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Hamiltonicity and the computational complexity of graph problems

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Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und eigenhändig sowie ohne unerlaubte fremde Hilfe und ausschließlich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe.

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Zusammenfassung

In der vorliegenden Arbeit untersuchen wir die algorithmische Komplexität klassischer NP-vollständiger Graphprobleme auf Subklassen Hamiltonischer Graphen. Die betrachteten Subklassen sind k-reguläre Graphen, dies sind Graphen, für welche jeder Knoten Grad k hat, und „k-ordered" Graphen, dies sind Graphen, die für jedes k-Tupel von Knoten einen Kreis enthalten, der diese in der gegebenen Reihenfolge durchläuft. Die von uns untersuchten Probleme sind Feedback Vertex Set, 3-Coloring, Independent Set, Clique und Treewidth. Es stellt sich heraus, dass die betrachteten Einschränkungen keinen wesentlichen Einfluss auf die algorithmische Komplexität der Probleme haben. Wir beweisen, dass alle genannten Probleme für jedes $k \geq 3$ auf „k-ordered" Hamiltonischen Graphen NP-vollständig bleiben. Desweiteren entscheiden wir für alle genannten Probleme bis auf Treewidth, und für jedes $k \geq 3$, ob sie auf k-regulären Hamiltonischen Graphen NP-vollständig bleiben oder polynomzeitlösbar sind, .

Abstract

In this thesis we analyze the computational complexity of classical NP-complete graph problems on subclasses of Hamiltonian graphs. The subclasses we study are k-regular graphs, which are graphs where every vertex has degree k, as well as k-ordered graphs, which are graphs that, given a k-tuple of vertices, contain a cycle that visits the vertices in that given order. The problems we analyze are Feedback Vertex Set, 3-Coloring, Independent Set, Clique and Treewidth. It turns out that the considered restrictions do not have an essential influence on the tractability of the mentioned problems. For all of the mentioned problems, we prove that they remain NP-complete when restricted to k-ordered Hamiltonian graphs, for every $k \geq 3$. Furthermore we decide for each of these problems except Treewidth, and for every $k \geq 3$, whether they remain NP-complete or become polynomial-time solvable when restricted to k-regular Hamiltonian Graphs.

Contents

1 Intro

In computational complexity theory, it is natural to study intractable graph problems restricted to finer subclasses. Hamiltonian graphs, which are graphs that contain a cycle passing through all their vertices, form a well-known and extensively studied graph class. This motivates our research towards understanding the influence of hamiltonicity on the tractability of graph problems. Problems involving Hamiltonian graphs can be traced back to the 9^{th} century, namely the *Knight's tour problem*, which is roughly speaking the problem of finding a Hamilton-cycle in a specific graph [Wil88]. During the past few decades, there have been many results on Hamiltonian graphs, and several sufficient and necessary conditions for Hamilton-cycles in graphs have been shown [CE72; DeL00; Dir52; HN65; Mey73]. Broersma [Bro02] published a survey gathering many results on Hamiltonian graphs such as open problems and conjectures in Hamiltonian graph theory. Hamiltonian graphs turn out to be of special interest in usability testing for computer systems, and arise from reliability considerations in network design [Wal17; WW84]. Moreover Hamilton-cycles play a fundamental role in the extensively studied and well-known Travelling Salesman Problem, as well as other folklore problems [BM+76; BN68].

In this thesis we focus on two major subclasses of Hamiltonian graphs. The first one is the class of *k-regular* Hamiltonian graphs, that are Hamiltonian graphs for which every vertex is of degree k. Regular graphs form a popular and broadly studied graph class, for which many subclasses, as for example cubic graphs, turn out to be of particular interest in several fields of graph theory and computer sciences [GP95]. The restriction to subclasses of regular graphs seems to be prevalent in computational complexity theory; there have been a large number of results for a broad and diverse collection of problems since the introduction of NP-completeness [FSS10; GJ90; LG83]. The second class we analyze is given by *k-Hamiltonian ordered* graphs, which are graphs that, for any k-tuple of vertices, contain a Hamilton-cycle with respect to the ordering given by the k-tuple. These graphs were introduced by Ng and Schultz [NS97] as a new strong Hamiltonian property that, by the time, have gained popularity. There have been several results on sufficient degree conditions and other properties of k-Hamiltonian ordered graphs [CGP04; KSS99]. Faudree [Fau01] published a survey gathering many results on k-Hamiltonian ordered graphs and highlighting relations to other well-known graph-classes. As our goal is to study the influence of hamiltonicity on the tractability of problems, studying the computational complexity of graph problems restricted to graphs with a strong Hamiltonian property seems natural.

Our Contributions.

	F.V. Set	3-Coloring	Ind. Set	Clique	Treewidth
Hamiltonian	NP-c	NP-c	NP-c	NP-c	NP-c
planar	NP-c	NP-c	NP-c	p-time	*unknown*
3-regular (planar)	p-time	p-time	NP-c	p-time	*unknown*
4-regular	NP-c	NP-c	NP-c	p-time	*unknown*
4-regular (planar)	NP-c	*unknown*	NP-c	p-time	*unknown*
5-regular planar	*unknown*	*unknown*	NP-c	p-time	*unknown*
k-regular, $k \geq 5$	NP-c	NP-c	NP-c	p-time	*unknown*
k-Ham. ordered, $k \geq 3$	NP-c	*unknown*	NP-c	NP-c	NP-c
k-ordered $k \geq 3$	NP-c	NP-c	NP-c	NP-c	NP-c
k-connected $k \in \mathbb{N}$	NP-c	NP-c	NP-c	NP-c	NP-c

Table 1.1: The Table surveying our results. Every mentioned graph class is also a subclass of Hamiltonian graphs if not mentioned. The problem-names are written in obvious abbreviations. NP-c stands for NP-complete, and p-time for polynomial-time solvable. For problems restricted to graph classes for which we do not know if they remain NP-complete or become polynomial-time solvable, we marked the respective cell in the table as *unknown*.

The problems that we will analyze throughout this thesis are the following:

- Feedback Vertex Set, that is, given a graph G and some integer k, decide whether there is a set of size k whose deletion leaves G acyclic.
- 3-Coloring, that is, given a graph G, decide whether G can be colored with three colors, such that no two vertices adjacent by an edge share a color.
- Independent Set, that is, given a graph G and some integer k, decide whether there is a set of size k, such that no two vertices in the given set are adjacent by an edge in G.
- Clique, that is, given a graph G and some integer k, decide whether there is a clique of size k.
- Treewidth, that is, given a graph G and some integer k, decide whether there is tree-decomposition of G with width k.

These are all graph-theoretic problems with many applications, making the analysis of their computational complexity crucial for the optimization of algorithm implementations. We will motivate and analyze each of the mentioned problems in their respective section.

In this thesis we prove that all of the mentioned problems remain NP-complete when restricted to Hamiltonian graphs. For every problem except 3-Coloring, we prove that they remain NP-complete on k-Hamiltonian ordered graphs, for every $k \geq 3$. Furthermore, we decide for each of the problems except for Treewidth, whether they remain

NP-complete or become polynomial-time solvable on k-regular Hamiltonian graphs, for every $k \geq 3$. A summary of our results is depicted in Table 1.1. As special cases, we prove that Feedback vertex set remains NP-complete when restricted to planar 4-regular Hamiltonian graphs, and Independent Set remains NP-complete when restricted to planar 5-regular Hamiltonian graphs. All of our NP-hardness proves will be done via polynomial-time many-one reductions.

Related work. As mentioned above, there have been many results on the computational complexity of several graph-problems restricted to Hamiltonian graphs. Fleischner, Sabidussi, and Sarvanov [FSS10] proved that independent set remains NP-complete when restricted to 3-regular planar Hamiltonian graphs, as well as to 4-regular planar Hamiltonian graphs. Fleischner and Sabidussi [FS03] proved that 3-coloring, remains NP-complete on 4-regular Hamiltonian graphs. A result due to Brooks [Bro41] proves that 3-Coloring is polynomial-time solvable on 3-regular Hamiltonan graphs. We are not aware of known results regarding the computational complexity of Feedback vertex set, Clique and treewidth on Hamiltonian graphs or subclasses thereof. However, Speckenmeyer [Spe88] shows that Feedback vertex set remains NP-complete on planar graphs of maximum degree four. Clique, as well as treewidth, have been proven to be NP-complete on general graphs by Karp [Kar72] and Arnborg, Corneil, and Proskurowski [ACP87] respectively.

2 Preliminaries

Graph theory. Let $G = (V, E)$ denote an undirected graph, where V denotes the set of vertices and $E \subseteq \{\{v, w\} \mid v, w \in V,\ v \neq w\}$ denotes the set of edges. We will also write $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G respectively. The order of any graph G will be denoted by $n_G := |V(G)|$ if not stated otherwise. For a vertex $v \in V(G)$, its *degree* is defined as $\deg(v) = |\{\{v, w\} \mid \{v, w\} \in E(G)\}|$. A *subgraph* $H \subseteq G$ is a graph for which it holds true that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $V' \subseteq V(G)$ be a set of vertices. Then the *induced subgraph* (on V'), denoted by $G[V']$, is defined by $V(G[V']) := V'$ and $E(G[V']) = \{\{u, v\} \mid \{u, v\} \in E(G) \text{ and } u, v \in V'\}$. Let $H \subseteq G$, then $G - H$ denotes the graph $G[V(G) \setminus V(H)]$. Let $V' \subseteq V(G)$, then we define $G - V' := G[V(G) \setminus V']$. Let $E' \subseteq E(G)$ and $V' := V(G) \setminus \{v, w \mid \{v, w\} \in E'\}$, then we define $G - E' := G[V']$.

For two graphs G_1 and G_2, the *union of two graphs* $G := G_1 \cup G_2$ is defined by $V(G) := V(G_1) \cup V(G_2)$ and $E(G) := E(G_1) \cup E(G_2)$. If furthermore $V(G_1) \cap V(G_2) = \emptyset$, then we write $G = G_1 \uplus G_2$ and call G the *disjoint union* of G_1 and G_2. Analogously we define $V(G_1) \uplus V(G_2)$ as the disjoint union of two vertex sets.

Let G denote a graph of order n_G. Then a bijective function $\Phi : \{0, \ldots, n_G - 1\} \to V(G)$ denotes an *enumeration* of V, where $\Phi(i) := v_{\sigma(i)}$ for some permutation $\sigma : \{0, \ldots, n_G - 1\} \to \{0, \ldots, n_G - 1\}$. For the sake of readability we will write $\Phi = (v_{\sigma(0)}, \ldots, v_{\sigma(n_G-1)}) \in V^{n_G}$, where again $\Phi(i) := v_{\sigma(i)}$ for every $i \in \{0, \ldots, n_G - 1\}$. By writing $V(G) = \{v_0, \ldots, v_{n_G-1}\}$ we implicitly give an enumeration of the vertices in G.

Let G be a graph of order n_G. Then a *path* $\mathcal{P} = (V_P, E_P)$, is a subgraph of G such that there is an enumeration $\Phi = (v_0, \ldots, v_k)$ of $\mathcal{P}$ for some $k < n_G$, satisfying $E_P = \{\{v_i, v_{(i+1)}\} \mid i \in \{0, \ldots, k - 1\}\}$. Subsequently we will write $\mathcal{P} = (v_0, \ldots, v_k)$ to denote a path from v_0 to v_k in G, where the v_i, v_j are pairwise disjoint for $i \neq j$. Its length is defined as $|P| := |V(P)| = k + 1$. If a vertex v is part of the path, we may write $v \in P$ instead of $v \in V(P)$. For two vertex disjoint paths $\mathcal{P}_1 = (v_0, \ldots, v_k)$ and $\mathcal{P}_2 = (v_j, \ldots, v_m)$, with $\{v_k, v_j\} \in E(G)$ and $k, j, m \in \{0, \ldots, n_G - 1\}$, we define the *concatenation of paths* as $\mathcal{P} := \mathcal{P}_1 \diamond \mathcal{P}_2 = (v_0, \ldots, v_k, v_j, \ldots, v_m)$. Note that this is again a path. Let $\mathcal{P}_3 = (v_k, \ldots, v_m)$ be another path such that $V(P_3) \cap V(P_1) = \{v_k\}$. Then, as an extension to the definition of *concatenation of paths*, we define $\mathcal{P}_1 \diamond \mathcal{P}_3 := (v_0, \ldots, v_k, \ldots, v_m)$. As an abbreviation, a single vertex v_k can be seen as a path of length 1. Given a path $\mathcal{P} = (v_0, \ldots, v_k)$ and two vertices $v_j, v_l \in P$ with $j > l$, we write $(v_j, \ldots, v_l)$ to denote the „sub-path“ $(v_j, \ldots, v_l)$.

A *cycle* $\mathcal{C} = (V_{\mathcal{C}}, E_{\mathcal{C}})$ is a subgraph of G such that there is an enumeration $\Phi = (v_0, \ldots, v_k)$ of $V_{\mathcal{C}}$ for some $k < n_G$, satisfying $E_{\mathcal{C}} = \{\{v_i, v_{(i+1)}\} \mid i \in \{0, \ldots, k - 1\}\} \cup \{v_k, v_0\}$. We will refer to Φ as the *enumeration induced by* $\mathcal{C}$. Subsequently, for the sake of readability, a cycle will be denoted as $\mathcal{C} = (v_0, \ldots, v_k, v_0)$, $k \leq n_G - 1$. The length of $\mathcal{C}$ is defined as $|C| = k + 1$. We denote by $\mathcal{P}^{\mathcal{C}} = (v_0, \dots, v_{n_G-1})$ the *path induced by the cycle* $\mathcal{C} = (v_0, \dots, v_{n_G-1}, v_0)$.

Let $G = (V, E)$ be a graph of order n_G. Then G is called *Hamiltonian*, if there is a cycle $\mathcal{C}$ with $V(\mathcal{C}) = V(G)$. A cycle $\mathcal{C}$ with the property that $V(\mathcal{C}) = V(G)$ will be referred to as *Hamilton-cycle*. If the Hamilton-cycle is part of the input, or was computed explicitly, we say that G is a Hamiltonian graph *with known Hamilton-cycle*.

A graph G is called *k-regular* for some $k \in \mathbb{N}$, if for every vertex $v \in V(G)$ it holds true that $\deg(v) = k$. Let $v \in V(G)$ be a vertex. Then we denote by $N(v) := \{w \mid \{w, v\} \in E\}$ the *neighborhood of* v. Furthermore let $H \subseteq G$ and $v \in H$. Then $N_H(v) := \{w \mid \{w, v\} \in E(H)\}$ denotes the neighborhood of v in H.

Let G be a graph of order n_G, and let $k \leq n_G$ be an integer. Then $Ord := (v_0, \dots, v_{k-1}) \in V(G)^k$ is called an *ordering*, if $v_i \neq v_j$ for $i \neq j$. Moreover we will write $v \in Ord$ if there is $i \in \{0, \dots, k-1\}$ with $v = v_i$. Note that due to our abuse of notation, cycles and paths are also orderings. We will refer to these as *orderings induced by a cycle (or path)*. The length of an ordering $Ord \in V(G)^k$ is defined by $|Ord| := k$. Let G be a graph of order n_G and $Ord \subset V(G)^k$ for some $k \leq n_G$, then we say that $\mathcal{C}$ is a cycle with respect to Ord, if $\mathcal{C}$ visits every vertex in Ord in its prescribed order.

For $k \in \mathbb{N}$, we call a graph G *k-Hamiltonian ordered*, if for every ordering of length k in G, there is a Hamilton-cycle that encounters the vertices in that given order. The notion of k-Hamiltonian ordered graphs was firstly introduced by Ng and Schultz [NS97].

Throughout this thesis, it will without loss of generality be assumed that for any graph G, its order is denoted by n_G and an enumeration $V(G) := \{v_0, \dots, v_{n_G-1}\}$ of the vertices is given unless stated otherwise. Note that relabelling the vertices in G can be done in linear time on G. Moreover if we are given a Hamilton-cycle $\mathcal{C}$ in G, the enumeration of the vertices in G will without loss of generality be assumed to be induced by the cycle, meaning that $\mathcal{C} = (v_0, \dots, v_{n_G-1}, v_0)$ unless stated otherwise.

3 The Feedback Vertex Set Problem on Hamiltonian Graphs

The problem of deciding whether a graph G contains a feedback vertex set of cardinality k for some integer k can be stated as follows.

Feedback Vertex Set
Input: An undirected graph G, an integer $k \in \mathbb{N}$.
Question: Is there a set $I \subseteq V(G)$ with $|I| = k$ such that $G[V(G) \setminus I]$ is acyclic?

The problem of computing a feedback vertex set in a graph has many applications; in the study of deadlock recovery, the solution to a minimum feedback vertex set problem determines the number of processes that need to be aborted in order to resolve the deadlocks [SGG18]. Applications of Feedback Vertex Set to constraint satisfaction problems and Bayesian systems have been highlighted by Bar-Yehuda et al. [Bar+98]. Over the past few decades, many variations of Feedback Vertex Set have been extensively studied from multiple angles. As one of the several NP-complete problems that have been studied by Karp [Kar72], the algorithmic complexity of Feedback Vertex Set, and variations thereof, has been analyzed on a multitude of restricted graph classes [Bra87; CTY07; Spe88]. In recent years, several polynomial-time approximation algorithms and exact exponential-time algorithms for finding minimum feedback vertex sets in different graph classes have been developed and analyzed [Fom+08; FP05; Hac97; RSS05].

In this chapter we will prove that Feedback Vertex Set remains NP-complete on general Hamiltonian graphs as well as some subclasses thereof—namely on k-regular Hamiltonian and k-Hamiltonian-ordered graphs, for every $k \geq 3$. As a special case, we will moreover prove that Feedback Vertex Set remains NP-complete on planar 4-regular Hamiltonian graphs, implying the NP-completeness on 4-regular planar graphs. In the first section we will show that Feedback Vertex Set remains NP-complete on Hamiltonian graphs. The second section is dedicated to the proof of the NP-completeness of Feedback Vertex Set on 4-regular planar Hamiltonian graphs. This result will then be used in the third section in order to prove that Feedback Vertex Set remains NP-complete on k-regular Hamiltonian graphs for every $k \geq 4$. We will conclude the chapter in the fourth section with a proof that Feedback Vertex Set remains NP-complete even when restricted to k-Hamiltonian ordered graphs, for every $k \geq 3$.

Throughout this chapter we will write $\mathrm{FVS}(G) \leq k$ if G has a feedback vertex set of maximal cardinality $k \in \mathbb{N}$, and $\mathrm{FVS}(G) \geq k$ if G has no feedback vertex set of cardinality less than k.

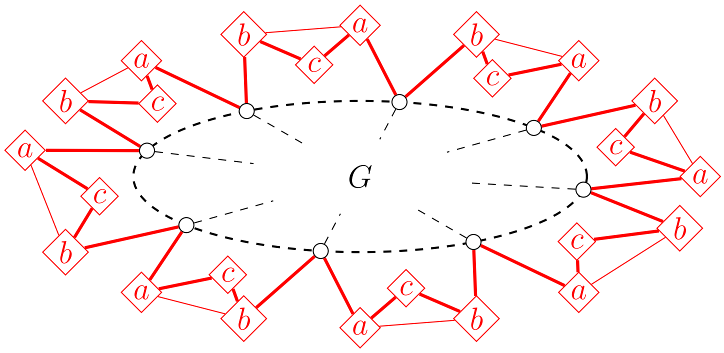


Figure 3.1: Envelope-shaped Hamiltonian closure of G. The (red) diamond-shaped vertices labelled with a, b, c represent V_{env}, and the (red) edges represent the edges of E_{env}, while the thick (red) edges highlight a Hamilton-cycle in G_{env}. Dashed edges, as well as the dashed ellipse, denote some of the possible edges in G.

3.1 General Hamiltonian graphs.

It is commonly known that FEEDBACK VERTEX SET is NP-complete on general (undirected) graphs [Kar72]. As a first result in this chapter, we will give a simple polynomial-time many-one reduction from FEEDBACK VERTEX SET on general graphs to FEEDBACK VERTEX SET on Hamiltonian graphs, yielding the following.

Theorem 3.1. FEEDBACK VERTEX SET *on Hamiltonian graphs with known Hamilton-cycle is NP-complete.*

For the polynomial-time many-one reduction in the proof of Theorem 3.1, we will need a construction that, given some graph G, constructs a Hamiltonian graph G'. The main idea behind our construction is to, given some graph G, enumerate its vertices by v_0 to v_{n_G-1}, and for each two consecutive vertices in the enumeration add a disjoint K_3 and connect both vertices to it, in order to guarantee the construction of a Hamilton-cycle through G. The choice of K_3 graphs will guarantee the hamiltonicity of the resulting graphs, and their influence on the cardinality of feedback vertex sets in G turns out to be easily understood. A schematic illustration of the construction is depicted in Figure 3.1.

