$ is a blocking pair. □

Theorem 2. Stable Roommate *for incomplete preferences with ties remains NP-complete, even if the preferences are single-peaked and single-crossing or single-peaked and tie-sensitive single-crossing.*

Proof. First, the problem is in NP since one can non-deterministically guess a matching and check the stability in polynomial time. To show NP-hardness, we reduce from the NP-complete Vertex Cover problem [18], which given an undirected graph $G = (U, E)$ and a non-negative integer k, asks whether there is a size-at-most k *vertex cover*, i.e. a subset $U' \subseteq U$ of size at most k such that



(a) The graph of a VERTEX COVER instance $(G, k = 2)$. The instance is a yes-instance and admits a vertex cover $\{u_2, u_4\}$, marked in light red.

(b) The acceptability graph of the corresponding STABLE ROOMMATE instance. It admits a stable matching, marked by thick dotted lines.

Fig. 4: An illustration of the hardness reduction for Theorem 2.

for each edge $e \in E$, it holds that $e \cap U' \neq \emptyset$. Let $(G = (U, E), k)$ be a VERTEX COVER instance with $p := |U|$. We assume w.l.o.g. that $k < p$. We will construct a STABLE ROOMMATE instance $\mathcal{P}$ with agent set V and show that G has a vertex cover of size at most k if and only if $\mathcal{P}$ admits a stable matching.

Main idea and the constructed agents. To explain the main idea of the reduction, we first describe the agent set V and the corresponding acceptability graph of $\mathcal{P}$ as illustrated through an example in Figure 4. For each vertex $u_i \in U$, we introduce a *vertex agent* u_i (for the sake of simplicity, we use the same symbol for the vertex and the corresponding agent). Additionally, there is a set of *selector agents* $S := \{s_1, \ldots, s_k\}$ as well as three sets of *collector agents* $A := \{a_1, a_2, \ldots, a_{p-k}\}$, $B := \{b_1, b_2, \ldots, b_{p-k}\}$, and $C := \{c_1, c_2, \ldots, c_{p-k}\}$. The agent set V is defined as $U \cup S \cup A \cup B \cup C$. For the acceptability graph, we have that every vertex agent u_i accepts every selector agent from S, every collector agent from A, and every vertex agent u_j that corresponds to a neighbor of u_j in the input graph G. For each $i \in \{1, 2, \ldots, p-k\}$, the collector agents a_l, b_i, and c_i pairwisely accept each other. We aim at constructing the agents' preferences such that in every stable matching only the selector agents from S and the collector agents from A can be matched to the vertex agents and the vertex agents matched to the selector agents correspond to a vertex cover (of size $|S| = k$). This property is given by the subsequent Claim 1.

Agent preferences. Now, we describe the preferences that realize the idea and the acceptability graph as described above:

$$\begin{array}{lll} \text{agent } u_i\colon & [S] \succ [N(u_i)] \succ a_1 \succ a_2 \succ \ldots \succ a_{p-k} & \forall 1 \leq i \leq p, \\ \text{agent } s_i\colon & u_1 \sim u_2 \sim \ldots \sim u_p & \forall 1 \leq i \leq k, \end{array}$$

$$\begin{array}{llll} \text{agent } a_i\colon & [U] \succ b_i \succ c_i, & & \\ \text{agent } b_i\colon & c_i \succ a_i, & \text{agent } c_i\colon \; a_i \succ b_i & \forall 1 \leq i \leq p-k. \end{array}$$

Herein, for each subset $X \subset S \cup U$, we denote by $[X]$ some arbitrary but fixed order (e.g. ordered wrt. the names or the indices), called the *canonical order*. This completes the construction and can clearly be performed in polynomial time.

Correctness of the construction. First of all, we claim the following:

Claim 1. *Every stable matching M for $\mathcal{P}$ satisfies the following two properties:*
1. *every vertex agent u_i is matched to either a selector agent from S or a collector agent from A: $M(u_i) \in S \cup A$, and*
2. no *two vertex agents that are both matched to a collector agent are adjacent.*

Proof (of Claim 1). Let M be a stable matching for $\mathcal{P}$. For the first statement, Lemma 1 immediately implies that for every collector agent $a_i \in A$, it holds that $M(a_i) \in U$. Thus, there are exactly k vertex agents left that are not matched to agents from A. Suppose towards a contradiction that some selector agent s_j is not matched to any vertex agent, implying that at least one vertex agent u_i is left with $M(u_i) \notin A \cup S$. This, however, implies that $\{s_j, u_i\}$ is a blocking pair for M—a contradiction. For the second statement, suppose towards a contradiction that there are two vertex agents u_i, u_j with $\{M(u_i), M(u_j)\} \subseteq A$ as well as $\{u_i, u_j\} \in E$. The preference orders of u_i and u_j immediately imply that agents u_i and u_j form a blocking pair—a contradiction. (of Claim 1) $\diamond$