Definition 3.2 (Envelope-shaped Hamiltonian closure.)**.** Let $G = (V, E)$ be a graph and $\Phi := (v_0, \ldots, v_{n_G-1})$ an enumeration of its vertices. Then the *envelope-shaped Hamiltonian closure of G (through Φ)*, denoted by $G_{\text{env}} = (V', E')$, is:

$$\begin{aligned}
V' &:= V \uplus V_{\text{env}}, \text{ where } V_{\text{env}} = \{a_i, b_i, c_i \mid i \in \{0, \ldots, n_G - 1\}\}, \text{ and} \\
E' &:= E \uplus E_{\text{env}}\,, \text{ where} \\
&\quad E_{\text{env}} := \{\{a_i, b_i\}, \{b_i, c_i\}, \{c_i, a_i\} \mid i \in \{0, \ldots, n_G - 1\}\} \cup \\
&\qquad\qquad \{\{v_i, a_i\}, \{b_i, v_{i+1 \bmod n_G}\} \mid i \in \{0, \ldots, n_G - 1\}\}.
\end{aligned}$$

The next two lemmata prove that the construction given by Definition 3.2 gives rise to Hamiltonian graphs, and that deciding whether G contains a feedback vertex set of cardinality k is equivalent to deciding whether G_{env} contains a feedback vertex set of cardinality $k + n_G$.

Lemma 3.3. *Let G be a graph. Then the envelope-shaped Hamiltonian closure G_{env} is Hamiltonian, and it can be constructed from G in $\mathcal{O}(|V(G)|)$ time after reading G.*

Proof. The following is a Hamilton-cycle in G_{env}:

$$\mathcal{C} = (v_0, a_0, c_0, b_0, v_1, a_1, \ldots, b_{n_G-2}, v_{n_G-1}, a_{n_G-1}, c_{n_G-1}, b_{n_G-1}, v_0),$$

since $\{v_i, a_i\}, \{b_i, v_{i+1 \bmod n_G}\} \in E(G_{\text{env}})$ and $G[\{a_i, b_i, c_i\}] \cong K_3$ for every $i \in \{0, \ldots, n_G - 1\}$ (see Figure 3.1). G_{env} can be constructed from G by a disjoint union with n_G many $K_3 \cong G_{\text{env}}[\{a_i, b_i, c_i\}]$ graphs, and subsequently connecting two distinct vertices $a_i, b_i \in V(K_3)$ to two consecutive vertices $v_i, v_{i+1 \bmod n_G} \in V(G)$ for every $i \in \{0, \ldots, n_G - 1\}$, given an enumeration of the vertices in G. This can clearly be done in $\mathcal{O}(|V(G)|)$ time after reading G. □

Lemma 3.4. *Let G be a graph and denote the envelope-shaped Hamiltonian closure of G by G_{env}. Then,*

$$\text{FVS}(G) \leq k \iff \text{FVS}(G_{\text{env}}) \leq k + n_G$$

for every $k \in \mathbb{N}$.

Proof. Let $V(G_{\text{env}}) = V \cup V_{\text{env}}$as in Definition 3.2. Define by $K_3^{(i)}$ the complete graph on $\{a_i, b_i, c_i\}$ for every $i \in \{0, \ldots, n_G - 1\}$, and note that by construction $K_3^{(i)}$ is a subgraph of G_{env} for every $i \in \{0, \ldots, n_G - 1\}$.

- $\Leftarrow$: Let $S \subset V(G_{\text{env}})$ be a feedback vertex set in G_{env} with $|S| \leq k + n_G$. Note that as $\text{FVS}(K_3^{(i)}) \geq 1$, there is at least one $v^i \in \{a_i, b_i, c_i\}$ such that $v^i \in S$, for every $i \in \{0, \ldots, n_G - 1\}$. As $V(K_3^{(i)}) \cap V(K_3^{(j)}) = \emptyset$ for all $i \neq j$, this yields that $\left|S \cap \left(\bigcup_{i=0}^{n_G-1} V(K_3^{(i)})\right)\right| \geq n_G$. Since moreover $V(K_3^{(i)}) \cap V = \emptyset$ for every $i \in \{0, \ldots, n_G - 1\}$, we conclude that $|S \cap V| \leq k$. As $S \cap V$ is a feedback vertex set in $G_{\text{env}}[V] = G$, it follows that $\text{FVS}(G) \leq k$.
- $\Rightarrow$: Let $S \subset V(G)$ be a feedback vertex set in G with $|S| \leq k$. We claim that $S' := S \cup \{a_i \mid i \in \{0, \ldots, n_G - 1\}\}$ is a feedback vertex set in G_{env}. By construction S' is a feedback vertex set in $G_{\text{env}}[V] = G$ and $G_{\text{env}}[V(K_3^{(i)})]$ for every $i \in \{0, \ldots, n_G - 1\}$. As moreover $V(K_3^{(i)}) \cap V(K_3^{(j)}) = \emptyset$ for every $i \neq j$, any remaining cycle $\mathcal{C}$ in $G_{\text{env}} - S'$ must contain at least one vertex $v \in V \setminus S'$ and one vertex $w \in \{b_i, c_i\}$ for some $i \in \{0, \ldots, n_G - 1\}$. Since in $G_{\text{env}} - S'$ it holds true that, for every $i \in \{0, \ldots, n_G - 1\}$, $\deg(b_i) = 2$, it follows that if $b_i \in V(\mathcal{C})$ also $c_i \in V(\mathcal{C})$ since $\{b_i, c_i\} \in E(G_{\text{env}} - S')$. But as $\deg(c_i) = 1$ in $G_{\text{env}} - S''$, there cannot be any cycle through c_i in $G_{\text{env}} - S'$ concluding that $b_i, c_i \notin V(\mathcal{C})$. □

We are now ready for the proof of Theorem 3.1.

Proof of Theorem 3.1. As FEEDBACK VERTEX SET on Hamiltonian graphs is trivially contained in NP, the proof follows immediately by Lemma 3.3, Lemma 3.4 and the fact that FEEDBACK VERTEX SET is known to be NP-complete on general graphs [Kar72]. □

3.2 4-regular planar Hamiltonian graphs.

Speckenmeyer [Spe88] showed that FEEDBACK VERTEX SET remains NP-complete on planar graphs with maximum degree four. We are not aware of a known result proving that FEEDBACK VERTEX SET remains NP-complete on 4-regular planar graphs. In this section we will prove an even stronger claim, namely that FEEDBACK VERTEX SET remains NP-complete when restricted to 4-regular planar Hamiltonian graphs.

Theorem 3.5. FEEDBACK VERTEX SET *on* 4*-regular planar Hamiltonian graphs with known Hamilton-cycle is* NP*-complete.*

The proof of Theorem 3.5 will be done in two major steps. As a first step we prove that FEEDBACK VERTEX SET remains NP-complete on 4-regular planar graphs, by two consecutive polynomial-time many-one reductions. The second step will be the proof of Theorem 3.5. The proof follows by a polynomial-time many-one reduction from the class of 4-regular planar graphs, that was inspired by a reduction given by Fleischner, Sabidussi, and Sarvanov [FSS10] and Fleischner and Sabidussi [FS03].

We emphasize that, throughout the rest of this chapter, our input graphs will be tacitly assumed to contain no degree-one vertex. The following observation shows that we can assume this without loss of generality.

Lemma 3.6. *Let* G *be a graph with* $\mathrm{FVS}(G) \leq k$ *for some* $k \in \mathbb{N}$ *and let* $v \in V(G)$ *with* $\deg(v) = 1$*. Then,*

$$\mathrm{FVS}(G) \leq k \iff \mathrm{FVS}(G - \{v\}) \leq k.$$

Thus we can construct a graph G' *from* G *in* $\mathcal{O}(n_G)$ *time, where* G' *has no degree-one vertex, without changing the size of a minimum feedback vertex set in* G*.*

Proof. Let $v \in V(G)$ with $\deg(v) = 1$. Then v cannot be part of any cycle in G. Let $S \subset V(G)$ be a feedback vertex set in G with $v \in S$. Then $S \setminus \{v\}$ is also a feedback vertex set in G. As we can determine the degree-one vertices in G in $\mathcal{O}(n_G)$ time, we can construct a graph G' without degree-one vertices in $\mathcal{O}(n_G)$ time from G such that $\mathrm{FVS}(G) \leq k \iff \mathrm{FVS}(G') \leq k$ for every $k \in \mathbb{N}$. □

As mentioned above, our first goal will be the proof of the following.

Theorem 3.7. FEEDBACK VERTEX SET *on* 4*-regular planar graphs is* NP*-complete.*

As an intermediate step towards a proof of Theorem 3.7, we will show that Feedback Vertex Set remains NP-complete on planar graphs with minimum degree three and maximum degree four. We denote the class of graphs with minimum degree three and maximum degree four by $\mathcal{G}_3^4 := \{G = (V, E) \mid 3 \leq \deg(v) \leq 4 \text{ for every } v \in V\}$. We will then get rid of the remaining degree-three vertices by introducing so-called H-insertion-chains (see Definition 3.19), which will keep the cardinality of feedback vertex sets in the resulting graph polynomially dependent on the input, and thus prove that Feedback Vertex Set remains NP-complete on 4-regular planar Hamiltonian graphs.

Theorem 3.8. Feedback Vertex Set *on planar graphs of minimum degree three and maximum degree four is NP-complete.*

Theorem 3.8 extends the following result due to Speckenmeyer [Spe88].

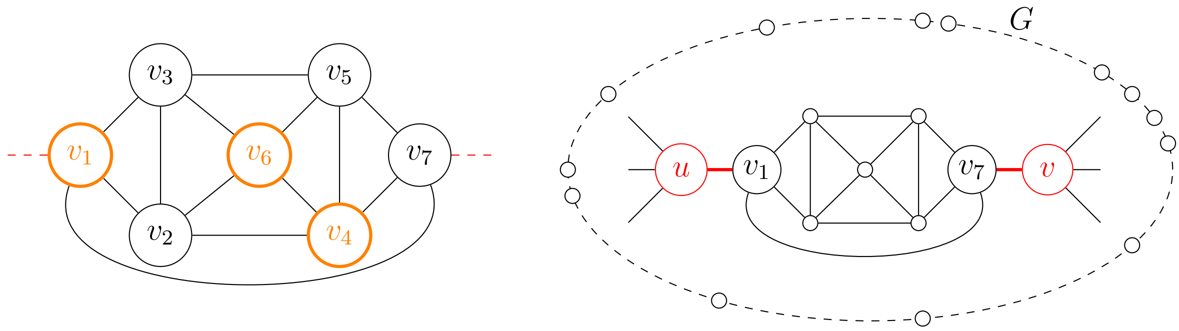
Theorem 3.9 (Speckenmeyer [Spe88])**.** Feedback Vertex Set *on planar graphs of maximum degree* 4 *is NP-complete.*

A key „ingredient" to most of the proofs in this section —i.e. the proof of Theorem 3.8 and most importantly the proof of Theorem 3.5—will be what we call *H-insertions*, where H will denote a specific graph (see Figure 3.2a). These H-insertions play a fundamental role in ensuring the regularity of our constructed graphs and getting rid of degree-two and degree-three vertices, while having a well understood and „not too heavy" impact on the cardinality of feedback vertex sets. Roughly speaking, an H-insertion on two given vertices u, v in some graph G results in a new graph G' by a disjoint union of G with what we call an *H-graph*, connecting u and v to H, and thus augmenting their degree in the graph G' (see Figure 3.2b). A formal definition of H-insertions reads as follows.

Definition 3.10 (H-insertion)**.** Let G be a graph and $u, v \in V(G)$ be two distinct vertices. Let H be the auxiliary graph on seven vertices as illustrated in Figure 3.2a. An *H-insertion on u, v* results in a graph G' by a disjoint union of G and the auxiliary graph H (maybe after relabelling the vertices in $V(H)$), and subsequently connecting v_1 with u and v_7 with v as illustrated in Figure 3.2b.

As we will use H-insertions in the construction of graphs $G' \in \mathcal{G}_3^4$ from our input graphs G (and later on in the construction of 4-regular graphs), it is of particular interest to understand the influence of H-insertions on the topology of the input graphs, and to analyze how they affect the cardinality of feedback vertex sets in G. The next lemma quantifies the impact of H-insertions on the degrees of the involved vertices, and shows that if two vertices $u, v \in V(G)$ lie on some common face given an embedding of G, then an H-insertion on u, v can be carried out in a way that the resulting graph stays planar.

Lemma 3.11. *Let G be a plane graph and let $u, v \in V(G)$ be two distinct vertices on a common face F of the given embedding of G with* $\deg(u) = k = \deg(v)$ *for some $k \in \mathbb{N}$. Let G' denote the graph resulting from an H-insertion on u, v in G. Then, G' can be constructed in constant time after reading G, and in G' it holds true that* $\deg(u) = k + 1 = \deg(v)$ *and* $\deg(v) = 4$ *for every $v \in V(H)$. Moreover, the H-insertion can be carried out in a way that G' stays planar.*



(a) Plane auxiliary graph H.

(b) H-insertion.

Figure 3.2: H-graph and an H-insertion. The dashed (red) edges in Figure 3.2a denote the edges that will connect the H-graph to G after an H-insertion as can be seen in Figure 3.2b (thick (red) edges connecting u, v_1 and v_7, v). The (orange) thickly drawn vertices highlight a feedback vertex set of size three in H. Figure 3.2b shows a detail of G after performing an H-insertion on two distinct degree-three vertices denoted by u, v. The dashed edges highlight some of the possible remaining edges in G.

Proof. As u and v have k neighbors in $G \subset G'$ and one new neighbor in H, namely v_1 and v_7 respectively, it follows easily that $\deg(u) = k + 1 = \deg(v)$. As u and v lie on a common face F given the planar embedding of G and since H, as can be seen in Figure 3.2a, is a planar graph, the H-insertion can be carried out by embedding H „inside“ the face F and thus keeping G' planar. The H-insertion can be carried out in constant time after reading G by adding seven new vertices and fifteen edges to G, namely $V(H)$, $E(H)$ and the two edges connecting H to G (after possible relabelling of the vertices in H. □

The next two lemmata analyze the cardinality of a minimum feedback vertex set in the H-graph, and quantify the impact of H-insertions on the cardinality of feedback vertex sets in G.

Lemma 3.12. *Let H be the graph as defined in Figure 3.2a. Then it holds true that* $\mathrm{FVS}(H) \geq 3$.

Proof. Note that $\{v_1, v_4, v_6\}$ is a feedback vertex set of size three in H as highlighted in Figure 3.2a by the (orange) thickly drawn vertices. We need to prove that $\mathrm{FVS}(H)$ cannot be less than three. To this end, note that (v_1, v_2, v_3, v_1) and (v_4, v_5, v_7, v_4) are two vertex-disjoint cycles in H. Thus at least one vertex in each of $H_1^0 := \{v_1, v_2, v_3\}$ and $H_1^1 := \{v_4, v_5, v_7\}$ has to be contained in a feedback vertex set of H, concluding that $\mathrm{FVS}(H) \geq 2$. The same argument leads to the fact that at least one vertex in each of $H_2^0 := \{v_1, v_2, v_4, v_7\}$ and $H_2^1 := \{v_3, v_6, v_5\}$, and one in each of $H_3^0 := \{v_1, v_3, v_5, v_7\}$ and $H_3^1 := \{v_2, v_6, v_4\}$ must be contained in a feedback vertex set of H. The only pairs of vertices in H satisfying these conditions, are $\{v_3, v_4\}$ and $\{v_2, v_5\}$. It can be easily verified that neither of both sets is a feedback vertex set in H, thus $\text{FVS}(H) > 2$. □

Lemma 3.13. *Let G be a graph and let $u, v \in V(G)$ be two distinct vertices. Let G' be the graph obtained from G by an H-insertion on u, v. Then,*

$$\text{FVS}(G) \leq k \iff \text{FVS}(G') \leq k + 3$$

for every $k \in \mathbb{N}$.

Proof. Recall that by Lemma 3.12 $\text{FVS}(H) \geq 3$ and that $\{v_1, v_4, v_6\}$ is a feedback vertex set of size three in H.

⇒: Let S be a feedback vertex set in G. We claim that $S' := S \cup \{v_1, v_4, v_6\}$ is a feedback vertex set in G'. Since S' is a feedback vertex set in $G'[V] = G$ and $G'[V(H)] \cong H$, any remaining cycle in $G' - S'$ must contain one vertex in $V(H) \setminus V(S') = \{v_2, v_3, v_5, v_7\}$ and one vertex in $V(G) \setminus V(S')$. Note that by construction only the vertex $v_7 \in \{v_2, v_3, v_5, v_7\}$ may have a neighbor in $V(G)$. As $G'[\{v_2, v_3, v_5, v_7\}]$ is isomorphic to a path of length four, any remaining cycle containing a vertex in $\{v_2, v_3, v_5, v_7\}$ must contain the whole path (v_7, v_5, v_3, v_2). But as $\deg(v_2) = 1$, v_2 cannot be part of any cycle, thus concluding that $G' - S'$ is acyclic, and hence S' is a feedback vertex set of G' with $|S'| \leq k + 3$.

⇐: Let S' be a feedback vertex set with $|S'| \leq k + 3$. By construction it holds true that $H \subset G'$ with $\text{FVS}(H) \geq 3$. Let $S_H := S' \cap H$. Since S' is a feedback vertex set of $H \subset G'$, it holds true that $|S_H| \geq 3$. Moreover $V(H) \cap V(G) = \emptyset$, thus $S := S' \setminus S_H$ must be a feedback vertex set in $G'[V(G)] = G$ with $|S| = |S'| - |S_H| \leq (k + 3) - 3 = k$. □

Iterative use of Lemma 3.13 yields the following.

Corollary 3.14. *Let G be a graph and let $m \in \mathbb{N}$ with $m < n_G$. Let G' be the graph resulting from G by m many H-insertions. Then,*

$$\text{FVS}(G) \leq k \iff \text{FVS}(G') \leq k + 3 \cdot m$$

for every $k \in \mathbb{N}$.

The main idea behind the proof of Theorem 3.8 is to connect two different degree-two vertices u, v in our input graph G by H-insertions, keeping the graph planar, and hence reducing the number of degree-two vertices in the input graph while keeping its maximal degree upper-bounded by four. To guarantee that the resulting graph stays planar, we need u and v to lie on a common face given an embedding of G (see Lemma 3.11). As in general the number of degree-two vertices is not even, and as they do not need to lie on a common face, we will need to add degree-two vertices to G without affecting the cardinality of feedback vertex sets in G. It turns out that Feedback Vertex Set is robust under „subdividing edges“, which enables it to add degree-two vertices to the input graph G lying on any desired face without affecting the cardinality of feedback vertex sets. This realization will play a key-role in the proofs of Theorem 3.7 and Theorem 3.5. We will prove an even stronger claim, namely that we can assume the feedback vertex sets of our input graph G to contain no degree-two vertex.

Lemma 3.15. *Let G be a connected graph with at least one vertex $v \in V(G)$ with $\deg(v) \geq 3$, and let $S \subset V(G)$ be a feedback vertex set of G. If there is $u \in S$ with $\deg(u) = 2$, then we can find $w \in V(G)$ in $\mathcal{O}(|V(G)| + |E(G)|)$ time with $\deg(w) \geq 3$ such that $(S \cup \{w\}) \setminus \{u\}$ is again a feedback vertex set in G.*

Proof. Let $u \in S$ with $\deg(u) = 2$. Let $N(u) = \{w_1, w_2\}$ for two distinct vertices $w_1, w_2 \in V(G)$. If $\deg(w_1) = 2 = \deg(w_2)$, then we set $S := (S \cup \{w_1\}) \setminus \{u\}$. Then S is still a feedback vertex set in G as any cycle through u has to pass through w_1. Relabel w_1 as u and vice-versa, and use the same argument iteratively until there is $w \in N(u)$ with $\deg(w) \geq 3$ which is guaranteed as G is connected and contains at least one vertex of degree at least three. By the same reasoning as above we can set $S := (S \cup \{w\}) \setminus \{u\}$. Then S is still a feedback vertex set in G as any cycle through u needs to pass through w. The proof moreover suggests an algorithmic approach to find w—namely start a *breadth-first search* from u, then the first vertex with degree larger than two in the breadth-first search will be w. □

A direct consequence of Lemma 3.15 is the following.

Corollary 3.16. Feedback Vertex Set *stays invariant under subdivision of edges.*

We are now ready to prove Theorem 3.8.

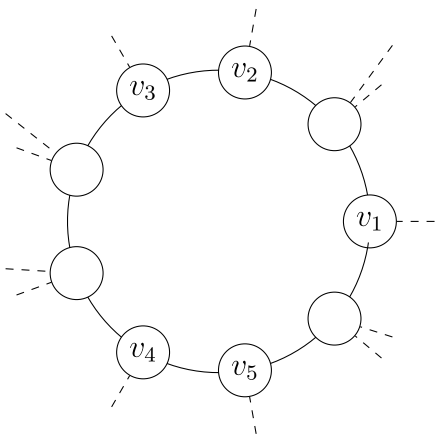
Proof of Theorem 3.8. Let G be a planar graph of maximum degree four. Then a (straight-line) planar embedding of G can be computed in polynomial time as shown by De Fraysseix, Pach, and Pollack [DPP90]. Denote by $D_2 := \{v \mid v \in V(G), \deg(v) = 2\}$ the set of degree-two vertices in G. Choose an arbitrary vertex $v \in D_2$ and let $\{v, u\} \in E(G)$ be an edge incident to v. Subdivide the edge by a new vertex z. Then $\deg(z) = 2 = \deg(v)$ and both vertices lie on some common face of G given a planar embedding. Note that the introduction of z does not affect the cardinality of feedback vertex sets in G due to Corollary 3.16. Now construct G' by an H-insertion on v, z. By Lemma 3.11, G' is a planar graph and $\deg(v) = 3 = \deg(z)$. Finally we can construct a planar graph $G' \in \mathcal{G}_3^4$ from G by $|D_2|$ subdivisions of edges and subsequent H-insertions as just described. The construction can be done in $\mathcal{O}(|D_2|)$ time after reading G, as H-insertions and subdividing edges can be done in constant time. The use of Lemma 3.11 and Corollary 3.14 together with Theorem 3.9 concludes the proof. □

We will use Theorem 3.8 together with a polynomial-time many-one reduction from the class of planar graphs of minimum degree three and maximum degree four to the class of 4-regular planar graphs in order to prove Theorem 3.7. The proof will be done in two steps. As a first step we will reduce the number of degree-three vertices per face (given some planar embedding of our input graph) to at most one, by consecutively connecting degree-three vertices in common faces by H-insertions. As a second step we will introduce the notion of *H-insertion chains* which will be used to connect degree-three vertices in our input graph G that do not share a common face, reducing the number of degree-three vertices in G while keeping the resulting graph planar. This will then result in a 4-regular planar graph.

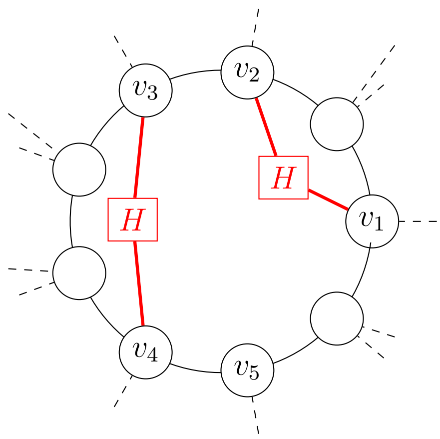
The next lemma proves that, given some planar graph $G \in \mathcal{G}_3^4$, we can construct a planar graph G' that has at most one degree-three vertex per face (that is not the canonical outer-face) while keeping the cardinality of feedback vertex sets in G' polynomially dependent on the input.

Lemma 3.17. *Let $G \in \mathcal{G}_3^4$ be a plane graph with known faces $F := \{F_1, \ldots, F_f\} \cup \{F_o\}$, where F_o denotes the canonical outer-face. Let $D_3 := \{v \mid v \in V(G), \deg(v) = 3\}$ denote the set of degree-three vertices in G. Suppose there is an $i \in \{1, \ldots, f\}$ with $|V(F_i) \cap D_3| \geq 2$. Then we can construct a planar graph $G' \in \mathcal{G}_3^4$ with $f + 8d$ many faces and $|D_3| - 2d$ many degree-three vertices, by d many H-insertions in polynomial time from G, where $d := |V(F_i \cap D_3)| \operatorname{div} 2$.*

Proof. Let G be a plane graph, and let $F_i \in F$, for some $i \in \{1, \ldots, f\}$, be a face of G with $\{v_1, \ldots, v_k\} \subseteq V(F_i) \cap D_3$ for some $1 < k \leq |V(F_i)|$; hence $\deg(v_j) = 3$ for every $j \in \{1, \ldots, k\}$. Suppose without loss of generality that $v_1, \ldots, v_k$ are in *cyclic order*, otherwise relabel them such that they are (see Figure 3.3a). For $F_i \neq F_o$ cyclic order means that there is a path through $V(F_i)$ that visits the vertices $v_1 \ldots, v_k$ in the given order $(v_1, \ldots, v_k)$. Since the minimum degree in G is three, the face F_i has to be a cycle in G, and thus a cyclic order must exist and can be calculated in polynomial time on G. To see this choose a vertex in $V(F_i)$ and relabel it as v_1 in G. Then run once through the cycle $V(F_i)$ and relabel the vertices in the order visited which can altogether be done in $\mathcal{O}(|V(F_i)| \cdot |E(G)|)$ time. Let $d := (k \operatorname{div} 2)$ and pair the vertices $v_1, \ldots, v_k$ up as follows: $\{(v_r, v_{r+1}) \mid r \in \{1, \ldots, k-1\} \cap \ (2\mathbb{N} + 1)\}$; meaning that we always pair up two consecutive vertices with respect to the cyclic order such that every vertex is part of exactly one of the d many pairs, except for maybe v_k, which is left out in the case of $k \bmod 2 = 1$ (see v_5 in Figure 3.3b). Now for each pair (v_r, v_{r+1}), $r \in \{1, \ldots, k-1\} \cap (2\mathbb{N} + 1)$ iteratively make altogether d many H-insertions on v_r, v_{r+1} in G. By iterative use of Lemma 3.11 the H-insertions can be carried out in a way that G' stays planar. To see this, note that after each iteration the remaining pairs still lie on a common face due to their choice respecting the cyclic order (see Figure 3.3b for planar embedded H-insertions given a cyclic order). After the H-insertions, Lemma 3.11 guarantees that $\deg(v_r) = 4 = \deg(v_{r+1})$ in G'. As H-insertions do not give rise to degree-two or degree-three vertices, the above described procedure produces a plane graph $G' \in \mathcal{G}_3^4$ from G reducing the number of degree-three vertices by $2d \geq 2$. The number of faces in the embedding grows by $8d$ since eight new faces arise for each H-insertion. To see this, note that each H-insertion splits the face F_i into two faces by connecting two vertices of F_i, and H itself adds seven new faces now embedded inside F_i (see Figure 3.2b and Figure 3.2a). By construction it is clear that the faces F_j for $j \neq i$ are not changed. This construction can be done in polynomial time as a single H-insertion can be carried out in constant time after reading G and the



(a) Cyclic Order in a face.



(b) H-insertion given cyclic order.

Figure 3.3: Cyclic order and H-insertions. Both Figures illustrate the same face in a graph G with five degree-three vertices. The (red) H-Box is a schematic representation of the plane H-graph where the (red) thick edges are the edges arising from the insertion as the (red) thick edges in Figure 3.2b. The dashed edges denote edges with vertices in other faces of G.

planar embedding can be updated in parallel to the H-insertions in polynomial time, as only $\mathcal{O}(n_G)$ many new faces arise. Alternatively, a planar embedding of G' can be computed in polynomial time on G' after completing the H-insertions as we know by construction that G' is planar [HT74; MM96]. □

Iterative use of Lemma 3.17 yields a planar graph G' with at most one degree-three vertex per face for some given embedding. Note that there might exist another embedding of G' where two vertices of degree-three might still lie on a common face. As Lemma 3.17 needed a planar embedding in the first place, it is worth noting that Hopcroft and Tarjan [HT74] presented an algorithm that tests whether a graph is planar in linear time, and suggested slight modifications to the algorithm, such that a planar embedding can be computed in linear time (see also [MM96]). We will thus assume without loss of generality that the planar graphs we are given are plane.

In a next step we will prove that we can connect degree-three vertices that do not share a common face, keeping the resulting graph planar and reducing the number of degree-three vertices. To this end, we will introduce what we will call *H-chains* and *H-insertion chains*. Our intermediate results towards this proof will need the planar embeddings of our graphs to be straight-line embeddings. The following result due to De Fraysseix, Pach, and Pollack [DPP90] shows that we may assume our embeddings to be straight-line embeddings, also known as *Fáry-embeddings*, without loss of generality, as they can be computed in polynomial time.

Theorem 3.18 (De Fraysseix, Pach, and Pollack [DPP90, Theorem 1])**.** *Any plane graph with n vertices has a straight-line embedding on the $2n$-4 by n-2 grid. The embedding* *can be computed in $\mathcal{O}(n^2)$ time.*

Additionally, we will need the y-coordinates of our embeddings to be pairwise distinct. To avoid the discussion of corner-cases we will moreover assume the vertices in the embeddings to be in *general position* in $\mathbb{R}^2$, meaning that no three vertices are on a line. Unfortunately we were not able to find literature proving that y-monotone graph drawings, or graph-drawings with vertices in general position, can be computed in polynomial time on G. At this point we want to emphasize that the assumption of a y-monotone drawing with vertices in general position only simplifies the argument in our proof, as we do not have to deal with case distinctions or corner-cases. We will come back to this at the point where the assumptions on the embeddings are needed, and discuss how they may have been omitted in the proofs.

From now on it will be assumed that the embeddings we are given are straight-line embeddings in $\mathbb{R}^2$, that the vertices are in general position, and that the drawings are y-monotone, meaning that the y-coordinates are pairwise distinct. Given some vertex v we will write $y(v)$ to refer to the y-coordinate of the embedding of v.

Definition 3.19 (H-insertion chain on u, v and H-chain.)**.** Let G be a plane graph with faces $F := \{F_1, \ldots, F_k\}$ and let $u \in F_1$, $v \in F_2$ be two different vertices. Denote by ℓ_u^v the line segment in $\mathbb{R}^2$ connecting u and v and by ℓ_e, the straight-line embedding of $e \in E(G)$ in $\mathbb{R}^2$. Let $E_c := \{e \mid \ell_u^v \cap \ell_e \neq \emptyset, e \in E(G)\}$ be the edges in G whose embedding intersects ℓ_u^v. Note that the intersection of $\ell_u^v \cap \ell_e$ cannot have the coordinates of another vertex in G that is already embedded in $\mathbb{R}^2$, as we supposed the vertices to be in general position. Moreover let $E_c = \{e_1, \ldots, e_m\}$ be sorted by increasing y-coordinates of the intersection points of $\ell_u^v \cap \ell_{e_i}$ for every $i \in \{1, \ldots, m\}$ and $m := |E_c|$. Now introduce a subdivision vertex z_i for each edge $e_i \in E_c$ such that the coordinates of z_i in the embedding of G in $\mathbb{R}^2$ are given by the coordinates of the intersection-point $\ell_u^v \cap \ell_{e_i}$ (see the (orange) diamond-shaped vertices in Figure 3.4).

An *H-insertion chain connecting* u, v in G gives rise to a graph G' by $m+1$ consecutive H-insertions on the vertices given by the pairs $(u, z_1), (z_1, z_2), \ldots, (z_{m-1}, z_m), (z_m, v)$. A schematic example of an H-insertion chain is depicted in Figure 3.4. If $E_c = \emptyset$ (thus there are no subdivision nodes z_i), the H-insertion chain is just an H-insertion on u, v. Let $V_H := \bigcup_{i=1}^{m+1} V(H_i) \cup \{z_1, \ldots, z_m\} \cup \{u, v\}$ where H_i denote the respective H-graphs needed for the $m+1$ many H-insertions. Then we call the induced graph $H^c := G'[V_H]$ the resulting *H-chain*.

The assumption that the vertices in the given embedding are in general position is only needed to avoid that the vertices z_i, introduced in Definition 3.19, would have coordinates coinciding with the coordinates of some other vertex; thus avoiding that ℓ_u^v and ℓ_{e_i} intersect in a vertex of e_i. Hence we could have omitted the assumption on general position and, if needed, shift the vertices of e_i by some amount $\epsilon > 0$ such that they do not lie on another edge and without affecting the planarity of our graph, which is possible as the number of vertices and edges of G is finite. Note that the definitions of H-insertion chains and the H-chain do not need the embedding of the graph to be y-monotone at first.

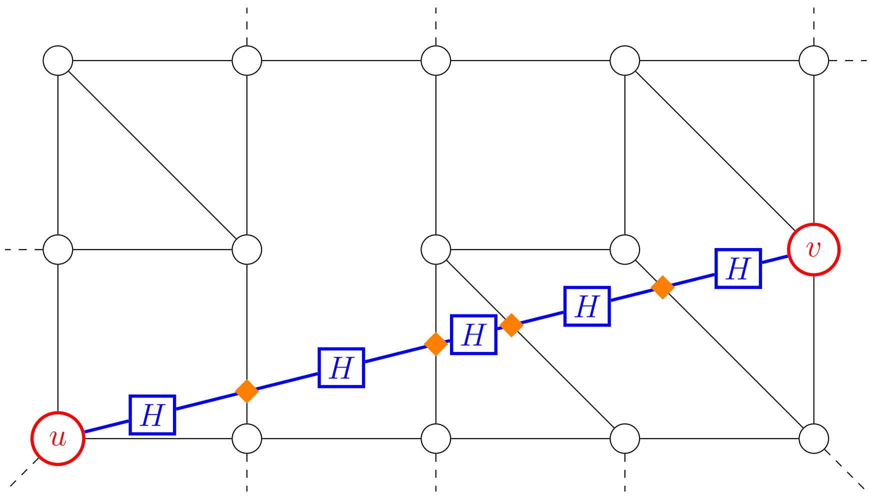


Figure 3.4: H-insertion chain on u, v. The figure shows a detail of some plane graph $G \in \mathcal{G}_3^4$. The (orange) diamond-shaped vertices correspond to the subdivision vertices z_i at the intersection points as explained in Definition 3.19. The (blue) H-boxes schematically denote plane embeddings of H-graphs arising from the H-insertions. The thickly drawn parts together with the (orange) diamond-shaped vertices induce the H-chain. The dashed edges denote remaining edges to other vertices of the graph.

The next two lemmata show that, given some input graph $G \in \mathcal{G}_3^4$, we can carry out H-insertion chains such that the resulting graph stays planar while reducing its number of degree-three vertices, which again does not need the assumption on the embedding to be y-monotone.

Lemma 3.20. *Let G be a plane graph with $f \in \mathbb{N}$ many faces, and let u, v be two distinct vertices in G. The graph G' resulting from an H-insertion chain on u, v is planar and has $f + 8d$ many faces, where $d \leq f$ denotes the number of H-insertions needed in the construction. Moreover G' can be constructed in polynomial time from G with $|V(G')| \in \mathcal{O}(n_G)$.*

Proof. Given an embedding of G, the coordinates of the „subdivision“ vertices needed in the construction of the H-insertion chain (the diamond-shaped (orange) vertices in Figure 3.4) can be computed in polynomial time on G. To see this note that their coordinates can be calculated by solving less than n_G many linear equation systems (consisting of two linear equations in two variables) and checking four inequalities (as we only have line segments) which is altogether solvable in $\mathcal{O}(n_G)$ time. Sorting the coordinates by increasing y-coordinates is feasible in $\mathcal{O}(|E(G)| \cdot \log |E(G)|)$ time. As at most f many H-insertions are needed to construct the H-insertion chain, the number of faces in the embedding of G' is upper-bounded by $f + 8f \in \mathcal{O}(n_G)$, as each H-insertion gives rise to eight new faces as explained in the proof ofLemma 3.17. An inductive use of Lemma 3.11 for each of the at most f many H-insertions in the definition of H-insertion chains, yields that the resulting H-chain can be embedded in such a way that G' is a plane graph (see Figure 3.4). As $f < n_G$, and as each of the at most f many H-insertions can be done in constant time after reading G, the H-insertion chain can be done in $\mathcal{O}(n_G)$ time. Finally, G' is altogether constructible in polynomial time from G with $V(G') = n_G + 7d + (d-1) \leq 9n_G$ where $7d$ vertices arise from the d many H-insertions and $d-1$ many „subdivision“ vertices are constructed. □

Lemma 3.21. *Let $G \in \mathcal{G}_3^4$ be a plane graph with faces $F := \{F_1, \dots, F_f\}$. Let $D_3 := \{v \mid v \in V(G), \deg(v) = 3\}$ denote the set of degree-three vertices in G. Suppose there are different $i, j \in \{1, \dots, f\}$ with $u \in F_i \cap D_3 \neq \emptyset$ and $v \in F_j \cap D_3 \neq \emptyset$. Then we can construct a planar graph $G' \in \mathcal{G}_3^4$ with $f \leq f' \leq f + 8f$ faces and $(|D_3| - 2)$ degree-three vertices by an H-insertion chain connecting u, v in polynomial time from G.*

Proof. Let $i, j \in \{1, \dots, f\}$ such that $V(F_i) \cap D_3 \neq \emptyset$ and $V(F_j) \cap D_3 \neq \emptyset$; hence there are $u_i \in V(F_i)$ and $u_j \in V(F_j)$ such that $\deg(u_i) = 3 = \deg(u_j)$. Let G' denote the graph resulting from G by an H-insertion chain connecting u and v. By Lemma 3.20 the graph G' is plane with $f \leq f' \leq f + 8f$ faces and can be constructed in polynomial time from G. Note that by construction for every „subdivision“ vertex z introduced by the H-insertion chain we have $\deg(z) = 4$ in G' (see Definition 3.19 and Figure 3.4). Now by Lemma 3.11 the vertices in the H-graphs have degree four in G' thus keeping the maximum degree of G' upper-bounded by four. As no other new vertices are introduced, no degree-three or degree-two vertex arises from an H-insertion chain. Finally $\deg(u) = 4 = \deg(v)$ in G' concluding that G' has $(|D_3| - 2)$ many degree-three vertices. □

As we want to give a polynomial-time many-one reduction from the class of planar graphs with minimum degree three and maximum degree four to the class of 4-regular planar graphs by a construction using H-chains, we need to ensure that the construction can be performed in polynomial time from G. As the number of H-insertions needed in an H-insertion chain depends on the number of faces in G, and as we want to use subsequent H-insertion chains, we need to prove that the resulting faces are altogether polynomially upper-bounded by the number of faces in our input graph G. Roughly speaking, we want to show that H-insertion chains can be carried out in a way that subsequent H-insertion chains do not need to pass through „many“ faces created by previously performed H-insertion chains. This is where the assumption that our embeddings are y-monotone will be helpful. Note that a naive iterative use of Lemma 3.21 would result in a graph with $\mathcal{O}(9^{n_G} \cdot f)$ faces.

The next lemma states that, given our assumptions on the embedding, two H-insertion chains on different vertices can be carried out in a way that they do not „cross“; meaning that the graph stays planar and the two H-chains are disjoint.

Lemma 3.22. *Let G be a plane graph with faces $F := \{F_1, \dots, F_f\}$. Let $u_i, v_i \in V(G)$ be different vertices for $i \in \{1, \dots, k\}$, $k \leq n_G$. Then (maybe after relabelling the u_i, v_i), we can construct a plane graph G' by k many H-insertion chains connecting u_i, v_i in G for every $i \in \{1, \dots, k\}$ in polynomial time on G, such that the resulting H-chains denoted by H_i^c are vertex disjoint and no two different H-chains have crossing edges. Moreover, the total number of resulting faces in G' is upper-bounded by $f + 8kf$, and $|V(G')| \in \mathcal{O}(n_G^2)$.*

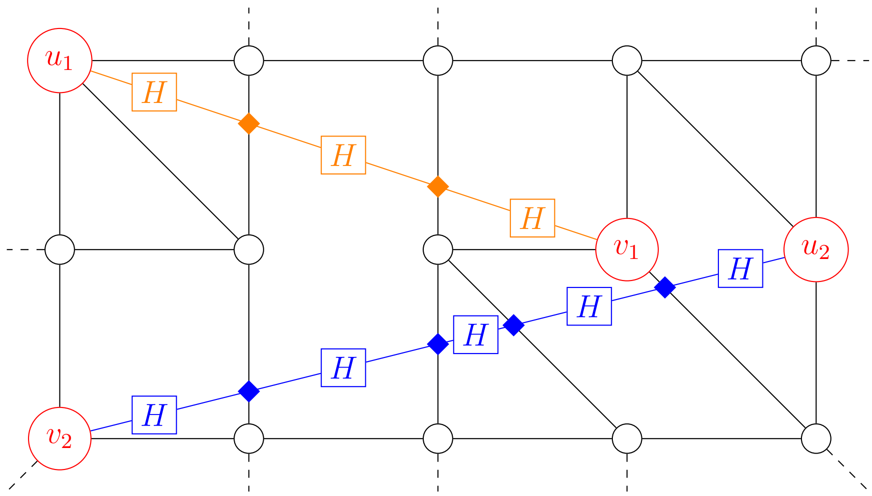


Figure 3.5: Two H-chains as in Lemma 3.22. The H-Boxes schematically denote H-graphs and the diamond-shaped vertices the subdivision vertices of the respective H-chains.