Now, we show that G has a vertex cover of size at most k if and only if $\mathcal{P}$ admits a stable matching. The "if" part follows immediately from Claim 1. For the "only if" part, suppose that $U' \subseteq U$ is a vertex cover of size k. Without loss of generality, assume that $U' = \{u_1, u_2, \ldots, u_k\}$ and further assume that the canonical order is $u_1 \succ u_2 \succ \cdots \succ u_n$. It is easy to verify that the following matching M is stable:
- for each $i \in \{1, 2, \ldots, k\}$ set $M(u_i) \coloneqq s_i$;
- for each $i \in \{1, 2, \ldots, p-k\}$ set $M(u_{i+k}) = a_i$;
- for each $i \in \{1, 2, \ldots, p\}$ set $M(b_i) = c_i$.

Single-peakedness and (tie-sensitive) single-crossingness. The constructed profile is single-peaked with respect to the following linear order $\rhd$:

$$[S] \rhd [U] \rhd a_1 \rhd a_2 \rhd \cdots \rhd a_{p-k} \rhd b_1 \rhd b_2 \rhd \cdots \rhd b_{p-k} \rhd c_1 \rhd c_2 \rhd \cdots \rhd c_{p-k}.$$

It is also single-crossing, since each preference order (after resolving all ties in favor of the canonical order as discussed when constructing the agent preferences) is a sub-order of one of two different preference orders, and two preference orders are always single-crossing. More specifically, the profile is single-crossing with respect to the order $\rhd$: After resolving all ties in the preferences of the selector agents in favor of the canonical order, the preference orders of the agents from $S \cup U \cup A$ are sub-orders of the linear order $[S] \succ [U] \succ a_1 \succ b_1 \succ c_1 \succ a_2 \succ b_2 \succ c_2 \succ \cdots \succ a_{p-k} \succ b_{p-k} \succ c_{p-k}$, and the preference orders of the agents from $B \cup C$ are sub-orders of the linear order $[S] \succ [U] \succ c_1 \succ a_1 \succ b_1 \succ c_2 \succ a_2 \succ b_2 \succ \cdots \succ c_{p-k} \succ a_{p-k} \succ b_{p-k}$.

The tie-sensitive single-crossing property also holds because ties only occur between pairs of agents from U and $\rhd$ contains first all agents with ties and then the agents with the same canonical order among agents from U. $\square$

The constructed profile in the proof of Theorem 2 cannot be extended to also satisfy the narcissistic property. However, we conjecture that Stable Roommate remains NP-complete even if the input preferences are also narcissistic.

5 Conclusion

We investigated Stable Roommate for preferences with popular structural properties, such as being narcissistic, single-peaked, and single-crossing. We showed the existence of stable matchings and managed to speed up the detection of such matchings when the preferences are complete, narcissistic, and single-peaked (or single-crossing). Some of the speed-up (Corollary 1) is even associated with a sublinear time algorithm. For incomplete preferences with ties, however, single-peakedness combined with single-crossingness does not help to lower the computational complexity—Stable Roommate remains NP-complete.

We conclude with some challenges for future research. First, considering the NP-completeness result, it would be interesting to study the parameterized complexity with respect to the "degree" of incompleteness of the input preferences, such as the number of ties or the number of agents that are in the same equivalence class of the tie-relation. Second, we were not able to settle the computational complexity for complete preferences that are also single-peaked and single-crossing and for incomplete preferences with ties that are also narcissistic and single-peaked. We conjecture, however, that the NP-hardness reduction by Ronn [31] can be (non-trivially) adjusted to also work for these restricted domains. Third, for incomplete preferences, we extended the concepts of single-peaked and single-crossing preferences. However, there are further relevant extensions in the literature [11, 15, 25], which deserve study within our framework. Finally, the algorithm of Bartholdi III and Trick [3] strongly relies on the fact that there are always two agents that consider each other most acceptable. It would be interesting to know which generalized structured preferences could guarantee this fact. For instance, the so-called worst-restricted property (i.e. no three agents exist such that each of them is least preferred by any agent) is a generalization of the single-peaked property. We could show that the narcissistic and worst-restricted properties are enough to guarantee this useful property.

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