Proof. As the y-coordinates of the vertices in the given embedding of G are pairwise distinct, we can order them in a strictly decreasing way in $\mathcal{O}(n_G \log n_G)$ time. Let $\Phi_y := (v_0, \dots, v_{n_G - 1})$ be an enumeration of the vertices with respect to the strict ordering, meaning that $y(v_i) > y(v_j)$ if $i < j$. Let $D_3 = \{u_1, v_1, \dots, u_k, v_k\} \subset \Phi_y$ be relabelled with respect to the strict ordering, meaning that $y(u_i) > y(v_i)$ for $i \in \{1, \dots, k\}$ and $y(v_i) > y(u_{i+1})$ for $i \in \{1, \dots, k-1\}$. Let G' be the graph resulting from G by k H-insertion chains on u_i, v_i in G for each $i \in \{1, \dots, k\}$ and denote the corresponding H-chains by H_i^c. Due to the strict ordering in D_3 we can subsequently embed every H_i^c in $\mathbb{R}^2$ in a way that $y(v_i) \leq y(v) \leq y(u_i)$ for every $v \in V(H_i^c)$ and every $i \in \{1, \dots, k\}$. As the edges of the H-chains can moreover be embedded as straight-lines, and by construction it holds true that $\min_{v \in V(H_i^c)}\{y(v)\} = y(v_i) > y(u_{i+1}) = \max_{v \in V(H_{i+1}^c)}\{y(v)\}$ for every $i \in \{1, \dots, k-1\}$, the embeddings of the resulting H-chains do not have crossing edges (see Figure 3.5). As a direct consequence, the subsequent embedding of H-chains H_i^c only affects the number of faces given by the initial embedding of G, meaning that the number of resulting faces arising from embedding an H-chain H_i^c does not depend on previously embedded H-chains H_j^c for any $j < i$. This argument together with Lemma 3.20 for each of the k many H-insertion chains yields that the number of faces in G' is upper-bounded by $f + 8kf$ and that the graph G' is planar embeddable. It holds moreover true by construction that $|V(G')| \leq n_G + 8k \in \mathcal{O}(n_G^2)$, and hence, after construction, a plane straight-line embedding of G' can be calculated in polynomial time on G by Theorem 3.18. Altogether, we conclude that G' is constructible in polynomial time from G. □

If we would not have a y-monotone embedding, then the above reasoning on why consecutive H-insertion chains are not affected by each other would not hold. We want to argue that the assumption is not necessary, yet useful as we do not have to make case-distinctions or deal with corner cases. To see this, assume that we are given a simple straight-line embedding. Then, order the degree-three vertices by decreasing y-coordinates and subsequently by decreasing x-coordinate in the cases where the y-coordinates are equal. Now proceed as in the proof of Lemma 3.22 with the only difference being that if the y-coordinates of two vertices u, v are equal, then we have to embed the H-graphs such that the y-coordinates of their vertices are bigger than $y(u)$ and $y(v)$, in order to guarantee that subsequent H-chains do not intersect the previous ones. Thus, one has to argue why this is always a possible construction. This would need an adapted definition of H-chains, as then we could not look at the intersections of the direct line ℓ_v^u with the embedding of edges as we have done in Definition 3.19. We would then again need to argue using „small ϵ-shifts". Note that such a proof would be rather technical, which has lead to our decision on the assumptions for the graph embeddings.

We are now ready to prove the construction needed in the proof of Theorem 3.7.

Lemma 3.23. *Let $G \in \mathcal{G}_3^4$ be a plane graph and let $F := \{F_1, \ldots, F_f\}$ denote its faces. Then we can construct a 4-regular planar graph G' with*

$$\mathrm{FVS}(G) \leq k \iff \mathrm{FVS}(G') \leq k + 3\frac{m}{2}$$

in polynomial time from G, where m denotes the number of H-insertions needed in the construction of G', which is polynomially dependent on G.

Proof. Let G be a planar graph. A planar straight-line embedding with known faces $F := \{F_1, \ldots, F_f\} \cup \{F_o\}$ can be computed in polynomial-time from G, where F_o denotes the canonical outer-face that may not be a cycle and does not need to be known explicitly [**thm˙strlineemb**; HT74; MM96]. Recall that we assumed the given embedding to be y-monotone and that the vertices are in general position. Given the embedding, use Lemma 3.17 to reduce the number of degree-three vertices for every face F_i with $i \in \{1, \ldots, f\}$ to one. Note that after each iteration no new faces with multiple degree-three vertices can arise, therefore it suffices to apply Lemma 3.17 for each of the f initial faces once, where $f \leq n_G$. The resulting graph G' is by construction planar and the number of faces in G' is upper-bounded by $f' \leq f + 8fn_G < 9n_G^2$. A planar straight-line embedding of G' can now be constructed in polynomial-time from G' thus in polynomial-time on G; altogether yielding that G', as well as a known planar embedding of G', can be constructed in polynomial time on G. As less than n_G many H-insertions are needed, we conclude that $|V(G')| \in \mathcal{O}(n_G^2)$.

Let $D_3 := \{v \mid \deg(v) = 3\} \subset G'$ denote the set of degree-three vertices in G'. Then by construction it holds true that $|F_i \cap D_3| \leq 1$ for every $i \in \{1, \ldots, f'\}$. Let $d_3 := |D_3'| \leq f'$ denote the number of degree-three vertices in G', and note that $d_3 \in 2\mathbb{N}$ as $G' \in \mathcal{G}_3^4$. Now order the vertices in $D_3 = \{v_1, \ldots, v_{d_3}\}$ by strictly decreasing y-coordinates in $\mathcal{O}(d_3 \log d_3)$ time. Apply Lemma 3.22 together with Lemma 3.21 in order to produce a 4-regular plane graph G'' in polynomial time from G'. As we need d_3 many H-insertion-chains, Lemma 3.22 yields that the number of faces in G'' is upper-bounded by $f' + 8d_3f' \in \mathcal{O}(n_G^3)$ and $|V(G'')| \in \mathcal{O}(n_{G'}^2) = \mathcal{O}(n_G^4)$, concluding that altogether G'' can be constructed in polynomial time from G. As finally the number of H-insertions needed to construct G'' from G is altogether upper-bounded by some $m \in \mathcal{O}(n^4)$, Corollary 3.14 concludes the proof. □

We have now gathered all the „machinery“ needed for a proof of Theorem 3.7. We have proven that, given a planar graph $G \in \mathcal{G}_3^4$, we can construct a planar 4-regular graph G', by connecting the degree-three vertices in G' by H-chains while keeping the cardinality of feedback vertex sets in G' polynomially dependent on the input. Thus we have given a polynomial-time many-one reduction from Feedback Vertex Set on planar graphs of minimum degree three and maximum degree four to Feedback Vertex Set on 4-regular planar graphs.

Proof of Theorem 3.7. By Theorem 3.8 we know that Feedback Vertex Set is NP-complete on planar graphs with mininum degree three and maximum degree four. By Lemma 3.23 we can construct a 4-regular planar graph G' from a planar graph $G \in \mathcal{G}_3^4$ in polynomial time by $f(n_G)$ inductive H-insertions such that $\text{FVS}(G) \leq k \iff \text{FVS}(G') \leq k + f(n_G)$ where $f \in \mathcal{O}(x^p)$ for some fixed $p \in \mathbb{N}$. Finally, the fact that Feedback Vertex Set on 4-regular planar graphs is contained in NP concludes the proof. □

In a last step we use Theorem 3.7 to prove the main result of this section—namely Theorem 3.5—stating that Feedback Vertex Set remains NP-complete on 4-regular planar Hamiltonian graphs.

The proof will be done via a polynomial-time many-one reduction from Feedback Vertex Set on 4-regular planar graphs; the reduction is highly inspired by the work of Fleischner and Sabidussi [FS03] and Fleischner, Sabidussi, and Sarvanov [FSS10]. The construction of the Hamiltonian graph will rely on what we will call *L-insertions*, which are very similar to H-insertions and turn out to have analogous properties.

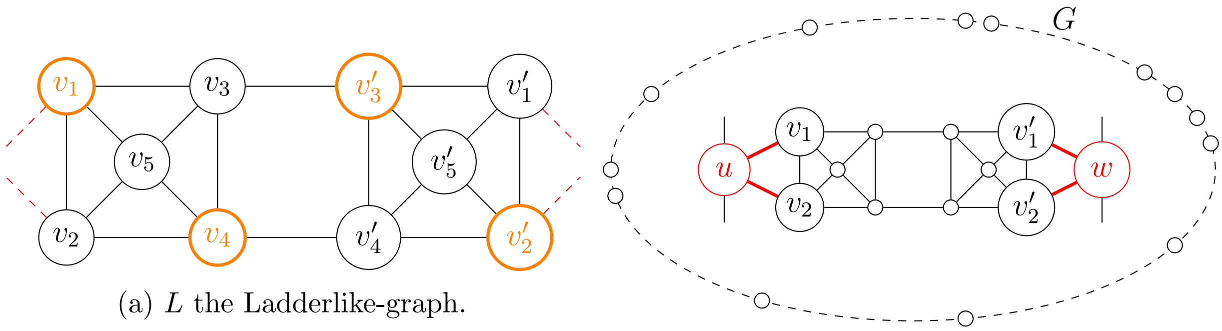
Definition 3.24 (Ladderlike-graph L and L-insertion on u, w)**.** We define the Ladderlike-graph L to be the graph on ten vertices as depicted in Figure 3.6a. Let G be a graph and let $u, w \in V(G)$ be two different vertices. Then an L-insertion on u, w results in a new graph G' by a disjoint union of G with L (maybe after relabelling the vertices in $V(L)$) and subsequently connecting u to both v_1 and v_2, and w to both v_1' and v_2' as illustrated in Figure 3.6b.

Analogously to the H-insertions, we will prove some properties on L-insertions quantifying their impact on the cardinality of feedback vertex sets and the topology of the input graphs.

Remark. Note that L is a planar graph, and given a plane graph G and two vertices v, w on a same face, an L-insertion on v, w can be carried out in such a way that the resulting graph G' stays planar by embedding L planar „inside“ that face.

Lemma 3.25. *Let G be a graph, and let L be the Ladderlike-graph from Definition 3.24. Let G' be the graph resulting from an L-insertion connecting two different vertices $u, v \in V(G)$. Then,*

$$\text{FVS}(G) \leq k \iff \text{FVS}(G') \leq k + 4$$



(a) L the Ladderlike-graph.

(b) L-insertion connecting u, w.

Figure 3.6: The Ladderlike-graph L from Definition 3.24 and an L-insertion on u, w. In Figure 3.6a the (orange) thickly drawn vertices denote a feedback vertex set in L and the (red) dashed vertices indicate the edges arising after an L-insertion (see the (red) thick edges in Figure 3.6b). In Figure 3.6b a detail of a graph G after an L-insertion on u, w is shown. The (red) thick edges show the new edges arising from the L-insertion while the dashed edges denote some of the possible edges in G.

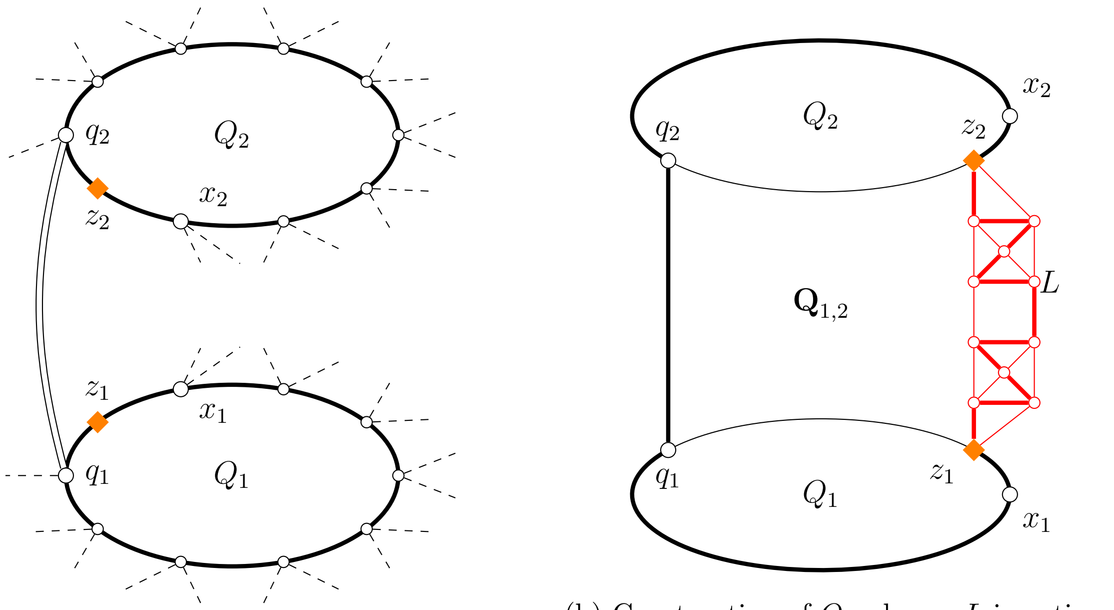
for every $k \in \mathbb{N}$.

Proof. First note that L has a feedback vertex set of cardinality at least four as the disjoint subgraphs induced by $v_1, \ldots, v_5$ and $v'_1, \ldots, v'_5$ each have a feedback vertex set of size at least two, respectively. It is easy to see that $I := \{v_1, v_4, v'_3, v'_2\}$ is a feedback vertex set of size four in L (see in Figure 3.6a the (orange) thickly drawn vertices). Now $L - I$ is exactly the two induced disjoint paths given by (v_2, v_5, v_3) and (v'_4, v'_5, v'_2).

⇒: Let $S \subset V(G)$ be a feedback vertex set in G with $|S| \leq k$ for some $k \in \mathbb{N}$. We claim that $S' := S \cup I$ is a feedback vertex set in G'. In order to see this, note that S' is a feedback vertex set in $G'[V(G)] = G$ and $G'[V(L)] \cong L$. Therefore, any remaining cycle $\mathcal{C}$ in $G' - S'$ must contain at least one vertex in $V(L)$. As $L - I$ is composed of two disjoint paths, either the path (v_2, v_5, v_3) or the path (v'_4, v'_5, v'_1) must be part of $\mathcal{C}$. As $\deg(v_3) = 1 = \deg(v'_4)$, neither path can be part of $\mathcal{C}$, thus S' is a feedback vertex set in G'.

⇐: Let S' be a feedback vertex set in G' with $|S'| \leq k + 4$. Since $|S' \cap V(L)| \geq 4$ as $\mathrm{FVS}(L) \geq 4$, and since $V(L) \cap V(G) = \emptyset$, we can conclude that $|S' \cap V(G)| \leq k + 4 - 4 = k$, and thus $S' \cap V(G)$ is a feedback vertex set of size k in G. □

The next lemma proves that, given a 4-regular planar graph, we can construct a 4-regular planar Hamiltonian graph by consecutive L-insertions in polynomial time. The ideas behind Lemma 3.26 and its proof are similar to, and inspired by the previous work done by Fleischner, Sabidussi, and Sarvanov [FSS10, Lemma 2.3]. We will first determine a 2-factor $Q = \{Q_1, \ldots, Q_p\}$ of G for some $p < n_G$, which is a set of disjoint 2-regular subgraphs $Q_i \subseteq G$ such that $\bigcup_{i=1}^{p} V(Q_i) = V(G)$. We will then extend Q inductively



(a) 2-factors Q_1 and Q_2.

(b) Construction of $Q_{1,2}$ by an L-insertion.

Figure 3.7: The thick edges denote the edges of the 2-factors Q_1, Q_2 and Q. The double edge represents the edge between q_1 and q_2 in G that will be part of Q. The vertices x_1 and x_2 are neighbors of q_1 and q_2 respectively, and lie on a common face. The (orange) diamond-shaped vertices z_1 and z_2 represent newly introduced subdivision vertices on which we make an L-insertion (see Figure 3.7b). The (red) edges and vertices next to L highlight the ladderlike-graph and new edges resulting from an L-insertion. Dashed edges represent all the remaining edges in G of the vertices that are part of either 2-factor.

to a 2-factor Q' having one component less than Q, by L-insertions as can be seen in Figure 3.7a. This will then result in a 4-regular Hamiltonian graph. Note that for 4-regular graphs such a 2-factor Q always exists [Mul92]. Moreover, determining a 2-factor (if it exists) can be done in $\mathcal{O}(n_G)$ time by Petersen's method [FS03, Proposition 2.1].

Lemma 3.26. *Let G be a 4-regular plane graph of order n_G. Then we can construct a 4-regular plane Hamiltonian graph G' from G with $d < n_G$ many L-insertions in polynomial-time from G. Moreover, it holds true that*

$$\mathrm{FVS}(G) \leq k \iff \mathrm{FVS}(G') \leq k + 4d$$

for every $k \in \mathbb{N}$.

Proof inspired by Fleischner, Sabidussi, and Sarvanov [FSS10, Lemma 2.3]. Determine a 2-factor $Q = \{Q_1, \ldots, Q_p\}$ of G for some $p \in \mathbb{N}$, which can be done in polynomial time on G. Now, if $p = 1$, then Q consists of only one component, namely Q_1. Since Q_1 is by construction 2-regular, it is a cycle; and since $V(Q) = V(G)$ it follows that G is Hamiltonian.

Otherwise let $Q_1, Q_2 \in Q$ be two different components such that there are $q_1 \in V(Q_1)$ and $q_2 \in V(Q_2)$ with $\{q_1, q_2\} \in E(G)$. In order to determine Q_1 and Q_2, construct $G_Q := (V_Q, E_Q)$ where $V_Q := \{t_i \mid Q_i \in Q\}$ and $E_Q := \{\{t_i, t_j\} \mid E(G) \cap \{q, q' \mid q \in V(Q_i), q' \in V(Q_j)\}\}$, the graph obtained by contracting each component $Q_i \in Q$ to a single vertex. One can think of G_Q as a graph having the components of the 2-factor as vertices, which are connected by an edge if the respective components are connected by an edge in G. Let E_T be a spanning tree of the graph G_Q. Now we may choose Q_1 and Q_2 such that the corresponding vertices in G_Q are connected by an edge, hence such that $\{t_1, t_2\} \in E_T$. This whole construction can be done in polynomial time on G as was shown by Fleischner, Sabidussi, and Sarvanov [FSS10, Lemma 2.3] (see also [FS03, Proposition 2.1]). As G is 4-regular, Q_1 and Q_2 must be cycles in G. Now let $\{q_1, x_1\} \in E(Q_1)$ and $\{q_2, x_2\} \in E(Q_2)$ be such that x_1, x_2 lie on a common face in G, and subdivide these edges by introducing respective new vertices z_1 and z_2 (see Figure 3.7a). Note that due to Corollary 3.16 this does not affect the cardinality of feedback vertex sets in G. Now construct a plane graph G' from G by an L-insertion on z_1, z_2 and recall that this yields $\deg(z_1) = 4 = \deg(z_2)$ and $\deg(v) = 4$ for every $v \in V(L)$; thus G' is again 4-regular. Update the 2-factor $Q' := (Q \cup \{Q_{1,2}\}) \setminus \{Q_1, Q_2\}$, where $Q_{1,2}$ is constructed as suggested by Figure 3.7b with $V(Q_{1,2}) = V(Q_1) \cup V(Q_2) \cup V(L)$. By construction Q' is a 2-factor of G' with $|Q'| = |Q| - 1$ and G' is a plane 4-regular graph. Repeat this procedure $|E_T| \le n_G$ many times until $|Q'| = 1$, while keeping G' 4-regular and plane. Finally G' is a 4-regular planar Hamiltonian graph obtained from G by $d := |E_T|$ many L-insertions. Inductive use of Lemma 3.25 yields $\mathrm{FVS}(G) \le k \iff \mathrm{FVS}(G') \le k + 4d$, concluding the proof. □

We are finally ready to prove Theorem 3.5.

Proof of Theorem 3.5. As we can find a planar embedding of a graph G in polynomial time (Theorem 3.18), the proof follows directly by a polynomial-time many-one reduction from Feedback Vertex Set on 4-regular planar graphs and Theorem 3.7 together with Lemma 3.26 and the fact that Feedback Vertex Set on 4-regular planar Hamiltonian graphs is trivially contained in NP. □

3.3 k-regular Hamiltonian graphs.

Having considered Feedback Vertex Set on 4-regular (planar) Hamiltonian graphs, a natural question would be whether Feedback Vertex Set remains NP-complete restricted to k-regular Hamiltonian graphs for every $k \in \mathbb{N}$. Li and Liu [LL99] showed that Feedback Vertex Set is polynomial-time solvable for 3-regular graphs.

Theorem 3.27 (Li and Liu [LL99]). Feedback Vertex Set *on 3-regular graphs is polynomial-time solvable.*

As the class of 3-regular Hamiltonian graphs is a subclass of the general 3-regular graphs we conclude the following.

Corollary 3.28. FEEDBACK VERTEX SET *on* 3*-regular Hamiltonian graphs is polynomial-time solvable.*

It turns out that the class of 3-regular graphs is a particular case, as we will show that FEEDBACK VERTEX SET is NP-complete on k-regular Hamiltonian graphs for every fixed $k \geq 4$. We will prove this result via inductive polynomial-time many-one reductions from FEEDBACK VERTEX SET on $(k+1)$-regular Hamiltonian graphs to FEEDBACK VERTEX SET on k-regular Hamiltonian graphs, knowing that FEEDBACK VERTEX SET is NP-complete on 4-regular Hamiltonian graphs as a special case of Theorem 3.5. We will prove a slightly stronger claim, namely that FEEDBACK VERTEX SET remains NP-complete on k-regular Hamiltonian graphs of even degree for every $k \geq 4$. The reason why we will be looking at graphs of even degree, is that the construction we want to use in the reductions needs the degree of the vertices to be even. This comes from the fact that we want to pair up vertices and subsequently connect them by introducing a „gadget“ similar to most of our previous proofs.

Theorem 3.29. FEEDBACK VERTEX SET *on k-regular Hamiltonian graphs of even order with known Hamilton-cycle is NP-complete for every fixed $k \geq 4$.*

The proof of Theorem 3.29 will be done in two major steps. As a first step we will prove the statement for the special case $k = 5$. This intermediate step is necessary, as in the second step we will inductively reduce FEEDBACK VERTEX SET on $(k+1)$-regular graphs of even order to FEEDBACK VERTEX SET on k-regular graphs of even order, which will then conclude the proof as any 5-regular graph is of even order. The proof of this special case uses already all of the needed ideas for the proof of Theorem 3.29, and it highlights why a case distinction into graphs of even and odd order is necessary in our construction.

Theorem 3.30. FEEDBACK VERTEX SET *is NP-complete on* 5*-regular Hamiltonian graphs with known Hamilton-cycle.*

The proof follows by a polynomial-time many-reduction from the class of 4-regular Hamiltonian graphs. The construction in the reduction will need a special case of graphs that we will call *parachute-like graphs* (subsequently denoted by P_k). Parachute-like graphs will play a key-role in the proof of Theorem 3.29. One can think of the parachute-like graph P_k as the graph resulting from two disjoint copies U and W of the complete graph K_k, such that each vertex in U has a 1-to-1 correspondence to a vertex in W, that is, they are connected by an edge. Moreover, each vertex in U is connected to an *apex-vertex* $\hat{u}$ and each vertex in W to another *apex-vertex* $\hat{w}$ (see the schematic Figure 3.8).

Definition 3.31 (Parachute-like graph of order k.)**.** A *parachute-like graph of order k,*

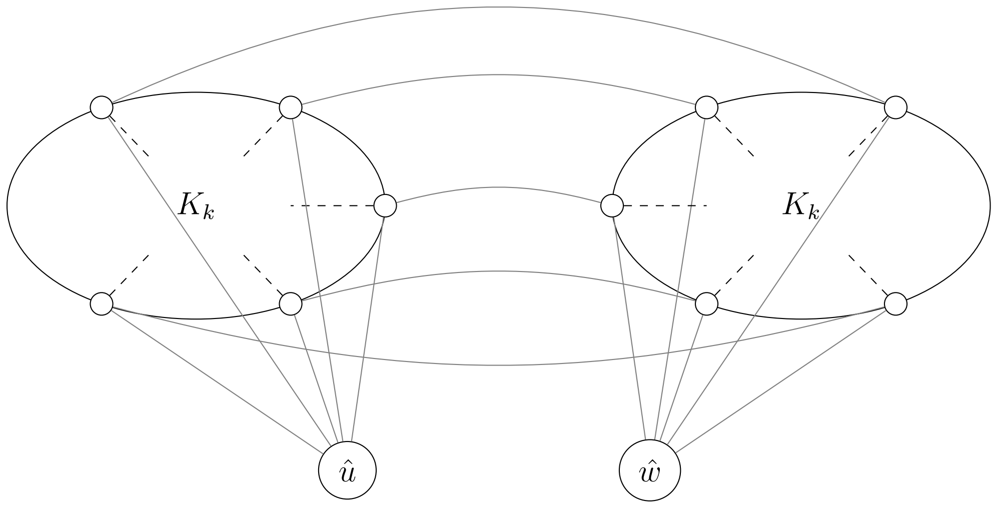


Figure 3.8: Parachute-like graph. The vertices $\hat{u}$ and $\hat{w}$ represent the apex-vertices. The ellipses schematically represent a K_k, where dashed edges indicate missing edges that have not been drawn explicitly. The bend edges between the two K_k's represent the edges between U and W in Definition 3.31 that are in 1-to-1 correspondence.

denoted by $P_k := (V^p, E^p)$, is a graph on $2k+2$ vertices given by:

$$\begin{aligned} &V^p := U \uplus W \uplus \{\hat{u}, \hat{w}\}, \text{ where} \\ &\quad U := \{u_i \mid i \in \{1, \dots, k\}\}, \text{ and } W := \{w_i \mid i \in \{1, \dots, k\}\}, \text{ and} \\ &E^p := E_U \uplus E_W \uplus E_{\hat{u}} \uplus E_{\hat{w}} \uplus E_{UW}, \text{ where} \\ &\quad E_U := \{\{u_i, u_j\} \mid u_i, u_j \in U, i \neq j\}, E_W := \{\{w_i, w_j\} \mid w_i, w_j \in W, i \neq j\}, \\ &\quad E_{\hat{u}} := \{\{\hat{u}, u_i \mid u_i \in U, i \in \{1, \dots, k\}\}, E_{\hat{w}} := \{\{\hat{w}, w_i \mid w_i \in W, i \in \{1, \dots, k\}\}, \text{ and} \\ &\quad E_{UW} := \{\{u_i, w_i\} \mid u_i \in U, w_i \in W, i \in \{1, \dots, k\}\}. \end{aligned}$$

The vertices $\hat{u}$ and $\hat{w}$ will be referred to as the *apex-vertices* of P_k.

Remark. It is clear by construction that for a parachute-like graph P_k of order $k \in \mathbb{N}$, every vertex $v \in V(P_k) \setminus \{\hat{u}, \hat{w}\}$ has degree $k+1$, and $\deg(\hat{u}) = k = \deg(\hat{w})$.

The main idea behind parachute-like graphs is that, given a Hamiltonian graph G and a Hamilton-cycle $\mathcal{C}$ in G, we can connect a parachute-like graph to two consecutive vertices in the cycle in order to augment their degree in the graph, while keeping the resulting graph Hamiltonian and controlling its maximal vertex degree. This procedure will be referred to as *parachute-apexing* (see Figure 3.9) and is defined as follows.

Definition 3.32 (Parachute-apexing on u, v given P_k)**.** Let G be a graph and $u, w \in V(G)$ be two different vertices. Let P_k be a parachute-like graph of order $k \in \mathbb{N}$ with

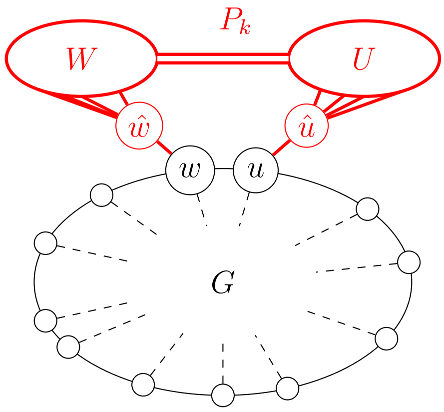


Figure 3.9: Parachute-apexing on u, w given P_k. The apex-vertices are denoted by $\hat{u}, \hat{w}$ and respectively connected by a thick (red) edge to every vertex in either of the two complete components U and W (as given in Definition 3.31) in P_k represented by thick (red) ellipses. The thick (red) double edge between the two ellipses in P_k denotes the one-to-one connection between vertices in U and W. Dashed edges represent some of the possible missing edges in G.

apex-vertices $\hat{u}, \hat{w}$. We define the graph G' resulting from *parachute-apexing on u, v given P_k* by:

$$V(G') := V(G) \uplus V(P_k), \text{ and}$$
$$E(G') := E(G) \uplus E(P_k) \uplus \{\{u, \hat{u}\}, \{w, \hat{w}\}\}.$$

See Figure 3.9 for a schematic illustration of parachute-apexing on u, w.

Analogously as for the H-insertions, we need to analyze the impact of parachute-apexing on the topology of the input graph G and the cardinality of feedback vertex sets in G. The following lemmata and their proofs are very similar to the ones in Section 3.2.

Lemma 3.33. *Let G be a Hamiltonian graph with known Hamilton-cycle $\mathcal{C}$ and $u, w \in V(G)$ such that u, w are neighbors in $\mathcal{C}$ with $\deg(u) = k = \deg(w)$ for some $k \in \mathbb{N}$. Let P_k be a parachute-like graph of order k with apex-vertices $\hat{u}, \hat{w}$. Let G' denote the graph obtained by parachute-apexing on u, w, then G' is Hamiltonian, $\deg(u) = k+1 = \deg(w)$ and $\deg(v) = k + 1$ in G' for each $v \in V(P_k)$. The construction can be done in $\mathcal{O}(k^2)$ time.*

Proof. Let $\mathcal{C} = (v_0, \ldots, u, w, \ldots, v_{n_G}, v_0)$ denote a Hamilton-cycle in G. Let $u_1, \ldots, u_k \in U \subset P_k$, as well as $w_1 \ldots, w_k \in W \subset P_k$ be as in Definition 3.31. We give an explicit Hamilton-cycle in G':

$$\mathcal{C}' := v_0 \mathcal{P}^{\mathcal{C}} u \diamond (\hat{u}, u_1, \ldots, u_k, w_k, \ldots, w_1, \hat{w}) \diamond w \mathcal{P}^{\mathcal{C}} v_0,$$

which is obviously a possible construction as the needed adjacency of vertices is given. Since $N_{G'}(u) \setminus N_G(u) = \{\hat{u}\}$, it follows that $\deg(u) = k + 1$, and analogously it follows thatdeg(w) $= k+1$, as well as $\deg(\hat{u}) = k+1$ and $\deg(\hat{w}) = k+1$. Note that for every $v \in V(P_k) \setminus \{\hat{u}, \hat{w}\}$ it holds true by construction that $\deg(v) = k+1$. The construction is feasible in $\mathcal{O}(k^2)$ time after reading G as $2k+2$ vertices and $\mathcal{O}(k^2)$ many edges are added to G, concluding the proof. $\square$

Lemma 3.34. *Let P_k be a parachute-like graph of order $k \in \mathbb{N}$ for some $k \geq 4$, and let $\hat{u}$ and $\hat{w}$ denote its apex-vertices. Then it holds true that* $\mathrm{FVS}(P_k) \geq 2(k-1)$.

Proof. Let $V(P_k) = U \cup W \cup \{\hat{u}, \hat{w}\}$ as given by Definition 3.31. As a first step, note that $P[U \cup \{\hat{u}\}] \cong K_{k+1}$. Together with $k \geq 3$, this yields that $\mathrm{FVS}(P_k[U \cup \{\hat{u}\}]) \geq k-1$. Analogously one can show that $\mathrm{FVS}(P_k[W \cup \{\hat{w}\}]) \geq k-1$, and since $U \cap W = \emptyset$, it follows that $\mathrm{FVS}(P_k) \geq 2(k-1)$. It is easy to verify that $I := \{\hat{u}, u_1, u_2, \ldots, u_{k-2}, \hat{w}, w_3, w_4, \ldots, w_k\}$ is a feedback vertex set in P_k of cardinality $2(k-1)$, as $P_k - I$ is composed of two disjoint paths of length two, namely (u_{k-1}, u_k) and (w_1, w_2). Note that the given feedback vertex set of order $2(k-1)$ needed the assumption that $k \geq 4$. $\square$

The next lemma quantifies the impact of parachute-apexing on the cardinality of feedback vertex sets in a G.

Lemma 3.35. *Let G be a graph, and let $u, w \in V(G)$ be two different vertices. Let P_k be a parachute-like graph of order $k \in \mathbb{N}$ for some $k \geq 4$, and let $\hat{u}, \hat{w}$ denote its apex-vertices. Let G' be the graph resulting from G by parachute-apexing on u, w. Then,*

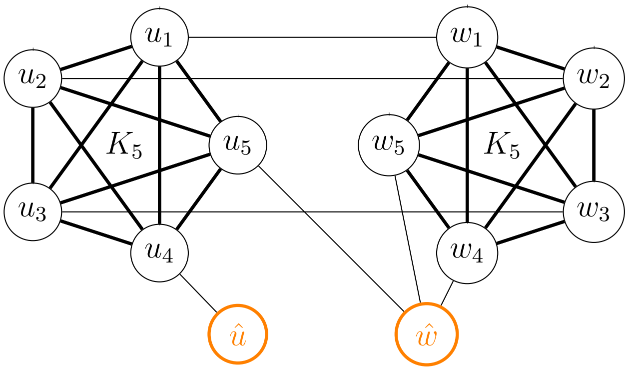
$$\mathrm{FVS}(G) \leq n \iff \mathrm{FVS}(G') \leq n + 2(k-1)$$

for every $n \in \mathbb{N}$.

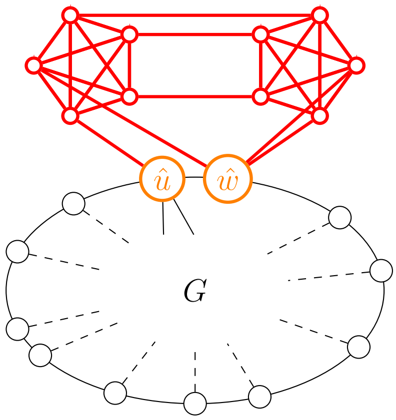
Proof. Let $V(P_k) = U \cup W \cup \{\hat{u}, \hat{w}\}$ as in Definition 3.31. As introduced in the proof of Lemma 3.34, let $I := \{\hat{u}, u_1, \ldots, u_{k-2}, \hat{w}, w_3, \ldots, w_k\}$ be a feedback vertex set of cardinality $2(k-1)$ in P_k.

- $\Rightarrow$: Let $S \subset V(G)$ be a feedback vertex set in G of size $n \in \mathbb{N}$. We claim that $S' := S \cup I$ is a feedback vertex set in G'. By construction S' is a feedback vertex set in $G'[V(G)]$ and in $G'[V(P_k)]$. Moreover, $\{\hat{u}, \hat{w}\} \subset V(G')$ separates $G'[V(P_k)]$ from $G'[V(G)]$ and thus there cannot be any cycle in $G' - I$ through both components, concluding that S' is a feedback vertex set in G' with $|S'| \leq |S| + 2(k-1)$.
- $\Leftarrow$: Let $S' \subset V(G')$ be a feedback vertex set in G' with $|S'| \leq n + 2(k-1)$. Then S' must contain at least $2(k-1)$ vertices in $V(P_k)$. Let $I' := S' \cap V(P_k)$, and let $S := S' \setminus I'$. Since $V(P_k) \cap V(G) = \emptyset$ it holds true that $S = S' \cap V(G)$. Therefore S is a feedback vertex set in $G'[V(G)] = G$ with $|S| = |S'| - |I'| \leq n$. $\square$

Note that parachute-apexing on two vertices u, v will augment their degrees by one, respectively. Hence, parachute-apexing on two vertices of degree k gives rise to two vertices of degree $k+1$. The main idea behind the proof of Theorem 3.30 (and Theorem 3.29) is to pair up the vertices in a given k-regular Hamiltonian graph, and conecutively perform



(a) Gad-graph.



(b) Gad-insertion.

Figure 3.10: The Gad-graph and a Gad-insertion on $\hat{u}, \hat{w}$. The thick (orange) vertices labelled with $\hat{u}, \hat{w}$ denote the common vertices to Gad and G. The thick edges in Figure 3.10a highlight isomorphisms to K_5. The thick (red) unlabelled vertices and edges in Figure 3.10b denote the vertices and edges in the Gad-graph that are distinct from the vertices in G. Dashed edges highlight some of the possible missing edges in G while all other edges denote edges that are part of the respective graphs.

parachute-apexing on the pairs in order to construct a $k+1$-regular Hamiltonian graph. Unfortunately, this is not possible if the graph has odd order, as then one vertex will be left out in the pairing procedure. We will thus introduce a derived version of the parachute-like graph of order five that will be referred to as the *Gad-graph*. The Gad-graph is a „gadget“ that will only be used to augment the vertex degree of the vertex that is left out during the pairing procedure.

Definition 3.36 (Gad-insertion on u, w)**.** Let the auxiliary graph Gad be defined as follows:

$$
\begin{aligned}
&V(\text{Gad}) := \{\hat{u}, \hat{w}, u_i, w_i, \mid i \in \{1,2,3,4,5\}\}, \text{ and}\\
&E(\text{Gad}) := \{\{u_i, u_j\}, \{w_i, w_j\} \mid i, j \in \{1,2,3,4,5\}, i \neq j\} \cup\\
&\quad \{\{u_1, w_1\}, \{u_2, w_2\}, \{u_3, w_3\}, \{\hat{u}, u_4\}, \{\hat{w}, u_5\}, \{\hat{w}, w_4\}, \{\hat{w}, w_5\}\}.
\end{aligned}
$$

Let G be a graph with vertices labelled such that $V(G) \cap V(Gad) = \{\hat{u}, \hat{w}\}$, where $\hat{u}, \hat{w}$ are two different vertices. Then a *Gad-insertion on* $\hat{u}, \hat{w}$ results in a new graph $G' := G \cup Gad$. The Gad-graph and a schematic Gad-insertion are depicted in Figure 3.10a and Figure 3.10b respectively.

Observation 1. It is clear by construction that for the Gad-graph from Definition 3.36 it holds that $\deg(w_i) = 5 = \deg(u_i)$ for every $i \in \{1,2,3,4,5\}$, as well as $\deg(\hat{w}) = 3$ and $\deg(\hat{u}) = 1$ (see Figure 3.10a).

Essentially the Gad-graph and the Gad-insertion share the same properties as the parachute-like graphs and parachute-apexing. We will again analyze the impact of Gad-insertions on the topology of a given graph G and the cardinality of a minimum feedback vertex set in G. The statements are similar, and their proofs are mostly analogous to the respective ones concerning parachute-like graphs.

Lemma 3.37. *Let G be a Hamiltonian graph and $\mathcal{C}$ a Hamilton-cycle in G. Let $\hat{u}, \hat{w} \in V(G)$ be vertices such that $\hat{u}$ and $\hat{w}$ are neighbors in $\mathcal{C}$ with* $\deg(\hat{u}) = 4$ *and* $\deg(\hat{w}) = 2$. *Let* Gad *be defined as in Definition 3.36 with $V(G) \cap V(\mathrm{Gad}) = \{\hat{u}, \hat{w}\}$. Let G' denote the graph obtained by a* Gad*-insertion on $\hat{u}, \hat{w}$ in G. Then G' is Hamiltonian,* $\deg(\hat{u}) = 5 = \deg(\hat{w})$ *and* $\deg(v) = 5$ *for each $v \in V(G'[V(\mathrm{Gad})])$. The construction can be done in constant time after reading G.*

Proof. Let $\mathcal{C} = (v_0, \ldots, \hat{u}, \hat{w}, \ldots, v_{n_G}, v_0)$ denote a Hamilton-cycle in G. Let $V(\mathrm{Gad}) = \{\hat{u}, \hat{w}, u_i, w_i \mid i \in \{1, 2, 3, 4, 5\}\}$ as given by Definition 3.36. We give an explicit Hamilton-cycle in G' by

$$\mathcal{C}' := v_0 \mathcal{P}^{\mathcal{C}} \hat{u} \diamond (\hat{u}, u_4, u_5, u_1, u_2, u_3, w_3, w_2, w_1, w_5, w_4, \hat{w}) \diamond \hat{w} \mathcal{P}^{\mathcal{C}} v_0,$$

which is obviously a possible construction. Since $N_{G'}(\hat{u}) \setminus N_G(\hat{u}) = \{u_4\}$, it follows that $\deg(\hat{u}) = 4 + 1 = 5$ and analogously it follows that $\deg(w) = 5$. The construction is feasible in constant time after reading G as exactly ten vertices and 27 edges need to be added. ☐

Lemma 3.38. *Let G denote a graph, let* Gad *denote the auxiliary graph from Definition 3.36 and let $\hat{u}, \hat{w} \in V(G) \cap V(\mathrm{Gad})$ be two different vertices (maybe after relabelling $V(G)$). Let G' be the resulting graph from a* Gad*-insertion on $\hat{u}, \hat{w}$. Then,*

$$\mathrm{FVS}(G) \leq k \iff \mathrm{FVS}(G') \leq k + 6$$

for every $k \in \mathbb{N}$.

Proof. Analogously to the argument in the proof of Lemma 3.35, note that Gad has two disjoint subgraphs isomorphic to K_5 and thus $\mathrm{FVS}(\mathit{Gad}) \geq 6$. Additionally it is easy to verify that $I := \{u_4, u_5, w_4, w_5, u_1, w_2\}$ is a feedback vertex set in Gad of size six, where the vertices u_i, w_i and edges between them are defined as in Definition 3.36. Note that the vertices $\hat{u}$ and $\hat{w}$ cannot be part of a minimum feedback vertex set in Gad as $\mathrm{Gad} - \{\hat{u}, \hat{w}\}$ still has two disjoint subgraphs isomorphic to K_5 and thus $\mathrm{FVS}(\mathrm{Gad} - \{\hat{u}, \hat{w}\}) \geq 6$.

- $\Rightarrow$: Let $S \subset V(G)$ be a feedback vertex set in G with $\mathrm{FVS}(S) \leq k$. We claim that $S' := S \cup I$ is a feedback vertex set in G'. By construction S' is a feedback vertex set in $G'[V(G)] \cong G$ and in $G'[V(\mathrm{Gad})] = \mathrm{Gad}$. Moreover, $\{u_4, u_5, w_4, w_5\}$ separates $G'[V(\mathrm{Gad})]$ from $G'[V(G)]$ and thus there cannot be any cycle in $G' - S'$ through both components. Hence S' is a feedback vertex set in G' with $|S'| \leq |S| + 6$.

$\Leftarrow$: Let $S' \subset V(G')$ be a feedback vertex set in G' with $|S'| \leq k+6$. Then S' contains at least six vertices in $V(\text{Gad}) \setminus \{\hat{u}, \hat{w}\}$. Let $I' := S' \cap V(\text{Gad})$ and let $S := S' \setminus I'$. Now as $V(\text{Gad}) \cap V(G) = \{\hat{u}, \hat{w}\}$, it holds true that $I' \cap V(G) = \emptyset$ and thus S must still be a feedback vertex set in $G'[V(G)] \cong G$ with $|S| = |S'| - |I'| \leq k$. $\square$

We are now ready to give the polynomial-time many-one reduction needed for the proof of Theorem 3.30. As mentioned above, the main idea lies in pairing up the vertices in a given Hamiltonian graph with respect to its Hamilton-cycle, and subsequently perform parachute-apexing on the constructed pairs. If the order of the graph is odd, then one vertex will be left-out in the pairing procedure. Only in that case we will make use of the Gad-insertion.

Lemma 3.39. *Let G be a 4-regular Hamiltonian graph and $\mathcal{C}$ a Hamilton-cycle in G. Then, we can compute a 5-regular Hamiltonian graph G' with*

$$\text{FVS}(G) \leq k \iff \text{FVS}(G') \leq f(k, n_G),$$

in polynomial time from G, where

$$f(k, n_G) := \begin{cases} k + 4n_G, & \text{if } n_G \bmod 2 = 0, \\ k + 4(n_G - 1) + 6, & \text{if } n_G \bmod 2 = 1. \end{cases}$$

Proof. Let $\mathcal{C} := (v_0, \ldots, v_{n_G-1}, v_0)$ be a Hamilton-cycle in G, and let $k \in \mathbb{N}$ with $\text{FVS}(G) \leq k$.

Case 1: $n_G \bmod 2 = 0$. Pair the vertices in $\mathcal{C}$ up in $\frac{n_G}{2}$ pairs of the form $P := \{(v_i, v_{i+1}) \mid i \in \{0, \ldots, n_G - 1\} \cap (2\mathbb{N})\}$. This can be done in $\mathcal{O}(n_G)$ time. We will refer to the pairs as $P := (p_1, \ldots, p_{\frac{n_G}{2}})$. For the sake of readability we relabel the pairs such that $p_i = (u_i, w_i)$ for every $i \in \{1, \ldots, \frac{n_G}{2}\}$. Let P_5^j be parachute-like graphs of order five for each $j \in \{1, \ldots, \frac{n_G}{2}\}$. We will construct the graph with the needed properties stated in Lemma 3.39 inductively. To this end, let $G_0 := G$ and for $1 \leq i < \frac{n_G}{2}$ suppose that a Hamiltonian graph G_i with $\deg(u_j) = 5 = \deg(w_j)$ and $\deg(v) = 5$ for every $v \in V(P_5^j)$ and every $j \in \{0, \ldots, i\}$ was already constructed. Now we construct G_{i+1} from G_i by parachute-apexing on $(u_{i+1}, w_{i+1}) = p_{i+1} \in P$ given P_5^{i+1} (see Definition 3.32). Lemma 3.33 guarantees that G_{i+1} is Hamiltonian (by inductive construction), as well as that $\deg(u_j) = 5 = \deg(w_j)$ and $\deg(v) = 5$ for every $v \in V(P_5^j)$ and for every for every $j \in \{0, \ldots, i+1\}$. Finally set $G' := G_{\frac{n_G}{2}}$. Then, the inductive use of Lemma 3.33 in the construction of G_i guarantees that G' is Hamiltonian and 5-regular. As G' was constructed by parachute-apexing on distinct vertices in G altogether less than n_G many times, inductive use of Lemma 3.35 yields that $\text{FVS}(G') \leq k + \frac{n_G}{2} \cdot 2(5-1) = k + 4n_G$. As parachute-apexing given parachute-like graphs of order five can be done in constant time after reading G (special case of Lemma 3.33), we can construct G' in $\mathcal{O}(n_G)$ time after reading G concluding the proof of *Case 1.*

Case 2: $n_G \bmod 2 = 1$. As in *Case 1*, pair up the vertices in $\mathcal{C}$ until v_{n_G} is the only vertex left that has not been paired. Let $P := \{(v_i, v_{i+1}) \mid i \in \{0, \ldots, n_G - 1\} \cap (2\mathbb{N})\}$ denote the pairs. As in *Case 1*, construct G_i inductively by parachute-apexing on the pairs given by P. Denote the resulting graph from the last induction step by $\hat{G}$. By the same reasoning as in *Case 1*, $\hat{G}$ is Hamiltonian, it can be constructed in $\mathcal{O}(n_G)$ time, and for each vertex $v \in V(\hat{G})$ with $v \neq v_{n_G}$ it holds true that $\deg(v) = 5$. Moreover,

$$\mathrm{FVS}(\hat{G}) \leq k + \frac{n_G - 1}{2} \cdot 2(5 - 1) = k + 4(n_G - 1). \tag{3.1}$$

By construction of $\hat{G}$ we implicitly constructed a Hamiltonian cycle $\hat{\mathcal{C}}$ in $\hat{G}$ (see the proof of Lemma 3.33), hence $\hat{\mathcal{C}}$ is a known Hamilton-cycle in $\hat{G}$. Now let $v \in V(\hat{G})$ be a neighbor of v_{n_G} with respect to the cycle $\hat{\mathcal{C}}$. Subdivide the edge $\{v, v_{n_G}\}$ by a new vertex z. By Corollary 3.16 this does not affect the feedback vertex set of $\hat{G}$. Now $\deg(z) = 2$ and $\deg(v_{n_G}) = 4$ and they are neighbors in $\hat{\mathcal{C}}$. Relabel z by $\hat{u}$ and v_{n_G} by $\hat{w}$ and let Gad be the graph as in Definition 3.36 such that $V(\mathrm{Gad}) \cap V(G) = \{\hat{u}, \hat{w}\}$. Construct G' by a Gad-insertion on $\hat{u}, \hat{w}$. By Lemma 3.37, the resulting graph G' is Hamiltonian with $\deg(\hat{u}) = 5 = \deg(\hat{w})$ and thus G' is 5-regular. By Lemma 3.38 it follows that $\mathrm{FVS}(G') \leq \mathrm{FVS}(\hat{G}) + 6$. Together with Inequality 3.1 this yields

$$\mathrm{FVS}(G') \leq k + 4(n_G - 1) + 6 \tag{3.2}$$

As the Gad-insertion can be done in constant time after reading $\hat{G}$ (and the cycle $\hat{\mathcal{C}}$), we can contruct G' in polynomial-time from G, concluding the proof of *Case 2*. □

We are now ready to prove Theorem 3.30.

Proof of Theorem 3.30. By Theorem 3.7, it is known that FEEDBACK VERTEX SET is NP-complete on 4-regular planar Hamiltonian graphs with known Hamilton-cycle. Together with the polynomial-time many-one reduction from Lemma 3.39, and the fact that FEEDBACK VERTEX SET is contained in NP for the class of 5-regular Hamiltonian graphs, we can conclude the proof. □

In a next step we will use Theorem 3.30 to inductively prove Theorem 3.29. The proof works analogously to the proof of *Case 1* in Lemma 3.39 as we will only be looking at graphs of even order, as can be seen by the following observations.

Observation 2. The parachute-like graph of order $k \in \mathbb{N}$, denoted by P_k, fulfills $|V(P_k)| \in 2\mathbb{N}$. Thus the number of vertices in P_k is even.

Observation 3. Let G be a graph with an even (odd) number of vertices and $v, w \in V(G)$ two different vertices, then the graph G' obtained by parachute-apexing on v, w given some parachute-like graph P_k of order $k \in \mathbb{N}$, has an even (odd) number of vertices.

Remark. Observation 3 states that the parity of the order of a graph is invariant under parachute-apexing.

The following lemma is a natural generalization of Lemma 3.39 to graphs of even order.

Lemma 3.40. *Let G be a k-regular Hamiltonian graph with $V(G) \in 2\mathbb{N}$ and $\mathcal{C}$ be a Hamilton-cycle in G for some $k \geq 5$. Let P_k^j denote parachute-like graphs of order k for each $j \in \{1, \dots, \frac{n_G}{2}\}$. Then a $(k+1)$-regular Hamiltonian graph G' can be constructed in polynomial-time from G by parachute-apexing given P_k^j such that*

$$\mathrm{FVS}(G) \leq p \iff \mathrm{FVS}(G') \leq p + n_G(k-1)$$

for every $p \in \mathbb{N}$.

Proof. As $V(G) \in 2\mathbb{N}$ the proof is analogous to the proof of *Case 1* in Lemma 3.39 (it follows by inductive use of Lemma 3.33 and Lemma 3.35) and can be readily generalized to this case. □

Observation 4. If the graph G in the construction of Lemma 3.40 has an even number of vertices, then it follows by Observation 3 that the resulting graph G' has an even number of vertices.

We are now ready to prove Theorem 3.29. The proof will be done by induction over $k \in \mathbb{N}$. As the same proof-scheme will be used in later chapters, we will now give a detailed proof that easily generalizes to the problems analyzed in the following chapters.

Proof. Proof of Theorem 3.29 We prove the statement by induction on $k \in \mathbb{N}$. *Induction base:* Let $k = 5$. Then by Theorem 3.30 Feedback Vertex Set is NP-complete on 5-regular Hamiltonian graphs of even order with known Hamilton-cycle.

Induction step: Let $k \geq 5$ and denote by $H_k^{\mathcal{C}}$ the set of tuples $(G, \mathcal{C}')$, where G is a k-regular Hamiltonian graph of even order and $\mathcal{C}'$ a Hamilton-cycle in G. Suppose that Feedback Vertex Set is NP-hard on $H_k^{\mathcal{C}}$. Let

$$\mathrm{Red} : H_k^{\mathcal{C}} \to H_{k+1}^{\mathcal{C}}, \quad G = (V, E) \mapsto G' = (V', E'),$$

where G' is the graph resulting from Lemma 3.40. By Lemma 3.40, the function Red is well-defined and can be evaluated in time polynomial in $|V(G)|$. Moreover, Lemma 3.40 proves that $\mathrm{FVS}(G) \leq k \iff \mathrm{FVS}(G') \leq f(k, G)$ for some polynomial-time function f. Thus it follows that Feedback Vertex Set is NP-hard on $H_{k+1}^{\mathcal{C}}$.

The claim now follows by induction on $k \geq 5$ and the fact that Feedback Vertex Set on k-regular Hamiltonian graphs of even order with known Hamilton-cycle is trivially contained in NP for every $k \in \mathbb{N}$. □

3.4 k-Hamiltonian-ordered graphs.

As mentioned in the introduction, the concept of k-Hamiltonian ordered graphs (originally named k-ordered Hamiltonian graphs), was introduced by Ng and Schultz [NS97] as a new strong Hamiltonian property. For completion and as a reminder, we give a definition of k-Hamiltonian ordered graphs.

Definition 3.41 (Ng and Schultz [NS97])**.** A graph G is called *k-Hamiltonian ordered* if for every sequence $v_1, \ldots, v_k$ of k distinct vertices in G, there exists a Hamilton-cycle that encounters $v_1, \ldots, v_k$ in that order.

A direct consequence from Definition 3.41 is that every Hamiltonian graph is also 3-Hamiltonian ordered. To see this, note that if G is Hamiltonian, there is a cycle $\mathcal{C}$ visiting every vertex in G, and note that every three points on a cycle can be visited in any desired order by varying the starting vertex of the cycle and the „direction" in which the cycle traverses the vertices.

Observation 5. Every Hamiltonian graph is 3-Hamiltonian ordered.

A result due to Ng and Schultz [NS97] shows that k-ordered graphs, and thus k-Hamiltonian ordered graphs, are $(k-1)$-connected. Hence our following results in the study of the computational complexity of Feedback Vertex Seton k-Hamiltonian ordered graphs hold true for $(k-1)$-connected Hamiltonian graphs as well.

Theorem 3.42 (Ng and Schultz [NS97])**.** *Let G be a k-ordered graph for some $k \in \mathbb{N}$ with $k \geq 3$. Then G is $(k-1)$-connected.*

In this section, we will prove that Feedback Vertex Set remains NP-complete when restricted to k-Hamiltonian ordered graphs for every fixed $k \geq 3$.

Theorem 3.43. Feedback Vertex Set *on k-Hamiltonian-ordered graphs with known Hamilton-cycle is NP-complete for every $k \geq 3$.*

Note that the property for a graph to be k-Hamiltonian ordered is way more restrictive than being Hamiltonian, and yet Theorem 3.43 shows that it does not affect the computational complexity of Feedback Vertex Set on a high level. The proof will be given by induction on k, reducing Feedback Vertex Set on $(k+1)$-Hamiltonian ordered graphs to Feedback Vertex Set on k-Hamiltonian ordered graphs, and using the fact that Feedback Vertex Set is known to be NP-complete on 3-Hamiltonian-ordered graphs as given by Theorem 3.1 together with Observation 5.

Ng and Schultz [NS97] have proven the following, which will play a key-role for the graph construction of our polynomial-time many-one reduction.

Theorem 3.44 (Ng and Schultz [NS97])**.** *Let G be a graph of order $n \geq 3$ and let k be an integer with $3 \leq k \leq n$. If*

$$\deg(x) + \deg(y) \geq n + 2k - 6$$

for every pair x, y of nonadjacent vertices in G, then G is k-Hamiltonian ordered.

The main idea in our construction will be, given some k-Hamiltonian ordered graph G, to construct a graph satisfying the needed properties in Theorem 3.44 for $k+1$, and thus guaranteeing it to be $(k+1)$-Hamiltonian ordered. The graph in the construction will be denoted by $\operatorname{chap}(G)$ and referred to as the *chapiteau-closure of G*. The idea for the name comes from the fact that the illustration given by Figure 3.11 resembles a „chapiteau", which is the french word for circus tent.

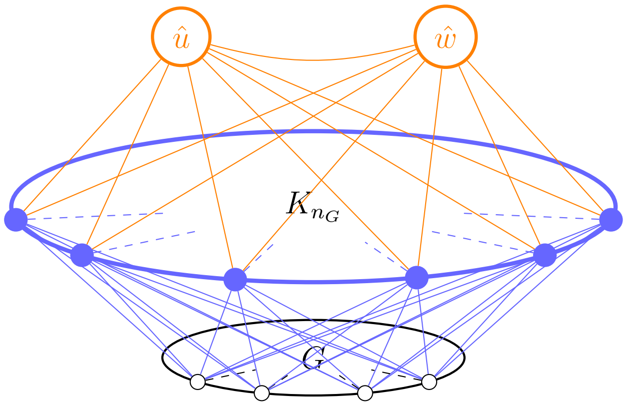


Figure 3.11: Chapiteau-closure of G. The thick (orange) vertices labelled with $\hat{u}, \hat{w}$ denote the apex-vertices given in Definition 3.45. The thick (light-blue) ellipse and the filled (light-blue) vertices denote vertices in K_{n_G}. The dashed edges in K_{n_G} and G indicate some of their missing edges. All other edges illustrate edges in $\text{chap}\,(G)$.

Definition 3.45 (Chapiteau-closure of G)**.** Let G be a graph of order n_G and let K_{n_G+2} denote the complete graph on $n_G + 2$ vertices. The *chapiteau-closure of* G, denoted by $\text{chap}\,(G)$, is defined by

$$\begin{aligned}
V(\text{chap}\,(G)) &:= V(G) \uplus V(K_{n_G}) \uplus \{\hat{u}, \hat{w}\}, \text{ and} \\
E(\text{chap}\,(G)) &:= E(G) \uplus E(K_{n_G}) \uplus \\
&\qquad \{\{x, v\}, \{\hat{u}, v\}, \{\hat{w}, v\}, \{\hat{u}, \hat{w}\} \mid x \in V(G), v \in V(K_{n_G})\},
\end{aligned}$$

and is schematically depicted in Figure 3.11. We will refer to $\hat{u}$ and $\hat{w}$ as the *apex-vertices* of $\text{chap}\,(G)$.

The following lemma proves that, given a k-Hamiltonian ordered graph G, the chapiteau-closure $\text{chap}\,(G)$ fulfills the prerequisites in order to apply Theorem 3.44. Thus, $\text{chap}\,(G)$ is $(k+1)$-Hamiltonian ordered.

Lemma 3.46. *Let G be a k-Hamiltonian ordered graph for some $k \in \mathbb{N}$ with $k \geq 3$. Then, $\text{chap}\,(G)$ is a $(k+1)$-Hamiltonian ordered graph and can be constructed in $\mathcal{O}(n_G^2)$ time from G.*

Proof. That $\text{chap}\,(G)$ is $(k+1)$-Hamiltonian ordered follows directly by Observation 5 and Theorem 3.44 as mentioned above. To construct $\text{chap}\,(G)$ it suffices to add a vertex-disjoint K_{n_G+2} to G, and add $\mathcal{O}(n_G^2)$ many edges respectively, which can altogether be done in $\mathcal{O}(n_G^2)$ time. $\square$

The next lemma quantifies the impact of the chapiteau-closure on the cardinality of feedback vertex sets in the input graph.

Lemma 3.47. *Let G be a graph of order $n_G \geq 3$, and let* $\mathrm{chap}\,(G)$ *denote its chapiteau-closure. Then,*

$$\mathrm{FVS}(G) \leq k \iff \mathrm{FVS}(\mathrm{chap}\,(G)) \leq k + n_G$$

for any $k \in \mathbb{N}$.

Proof. Recall that by construction $\mathrm{chap}\,(G) - V(G) \cong K_{n_G+2}$. Moreover, it holds true that $\mathrm{FVS}(K_{n_G+2}) \geq n_G$ as $n_G \geq 3$. Denote the apex-vertices of $\mathrm{chap}\,(G)$ by $\hat{u}$ and $\hat{w}$ respectively.

- $\Rightarrow$: Let $S \subset V(G)$ be a feedback vertex set in G with $|S| \leq k$. Let $S' := \{v \mid v \in V(K_{n_G+2}) \setminus \{\hat{u}, \hat{w}\}\}$. Then S' is by construction a feedback vertex set of size n_G in $\mathrm{chap}\,(G)[V(K_{n_G+2})]$. It is easy to verify that $S \cup S'$ is a feedback vertex set in $\mathrm{chap}\,(G)$ as $\hat{u}$ and $\hat{w}$ are not connected to any vertex in $V(G)$. Thus, we conclude that $\mathrm{FVS}(\mathrm{chap}\,(G)) \leq |S| + |S'| \leq k + n_G$.

- $\Leftarrow$: Let $S' \subset V(\mathrm{chap}\,(G))$ be a feedback vertex set in $\mathrm{chap}\,(G)$ with $|S'| \leq k+n_G$. The set S' is by construction a feedback vertex set in $\mathrm{chap}\,(G)[V(G)] = G$. As there has to be $U = S' \cap V(K_{n_G+2})$ with $|U| \geq n_G$ and $U \cap V(G) = \emptyset$, it holds true that $S := S' \setminus U$ is a feedback vertex set in $\mathrm{chap}\,(G)[V(G)] = G$ with $|S| \leq k$. □

We are now able to prove Theorem 3.43. The proof follows by induction on $k \geq 3$ and is analogous to the proof of Theorem 3.29.

Proof of Theorem 3.43. The proof follows by induction on $k \geq 3$. To see this, note that if Feedback Vertex Set on k-Hamiltonian-ordered graphs is known to be NP-hard, then Lemma 3.46 and Lemma 3.47 give rise to a polynomial-time many-one reduction from Feedback Vertex Set on k-Hamiltonian ordered graphs to Feedback Vertex Set on $(k+1)$-Hamiltonian ordered graphs, concluding that the latter problem is NP-hard. As Feedback Vertex Set on k-Hamiltonian ordered graphs is trivially contained in NP, the induction start for $k = 3$, proven as Theorem 3.1, together with Observation 5 concludes the proof. □

As mentioned in the beginning of this section, a direct corollary to Theorem 3.43 due to Theorem 3.42 is given by the following.

Corollary 3.48. Feedback Vertex Set *is NP-complete on k-connected Hamiltonian graphs with known Hamilton-cycle, for every $k \in \mathbb{N}$.*

4 The 3-Coloring problem on Hamiltonian graphs

The problem of deciding whether a graph can be colored with at most k colors for some integer k, such that no two adjacent vertices share a same color, can be traced back to the 19^{th} century and is claimed to be one of the oldest known graph theoretic problems [Kub04]. Since then, graph-coloring problems can be found in many different applications: Marx [Mar04] highlights a deep connection to scheduling problems, whereas Chaitin [Cha82] makes use of graph coloring techniques for register allocations. Lewis [Lew15] highlights numerous other applications. Over the past decades many variations of graph coloring problems have been thoroughly analyzed on a broad variety of graph classes. The problem of deciding whether a graph G is 3-colorable turned out to be of particular interest, and can be stated as follows.

3-Coloring
Input: An undirected graph G.
Question: Is there a coloring $c : V(G) \to \{1, 2, 3\}$ such that for all $\{v, w\} \in E(G) \Rightarrow c(v) \neq c(w)$?

The 3-Coloring problem was one of the first problems that have been proven to be NP-complete [GJ90]. Many results analyzing the algorithmic complexity of graph-coloring problems on a multitude of graph-classes followed [Abo+17; Dai80; FS03; Krá+01], as well as exact exponential-time algorithms solving them [BE05; BK97].

In this chapter we will analyze the computational complexity of 3-Coloring on Hamiltonian graphs and restricted subclasses thereof—namely k-regular Hamiltonian graphs and k-ordered Hamiltonian graphs. In the first section we will prove that 3-Coloring remains NP-complete on Hamiltonian graphs. In the second section we will prove that 3-Coloring remains NP-complete on k-regular Hamiltonian graphs for every $k \geq 4$ by an inductive use of polynomial-time many-one reductions. We will conclude the chapter in the third section with a proof that 3-Coloring remains NP-complete when restricted to k-Hamiltonian ordered graphs, for every $k \geq 3$. Note that the presented results are readily generalised to the general Coloring, and k-Coloring problems with $k \geq 3$.

4.1 General Hamiltonian graphs.

It is commonly known that 3-Coloring is NP-complete on general graphs [GJ90]. As a first result in this chapter we prove that 3-Coloring remains NP-complete on (planar) Hamiltonian graphs, via a polynomial-time many-one reduction from 3-Coloring

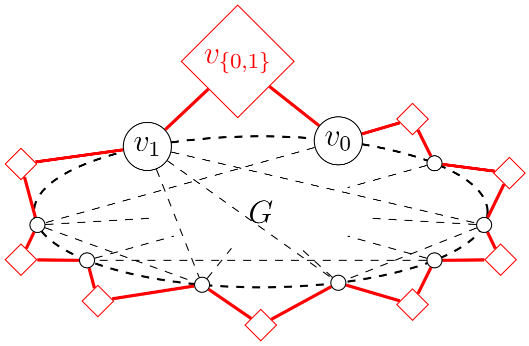


Figure 4.1: A schematic Hamiltonian closure of a graph G. The diamond-shaped (red) vertices denote the vertices in V_{edges}. The thick (red) edges denote the edges in $E_{\mathcal{C}}$ as in Definition 4.2, and they highlight a Hamilton-cycle in the Hamilton closure of G. The dashed edges and the dashed ellipse represent some of the possible edges in G.

on general graphs. Note that in the reduction construction, a Hamilton-cycle will be explicitly constructed.

Theorem 4.1. 3-COLORING *on Hamiltonian graphs with known Hamilton-cycle is NP-complete.*

The idea behind the reduction is similar to the one provided in the proof of Theorem 3.1; its proof, as well as the proofs of the necessary intermediate lemmata, work analogously to the ones in Chapter 3. Given a graph G and a fixed enumeration of its vertices, we will pair up the vertices and construct a graph G', by adding a vertex for each pair, and connecting it to the both vertices of the pair. This will naturally guarantee G' to be Hamiltonian (see Figure 4.1). We will refer to the resulting graph G' as the *Hamiltonian closure of* G, as its construction seems to be a canonical way to extend a graph such that it becomes Hamiltonian. The Hamiltonian-closure of G will then by construction be 3-colorable if and only if the underlying graph G is.

Definition 4.2 (Hamiltonian closure of a graph)**.** Let $G = (V, E)$ be a connected graph, and let $\Phi := (v_0, \ldots, v_{n_G-1})$ with $v_i \in V$ for all $i \in \{0, \ldots, n_G - 1\}$, and $v_i \neq v_j$ for every $i \neq j$, denote an enumeration of V. The *Hamiltonian closure of G (through Φ)*, denoted by $G' = (V', E')$, is :

$$\begin{aligned}
V' =& V \uplus V_{\text{edges}}, \text{ where} \\
&V_{\text{edges}} := \{v_{\{i,i+1\} \bmod n_G} \mid i \in \{0, \ldots, n_G - 1\}\} \text{ and,} \\
E' =& E \uplus E_{\mathcal{C}}, \text{ where} \\
&E_{\mathcal{C}} := \{\{v_i, v_{\{i,i+1\}}\}, \{v_{\{i,i+1\}}, v_{i+1}\} \mid i \in \{0, \ldots, n_G - 2\}\} \cup \\
&\qquad\qquad \{\{v_{n_G-1}, v_{\{n_G-1,0\}}\}\{v_{\{n_G-1,0\}}, v_0\}\}.
\end{aligned}$$

A schematic construction of G' from G is visualized in Figure 4.1.

The proof of Theorem 4.1 now follows immediately from the following two lemmata that are similar to Lemma 3.3 and Lemma 3.4,and whose statements have already been discussed above.

Lemma 4.3. *The Hamiltonian closure of a graph $G = (V, E)$, denoted by $G' = (V', E')$, is Hamiltonian, and can be constructed in polynomial time from G.*

Proof. We claim that $\mathcal{C} := (v_0, v_{\{0,1\}}, \ldots, v_{\{n_G-1,v_0\}}, v_0)$ is by construction a Hamilton-cycle in G'. To see this, note that $\{v_i, v_{\{i,i+1\} \bmod n_G}\} \in V(G')$ for every $i \in \{0, \ldots, n_G - 1\}$, proving that $\mathcal{C}$ is a cycle in G'. As moreover $V' = V(\mathcal{C})$, the cycle $\mathcal{C}$ visits every vertex in V' exactly once. It is clear that the construction in Definition 4.2 can be realized in polynomial time in $|V(G)|$ as G' can be constructed from G by adding n_G many new vertices and $2n_G$ many new edges. □

Lemma 4.4. *Let $G = (V, E)$ be a graph. Then G is 3-colorable if and only if the Hamiltonian closure of G denoted by $G' = (V', E')$ is 3-colorable.*

Proof. Let $V' = V \cup V_{\text{edges}}$ as in Definition 4.2.

⇒: Suppose G is 3-colorable and let $c : V \to \{1, 2, 3\}$ be a 3-coloring of G. Then we claim that $c' : V' \to \{1, 2, 3\}$ defined by
$c'(v) = c(v)$ if $v \in V$, and
$c'(v_e) \in \{1, 2, 3\} \setminus \{c(w) \mid w \in N(v_e)\}$ if $v_e \in V_{\text{edges}}$,
is a well-defined 3-coloring on G'. Since c' is a 3-coloring on the induced graph $G'[V]$ with $V \subset V'$, and since every vertex $v_{\{i,i+1\} \bmod n_G} \in V_{\text{edges}}$ has exactly two neighbors, namely $v_i, v_{(i+1) \bmod n_G} \in V$, the set $\{1, 2, 3\} \setminus \{c(v) \mid v \in N(v_{\{i,i+1\} \bmod n_G})\}$ is not empty; hence c' is well-defined. Moreover, this concludes that the extension of c' on V_{edges} remains a 3-coloring by construction, proving that c' is a 3-coloring on G'.

⇐: Suppose G' has a 3-coloring c'. By construction G is a subgraph of G', hence the restricted coloring $c := c'_{|V}$ is a 3-coloring on the induced subgraph $G'[V] = G$, concluding the proof. □

We are now ready for the proof of Theorem 4.1.

Proof of Theorem 4.1. By Lemma 4.3 and Lemma 4.4, together with the fact that 3-Coloring is known to be NP-hard on general graphs (see [GJ90]), prove that 3-Coloring is NP-hard on Hamiltonian graphs. Since 3-Coloring on Hamiltonian graphs is trivially contained in NP, it follows that 3-Coloring on Hamiltonian graphs is NP-complete. □

Having shown that 3-Coloring remains NP-complete on Hamiltonian graphs, we will now prove that 3-Coloring remains NP-complete even when restricted to planar Hamiltonian graphs. Dailey [Dai80] showed that 3-Coloring remains NP-complete when restricted to 4-regular planar graphs. We will use this result in order to prove that 3-Coloring remains NP-complete on planar Hamiltonian graphs, via a polynomial-time many-one reduction.

Theorem 4.5. 3-Coloring *is NP-complete on planar Hamiltonian graphs.*

The reduction will use an analogous construction and an analogous argumentation to the reduction in the proof of Lemma 3.26, that we needed for the proof of Theorem 3.5. We emphasize again that the ideas behind the proof are inspired by the work due to Fleischner, Sabidussi, and Sarvanov [FSS10, Lemma 2.3]. The main idea of the proof is to determine a 2-factor Q of G, and iteratively „shrink“ it to a 2-factor Q' consisting of only one component. This then implies that the underlying graph is Hamiltonian. The used construction has to leave the planarity and 3-colorability of G invariant, guaranteeing that the resulting graph will be a planar Hamiltonian graph.

The proof of Theorem 4.5 follows at once from the following lemma and the fact that 3-Coloring is known to be NP-complete on 4-regular planar graphs [Dai80].

Lemma 4.6. *Let G be a 4-regular planar graph. Then we can construct a planar Hamiltonian graph G', such that G' is 3-colorable if and only if G is 3-colorable, in polynomial time from G.*

As the proof of this lemma works analogously to the proof of Lemma 3.26, we will only give an outline and spare some details that may be found in the proof of Lemma 3.26 and the proof due to Fleischner, Sabidussi, and Sarvanov [FSS10, Lemma 2.3].

Proof. As in the proof of Lemma 3.26: determine a 2-factor Q of G, and let $Q_1, Q_2 \in Q$ be two different components such that there are $q_1 \in V(Q_1)$ and $q_2 \in V(Q_2)$ sharing an edge in G (see Figure 3.7a). Now let $x_i \in V(Q_i)$ such that $\{q_i, x_i\} \in E(Q_i)$ for each $i \in \{1, 2\}$ and such that x_1 and x_2 lie on a common face given an embedding of G. Note that until now the construction is identical to the one in the proof of Lemma 3.26. Now let G' be the graph constructed from G by adding one vertex z to G, and subsequently connecting z to x_1 and x_2. Note that G' is 3-colorable if and only if G is 3-colorable, as $\deg(z) = 2$. Update the 2-factor: $Q' := (Q \cup \{Q_{1,2}\}) \setminus \{Q_1, Q_2\}$, where $V(Q_{1,2}) := V(Q_1) \cup V(Q_2) \cup \{z\}$ and $E(Q_{1,2}) := (E(Q_1) \cup E(Q_2) \cup \{\{z, x_1\}, \{z, x_2\}\}) \setminus \{\{x_1, q_1\}, \{x_2, q_2\}\}$. By construction, Q' is a 2-factor of G' with one component less than Q. Repeating this procedure $|Q|$ many times leaves a planar Hamiltonian graph G' that is 3-colorable if and only if G is. As the whole construction can be done in polynomial time (see Lemma 3.26), this concludes the proof. □

Note that the only difference to the proof of Lemma 3.26 is that we do not need to introduce subdivision vertices, and that the L-insertion is replaced by adding a single vertex connecting two adjacent components in G that are given by the Q-factor. Also note that we needed the graphs in the construction to be 4-regular, as this guarantees the existence of a 2-factor as was shown by Petersen (see Mulder [Mul92, Theorem 2] for a reference), and is a needed property in the proof of Lemma 3.26 and the proof due to Fleischner, Sabidussi, and Sarvanov [FSS10, Lemma 2.3].

4.2 k-regular Hamiltonian graphs

There have been several previous results on the computational complexity of 3-Coloring on subclasses of regular graphs [Dai80; FS03]. By Brooks' Theorem [Bro41] it follows that 3-Coloring is polynomial-time solvable on 3-regular graphs.

Theorem 4.7 (Brooks' Theorem [Bro41])**.** *Every graph G with maximum degree Δ has a* Δ-Coloring *unless either (i) G contains $K_{\Delta+1}$ or (ii) $\Delta = 2$ and G contains an odd cycle.*

As $|V(K_4)| = 4$, and as the graphs that are isomorphic to K_4 are the only 3-regular graphs on four vertices, we can identify whether a given graph G is a K_4 by verifying whether $V(G) = 4$. Hence a direct consequence to Theorem 4.7 yields the following.

Corollary 4.8. 3-Coloring *on* 3*-regular Hamiltonian graphs is polynomial-time solvable.*

Fleischner and Sabidussi [FS03] showed that 3-Coloring remains NP-complete on 4-regular Hamiltonian graphs.

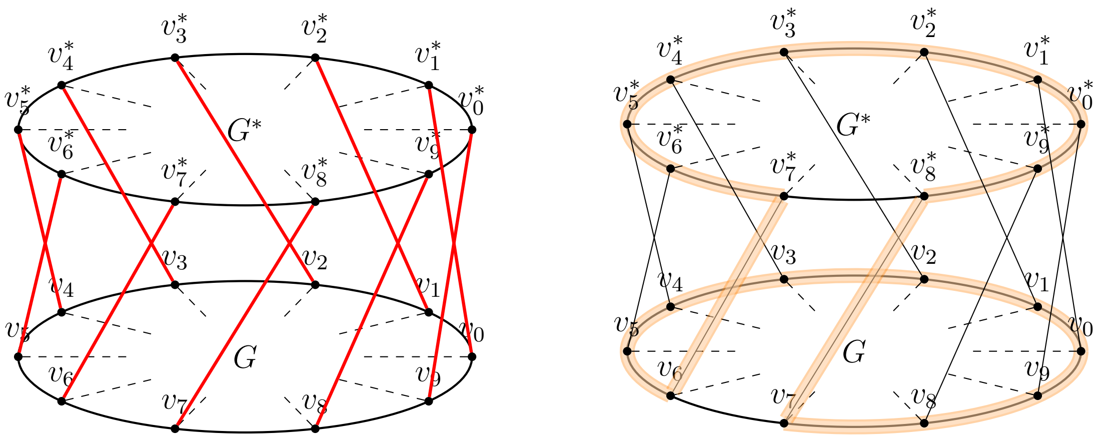
Theorem 4.9 (Fleischner and Sabidussi [FS03, Proposition 2.1])**.** 3-Coloring *is NP-complete on* 4*-regular Hamiltonian graphs with known Hamilton-cycle.*

In this section we follow the spirit of these results, and prove that, for any fixed integer $k \geq 4$, 3-Coloring remains NP-complete on k-regular Hamiltonian graphs. This shows that restricting k-regular graphs to be Hamiltonian in addition, does not affect the tractability of 3-Coloring. We will use Theorem 4.9 as the base case for inductive polynomial-time many-one reductions from 3-Coloring on k-regular Hamiltonian graphs to 3-Coloring on $(k+1)$-regular Hamiltonian graphs, proving the following.

Theorem 4.10. 3-Coloring *on k-regular Hamiltonian graphs with known Hamilton-cycle is NP-complete for every fixed $k \geq 4$.*

For the reductions we will need a general construction that, given a k-regular Hamiltonian graph, constructs a $(k+1)$-regular Hamiltonian graph, while leaving the 3-colorability property invariant. For this purpose, given some graph G, we introduce what we will call the *stacked graph of G*, that turns out to have the desired properties. An intuitive but informal construction of a stacked graph can be thought of as follows: Given some Hamiltonian graph G that is embedded with its vertices lying on an „outer-cycle", the stacked graph G' results from two disjoint copies of G, namely G and G^*, „stacking" G^* on top of G, rotating it counter-clockwise, and then connecting the vertices in the cycle of G^* to the vertices in the cycle of G embedded straight below them (see Figure 4.2a).

Definition 4.11 (Stacked graph of G)**.** Let $G = (V, E)$ be a Hamiltonian graph and without loss of generality let $V(G) := \{v_0, \ldots, v_{n_G-1}\}$ be an enumeration of the vertices induced by the known Hamilton-cycle $\mathcal{C}$, thus $\mathcal{C} = (v_0, \ldots, v_{n_G-1}, v_0)$. Construct $G' :=$



(a) Stacked graph of G

(b) Hamilton-cycle in the stacked graph.

Figure 4.2: Stacked graph G' of a graph G, and a Hamilton-cycle in G'. The subgraph denoted by G^* is a copy of G. The dashed edges denote possible edges in G and G^*, while the (black) edges forming the ellipses highlight the Hamilton-cycles in G and G^* respectively. The thick (red) edges in Figure 4.2a represent the edges in E_{stack}. In Figure 4.2b a Hamilton-cycle in the stacked graph is highlighted by a tube (orange) marking the edges of the cycle.

(V', E') from G as follows:

$$\begin{aligned} V' &= V \uplus V^*, \text{ where} & V^* &:= \{v^* \mid v \in V\} \text{ and,} \\ E' &= E \uplus E^* \uplus E_{\text{stack}}, \text{ where} & E^* &:= \{e^* \mid e \in E\} \text{ and,} \\ & & E_{\text{stack}} &:= \{\{v_i, v^*_{i+1 \bmod n_G}\} \mid i \in \{0, \dots, n_G - 1\}\}. \end{aligned}$$

The resulting graph G' will be referred to as the *stacked graph of* G, and is illustrated in Figure 4.2a.

Similar to the scheme of our previous results, we will now prove that the stacked graph inherits our desired properties: that the stacked graph of a k-regular Hamiltonian graph G is indeed $(k+1)$-regular Hamiltonian, that it can be constructed in polynomial time from G, and that the stacked graph of G is 3-colorable if and only if G is.

Lemma 4.12. *Let $G = (V, E)$ be a k-regular Hamiltonian graph with $k \in \mathbb{N}$ such that $k \geq 4$. Denote by $\mathcal{C}$ a Hamilton-cycle in G. Then, the stacked graph $G' = (V', E')$ of G is a $(k+1)$-regular Hamiltonian graph and can be constructed in polynomial time from G.*

Proof. Let $V' = V \cup V^*$ as described in Definition 4.11. The construction in Definition 4.11 of G' from G can be performed in $\mathcal{O}(\max\{|V(G)|, |E(G)|\})$ time, as the vertex set V^* is a relabelled copy of V and the construction of the edges E^* and E_{stack} can be done in $\mathcal{O}(|E(G)|)$ and $\mathcal{O}(|V(G)|)$ time respectively.

Now let $V(G) = \{v_0, \ldots, v_{n_G-1}\}$ be the enumeration induced by $\mathcal{C}$, meaning that the cycle can be written as $\mathcal{C} := (v_0, \ldots, v_{n_G-1}, v_0)$. Then $\mathcal{C}$ is also a Hamilton-cycle in $G'[V] = G$. Denote by $\mathcal{C}^* := (v_0*, \ldots, v^*_{n_G-1}, v^*_0)$ the respective Hamilton-cycle in the induced (copied) graph $G'[V^*] \cong G$. Then,

$$\mathcal{C}' = (v_0, v_1, \ldots, v_{n_G-1}, v^*_0, v^*_1, \ldots, v^*_{n_G-1}, v_0)$$

is a Hamilton-cycle in the stacked graph G' (see Figure 4.2b). In order to see this, note that $\{v_{n_G-1}, v^*_0\}, \{v^*_{n_G-1}, v_0\} \in E(G')$. Moreover as $G'[V] = G$ and $G'[V^*] \cong G$ are both k-regular, and since for every vertex $v \in V$ there exists exactly one vertex $w^* \in V^*$ (and vice-versa) with $\{v, w^*\} \in E(G')$, it follows that $\deg(v) = k+1$ for every $v \in V'$, concluding that G' is $(k+1)$-regular. □

Lemma 4.13. *Let $k \in \mathbb{N}$ and let $G = (V, E)$ be a k-regular graph of order n_G. Then, G is 3-colorable if and only if the stacked graph $G' = (V', E')$ is 3-colorable.*

Proof. Let $V' = V \cup V^*$ and $E' = E \cup E^* \cup E_{\text{stacked}}$ be as in Definition 4.11.

⇒: Suppose G is 3-colorable and let $c : V \to \{1, 2, 3\}$ be a 3-coloring of G. We claim that c' is a 3-coloring in G' where $c' : V' \to \{1, 2, 3\}$ is defined as follows:

$$c'(v) = c(v), \quad \text{for all } v \in V, \text{ and}$$
$$c'(v^*) = c(v), \quad \text{for all } v^* \in V^*.$$

By construction, c' is a 3-coloring in the induced graphs $G'[V] = G$and $G'[V^*] \cong G$. For every edge $\{v, w^*\} \in E_{\text{stack}}$ it holds true that $c'(v) \neq c'(w^*)$, which concludes that c' is a 3-coloring on G'. To see this, note that otherwise $c'(v) = c'(w^*)$ would imply that $c'(v) = c'(w)$. As $\{v, w\} \in E(G) \subset E(G')$, this would be a contradiction to c' being a 3-coloring on $G'[V]$.

⇐: Suppose that G' is 3-colorable, and let c' be a 3-coloring of G'. By construction, the graph G is a subgraph of G'. Therefore, the restricted coloring $c := c'_{|V}$ is a 3-coloring on the induced graph $G'[V] = G$, concluding the proof. □

Recall that Definition 4.11 is applicable to general graphs, and transforms k-regular Hamiltonian graphs into $(k+1)$-regular Hamiltonian graphs for arbitrary but fixed $k \in \mathbb{N}$. This will be the key to the inductive proof of Theorem 4.10. Also note that the construction of a stacked graph needs a known Hamilton-cycle in the original graph G. The proof is analogous to the inductive proof of Theorem 3.29 and can readily be adapted to this case. For completion we will give a general outline of the proof.

Proof of Theorem 4.10. Note that 3-Coloring on k-regular Hamiltonian graphs is trivially contained in NP. The statement for $k = 4$ forms the base case of our induction and follows by Theorem 4.9. Let $k \geq 4$ and define

$$\text{Red} : H_k^{\mathcal{C}} \to H_{k+1}^{\mathcal{C}}, \quad G = (V, E) \mapsto G' = (V', E'),$$

where G' is the stacked graph of G using similar notation as in the proof of Theorem 3.29. By Lemma 4.12, the function Red is well-defined and can be evaluated in time polynomial in $|V(G)|$. The proof now follows by Lemma 4.13 and induction on k. □

4.3 k-ordered Hamiltonian graphs.

In this section we will prove that 3-Coloring remains NP-complete when restricted to k-ordered Hamiltonian graphs for every fixed $k \geq 3$.

Theorem 4.14. 3-Coloring *is NP-complete on k-ordered Hamiltonian graphs with known Hamilton-cycle for arbitrary but fixed $k \in \mathbb{N}$.*

Unfortunately, we were not able to prove that 3-Coloring remains NP-complete when restricted to k-Hamiltonian ordered graphs, as opposed to the other problems that we analyze throughout thesis. Recall that, as stated in Chapter 3, a result due to Ng and Schultz [NS97] shows that every k-Hamiltonian ordered graph is $(k-1)$-connected (see also Theorem 3.42). Note moreover that k-Hamiltonian ordered graphs are trivially k-ordered and Hamiltonian. Hence, the class of k-Hamiltonian ordered graphs is a subclass of the $(k-1)$-connected and k-ordered Hamiltonian graphs. Aboulker et al. [Abo+17] proved that 3-Coloring remains NP-complete on k-connected graphs, and we prove that 3-Coloring remains NP-complete on k-ordered Hamiltonian graphs. This raises the question whether 3-Coloring remains NP-complete, or becomes polynomial-time solvable on the finer class of k-Hamiltonian ordered graphs,.

The proof of Theorem 4.14 will be done by induction on $k \in \mathbb{N}$ that, on a high level, will be analagous to the proof of Theorem 3.43. We will give a polynomial-time many-one reduction from 3-Coloring on k-ordered Hamiltonian graphs to 3-Coloring on $(k+1)$-ordered Hamiltonian graphs, using the statement for 3-Coloring on 3-ordered Hamiltonian graphs as induction base. To this end, recall that by Observation 5 every Hamiltonian graph is 3-Hamiltonian-ordered (and hence 3-ordered Hamiltonian), which combined with Theorem 4.1 yields the following.

Corollary 4.15. 3-Coloring *is NP-complete on* 3*-Hamiltonian-ordered (and thus* 3*-ordered Hamiltonian) graphs with known Hamilton-cycle.*

The just mentioned reductions will rely on a construction that, when applied to k-ordered Hamiltonian graphs, gives rise to $(k+1)$-ordered Hamiltonian graphs. Proving that a graph G is $(k+1)$-ordered by verifying that, given an arbitrary ordering of length $(k+1)$, there is a cycle in G with respect to the ordering, turned out to be very fastidious. This is why we will give an „indirect“ proof. It turns out that the property of being k-ordered is qualitatively related to the so-called *k-linkage* property as highlighted in a survey done by Faudree [Fau01]. Apparently, for our purposes, k-linkage is a graph-property that is easier to verify. This motivates the introduction of *k-linked* graphs.

Definition 4.16 (k-linked [Fau01, Definition 3])**.** For any $1 \leq k \leq \frac{n_G}{2}$, a graph G of order n_G is *k-linked* if given any collection of k pairs of vertices $L = \{\{x_i, y_i\} \mid 1 \leq i \leq k\}$, there are k vertex disjoint paths (except possibly for endvertices) P_i such that P_i is a path from x_i to y_i.

The three following results that were listed by Faudree [Fau01, Theorem 15], prove that the properties of being k-ordered, k-connected or k-linked are qualitatively related. Note

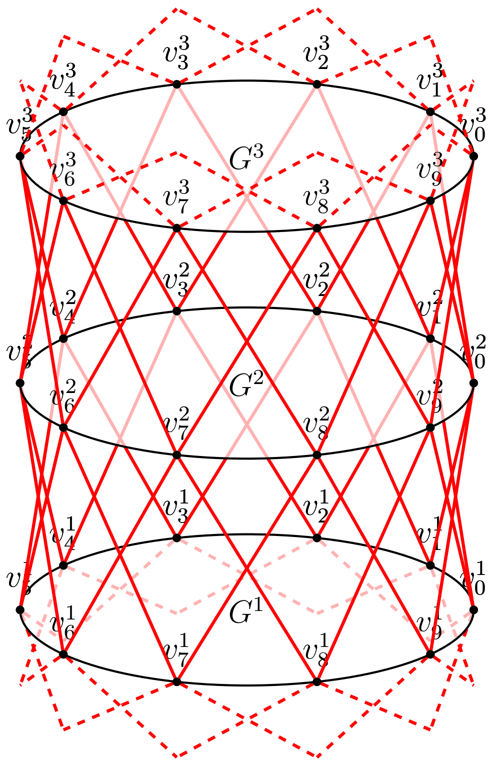


Figure 4.3: A 3-stacked graph $G^{\boxminus^3}$. The three ellipses denote the three disjoint copies of G—namely $G^{\boxminus^3}[V^j]$—here denoted by G^j for $j \in \{1, 2, 3\}$. The thick (red) edges are the edges in E_{stack} between G^1 and G^2, as well as G^2 and G^3. For clarity reasons, the edges between G^1 and G^3 are drawn as disconnected dashed (red) edges „through infinity".

that the first of the three following statements was already encountered as Theorem 3.42 in Chapter 3; for completion we will recall it.

Theorem. *Let G be a k-ordered graph for some $k \in \mathbb{N}$, $k \geq 3$. Then G is $(k-1)$-connected.*

Theorem 4.17 (Bollobás and Thomason [BT96])**.** *Let G be a $22k$-connected graph for some $k \in \mathbb{N}$, $k \geq 3$. Then G is k-linked.*

Theorem 4.18. *Let G be a k-linked graph for some $k \in \mathbb{N}$, $k \geq 3$. Then G is k-ordered.*

If, given a k-ordered graph G, we construct a $22(k+1)$-connected graph G', Theorem 4.17 together with Theorem 4.18 conclude that G' is $(k+1)$-ordered. This motivates the definition of what we will call *p-stacked graphs* (see Figure 4.3), as they turn out to be highly-connected depending on $p \in \mathbb{N}$, and are thus guaranteed to be highly-ordered by construction. Intuitively, the *p-stacked graph* is obtained by taking p disjoint copies of a given graph G, and connecting vertices in different copies by an edge if the „relative vertices" in the original graph are connected by an edge.

Definition 4.19 (p-stacked graph)**.** Let $G = (V, E)$ be a graph of order n_G, and let $p \in \mathbb{N}_{>0}$. Then, the *p-stacked graph*, denoted by $G^{\boxminus p}$, is:

$$V^{\boxminus p} := \bigcup_{i=1}^{p} V^i, \text{ where}$$
$$V^i := \{v^i \mid v \in V\} \text{ for each } i \in \{1, \ldots, p\}, \text{ and}$$
$$E^{\boxminus p} := \bigcup_{i=1}^{p} E^i \cup E_{\text{pstack}}, \text{ where}$$
$$E^i := \{\{u^i, v^i\} \mid \{u, v\} \in E\} \text{ for each } i \in \{1, \ldots, p\}, \text{ and}$$
$$E_{\text{pstack}} := \{\{v^i, w^j\} \mid \{v, w\} \in E, i, j \in \{1, \ldots, p\}, i \neq j\}.$$

Figure 4.3 gives a schematic example of a 3-stacked graph $G^{\boxminus 3}$.

We will now prove that the construction of p-stacked graphs gives rise to polynomial-time many-one reduction from 3-Coloring on k-ordered Hamiltonian graphs to 3-Coloring on $(k+1)$-ordered Hamiltonian graphs. In a first step we prove that, given a Hamiltonian graph G, $G^{\boxminus p}$ is again Hamiltonian, and that it holds true that $G^{\boxminus p}$ is 3-colorable if and only if G is. Again, we would like to mention that we followed this same scheme of intermediate proof steps in several of our previous proofs.

Lemma 4.20. *Let $G = (V, E)$ be a k-ordered Hamiltonian graph with known Hamilton-cycle $\mathcal{C}$ for some $k \geq 3$. Then the p-stacked graph $G^{\boxminus p}$ is Hamiltonian and can be constructed in $\mathcal{O}(p \cdot (|V| + |E|))$ time from G.*

Proof. That $G^{\boxminus p}$ is again Hamiltonian is obvious by construction (see Figure 4.3 for a 3-stacked graph, and see Figure 4.2b for a similar construction with highlighted Hamilton-cycle). Let $\mathcal{C}^0 := (v_0, \ldots, v_{n_G-1}, v_0)$ be a cycle in G. Then, the following is a Hamilton-cycle in $G^{\boxminus p}$:

$$\mathcal{C} := (v_0^1, \ldots, v_{n_G-1}^1, v_0^2, \ldots v_{n_G-1}^{p-1}, v_0^p \ldots, v_{n_G-1}^p, v_0^1),$$

as can be easily verified (see Lemma 4.12 for a similar argumentation). To construct $G^{\boxminus p}$ from G, construct p disjoint copies of G as suggested in Definition 4.19, and add the edges $\{u^i, v^j\}$ for every $\{u, v\} \in E(G)$ and every $i, j \in \{0, \ldots, p\}$ with $i \neq j$, which can be altogether performed in $\mathcal{O}(p \cdot (|V| + |E|))$ time. □

Lemma 4.21. *Let G be a graph and let $p \in \mathbb{N}_{>0}$ be some integer. Then, G is 3-colorable if and only if $G^{\boxminus p}$ is 3-colorable.*

Proof. $\Rightarrow$: Let $c : V(G) \to \{1, 2, 3\}$ be a 3-coloring of G. Define $c' : V(G^{\boxminus p}) \to \{1, 2, 3\}$ as follows:

$$c'(v^i) = c(v) \text{ for } i \in \{1, \ldots, p\}, \text{ and the respective } v \in V(G).$$

We claim that c' is a 3-coloring of $G^{\boxminus p}$. To see this, note that $G^{\boxminus p}[V^i] \cong G$ for every $i \in \{1, \ldots, p\}$, and $c'_{|V^i} = c$, which means that c' induces a 3-coloring on the p many disjoint subgraphs $G^{\boxminus p}[V^i]$. Hence it suffices to prove that the coloring of the vertices forming the edges in E_{pstack} does not violate the 3-coloring property, where E_{pstack} is defined as in Definition 4.19. To this end, let $v^i \in V^i$ and $w^j \in V^j$ with $\{v^i, w^j\} \in E_{\text{pstack}}$ for some $i, j \in \{1, \dots, p\}$. By construction, it holds true that $\{v^i, w^i\}, \{v^j, w^j\} \in E(G^{\boxminus p})$, with respective $v^j, w^i \in V(G^{\boxminus p})$. Since $c'(w^i) = c'(w^j)$ and $c'(v^i) \neq c'(w^i)$, as well as $c'(v^j) \neq c'(w^j)$, it holds true that $c'(v^i) \neq c'(w^j)$. Roughly speaking, this means that the edge $\{v^i, w^j\}$ does not violate the 3-coloring property. Since the edge was an arbitrary edge in E_{pstack}, c' is a 3-coloring on $G^{\boxminus p}$.

$\Leftarrow$: This follows immediately from the fact that G is isomorphic to an induced subgraph of $G^{\boxminus p}$, namely $G \cong G^{\boxminus p}[V^1]$. $\square$

To conclude Theorem 4.14, we need to prove that, given a k-ordered graph, $G^{\boxminus p}$ is $(k+1)$-ordered.

Lemma 4.22. *Let G be a k-ordered graph for some $k \in \mathbb{N}$, $k \geq 3$. Then $G^{\boxminus p}$ is $(k+1)$-ordered.*

We will prove Lemma 4.22 by first proving that, given a k-ordered graph G, $G^{\boxminus p}$ is $p(k-1)$-connected using Menger's Theorem (see Theorem 4.23) [Die12]. Then, for p large enough, $G^{\boxminus p}$ is $(k+1)$-linked as guaranteed by Theorem 4.17. We will finally use Theorem 4.18, stating that k-linked graphs are known to be k-ordered, to conclude the proof.

Theorem 4.23 (Menger's Theorem [Die12])**.** *Let G be a graph. Then, G is k-connected if and only if for any two distinct vertices $x, y \in V(G)$ there exist k pairwise vertex-disjoint paths (except for maybe the end-vertices) connecting x and y in G.*

We will now show that, given a k-ordered graph G, there are $p(k-1)$ many disjoint paths between any two vertices in $G^{\boxminus p}$, which then combined with Menger's theorem yields the following.

Lemma 4.24. *Let G be a k-ordered graph. Then, $G^{\boxminus p}$ is $p(k-1)$-connected.*

Proof. As G is k-ordered, Theorem 3.42 yields that G is $(k-1)$-connected. We prove that given two distinct vertices $x, y \in V(G^{\boxminus p})$, there are $p(k-1)$ disjoint paths connecting x and y in $G^{\boxminus p}$. By Theorem 4.23, we conclude that $G^{\boxminus p}$ is $p(k-1)$-connected. There are two cases for x and y to consider.

Case 1: $x = v_i^j$ and $y = v_i^\ell$ for some $i \in \{0, \dots, n_G - 1\}$ and $j, \ell \in \{1, \dots, p\}$ with $j \neq \ell$. Intuitively, this means that x and y represent the „same vertex in a different copy of G“, which are by construction not connected by an edge. Without loss of generality we may assume that $i = 0$, hence $x = v_0^j$ and $y = v_0^\ell$. By construction of $G^{\boxminus p}$ it holds true that $N(v_0^j) = N(v_0^\ell)$. As by Theorem 3.42, $G^{\boxminus p}[V^m] \cong G$ is $(k-1)$-connected for every $m \in \{1, \dots, p\}$, we conclude that $|N_G(v)| \geq (k-1)$ for every $v \in V(G)$, and thus $|N(w)| \geq p(k-1)$ for every $w \in V(G^{\boxminus p})$ by construction of $G^{\boxminus p}$. As $N(v_0^j) = N(v_0^\ell)$, this concludes that there are at least $p(k-1)$ disjoint paths connecting v_0^j and v_0^ℓ.

Case 2: $x = v_i^j$ and $y = v_m^\ell$ for some $i, m \in \{0, \ldots, n_G - 1\}$ and $j, \ell \in \{1, \ldots, p\}$ with $i \neq m$. Without loss of generality we may assume that $i = 0$, $m = 1$ as well as $j = \ell$; hence $x = v_0^j$ and $y = v_1^j$. To see this, note that as $N(v_1^\ell) = N(v_1^j)$ we may as well „exchange“ v_1^ℓ and v_1^j by relabelling them accordingly, and then look at v_1^j instead. We know that $G^{\boxminus p}[V^j] \cong G$ is $(k-1)$-connected, thus there are at least $(k-1)$ disjoint paths in $G^{\boxminus p}[V^j]$ connecting v_0^j and v_1^j. Analogously there are $(k-1)$ disjoint paths $P_1^q, \ldots, P_{k-1}^q$ connecting v_0^q to v_1^q in $G^{\boxminus p}[V^q]$ for every $q \in \{1, \ldots, p\}$. Note that by construction P_i^q and P_m^ℓ are disjoint for every $i, m \in \{1, \ldots, k-1\}$, and $q, \ell \in \{1, \ldots, p\}$ except if $i = m$ and $q = \ell$ both hold true, since then P_i^q and P_m^ℓ are equal. As $N(v_0^j) = N(v_0^q)$ and $N(v_1^j) = N(v_1^q)$ for every $q \in \{1, \ldots, p\}$, we may exchange the end-vertices of every path P_i^q to be v_0^j and v_1^j respectively. Thus we have found alltogether $p(k-1)$ disjoint paths (except for the end-vertices) connecting v_0^j and v_1^j. By Theorem 4.23 this concludes the proof. □

As mentioned above, we need p to be large enough to ensure that $G^{\boxminus p}$ is $(k+1)$-linked. The following observation states that we can compute the needed integer p in linear time, which is obviously true as we only need to solve the inequality $p(k-1) \geq 22(k+1)$ to get the desired p.

Observation 6. Given a k-ordered graph G for some $k \geq 3$, we can compute $p_k \in \mathbb{N}_{>0}$ in linear time in k, such that $p_k(k-1) \geq 22(k+1)$. Then, by Lemma 4.24, $G^{\boxminus p_k}$ is $22(k+1)$-connected.

The proof of Lemma 4.22 is a direct consequence of Observation 6 together with Theorem 4.17 and Theorem 4.18.

Proof of Lemma 4.22. Let G be a k-ordered graph. By Observation 6, we can compute $p_k \in \mathbb{N}_{>0}$ in linear time on k such that $G^{\boxminus p_k}$ is $22(k+1)$-connected. Applying Theorem 4.17 yields that $G^{\boxminus p_k}$ is $(k+1)$-linked. By Theorem 4.18, $G^{\boxminus p_k}$ is $(k+1)$-ordered. □

We are now ready to prove Theorem 4.14. The proof goes analogously to the proof of Theorem 3.29, which is why we only give an outline of the proof and spare some details.

Proof. For the special case of $k = 3$, the statement follows by Corollary 4.15. Let $k \geq 3$ be arbitrary but fixed, and assume that 3-Coloring is known to be NP-hard on k-ordered Hamiltonian graphs. Given a k-ordered Hamiltonian graph G, we may compute $p_k \in \mathbb{N}_{>0}$ in linear time on $k \in \mathbb{N}$ by Observation 6, and subsequently construct a $(k+1)$-ordered Hamiltoninan graph $G^{\boxminus p_k}$ in polynomial-time from G by Lemma 4.20 and Lemma 4.22. By Lemma 4.21, it holds true that $G^{\boxminus p_k}$ is 3-colorable if and only if G is 3-colorable. Thus, 3-Coloring remains NP-hard on $(k+1)$-ordered Hamiltonian graphs. The proof now follows by induction on k and the fact that 3-Coloring on k-ordered Hamiltonian graphs is trivially contained in NP. □

A direct corollary to Theorem 4.14 is the following.

Corollary 4.25. 3-Coloring *is NP-complete on k-connected Hamiltonian graphs with known Hamilton-cycle, for every $k \in \mathbb{N}$.*

5 The Independent Set Problem on Hamiltonian Graphs

The problem of deciding whether a graph contains an independent set of size k for some integer k can be stated as follows.

INDEPENDENT SET
Input: An undirected graph G, an integer $k \in \mathbb{N}$.
Question: Is there a set $I \subseteq V(G)$ with $|I| = k$ such that for all $v, w \in I$, it holds true that $\{v, w\} \notin E(G)$?

Many variations of INDEPENDENT SET, such as MAXIMUM INDEPENDENT SET, which is the problem of finding an independent set of maximum size in a graph, have many different practical applications. Van Bevern et al. [Van+15] mention relations of the INDEPENDENT SET problem to several scheduling-problems, and Verweij and Aardal [VA99] highlight applications in map labelling. Over the time, there have been many different results regarding INDEPENDENT SET. There have been a multitude of results giving upper-bounds for maximum independent sets in numerous graph classes [Für87; Min80]. As an NP-complete problem [Kar72], INDEPENDENT SET is widely believed to be intractable; many polynomial-time approximation algorithms [Bak83; FRS94], as well as exact exponential-time algorithms [JYP88; TT77] have been developed. The computational complexity of INDEPENDENT SET has been extensively studied on numerous different graph classes [Ale+08; AP89; FSS10; Kar72].

In this chapter we will analyze the computational complexity of INDEPENDENT SET on Hamiltonian graphs and restricted subclasses thereof—namely k-regular Hamiltonian graphs and k-ordered Hamiltonian graphs. We will prove that INDEPENDENT SET remains NP-complete on k-regular Hamiltonian graphs, as well as on k-Hamiltonian ordered graphs for every fixed $k \geq 3$. As a special case we will prove that INDEPENDENT SET remains NP-complete even when restricted to planar 5-regular Hamiltonian graphs. In the first section, we prove that INDEPENDENT SET remains NP-complete on general Hamiltonian graphs. The second section is dedicated to the proof of the NP-completeness of INDEPENDENT SET on k-regular Hamiltonian graphs for $k \geq 3$.. We will conclude the chapter in the third section with a proof that FEEDBACK VERTEX SET remains NP-complete on k-Hamiltonian ordered graphs for every $k \geq 3$.

Throughout this chapter we will write $\alpha(G) \geq k$ if G has an independent set of size at least $k \in \mathbb{N}$.

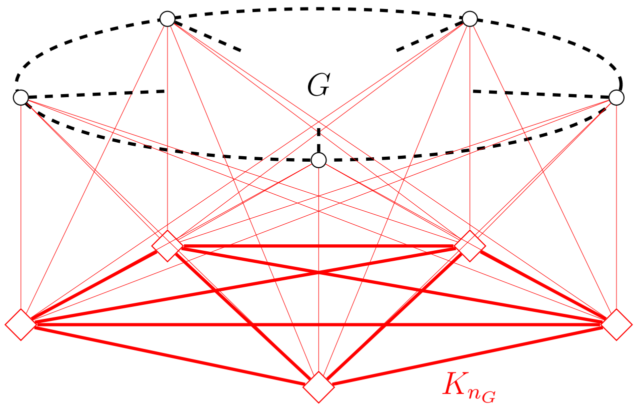


Figure 5.1: Fully-connected Hamiltonian closure of G. The graph G is represented by a dashed ellipse where the dashed lines denote possible edges in G. The (red) diamond-shaped vertices represent the vertices in V_{edges} and the (red) thin and thick edges represent the edges in E_{fc}. The edges between vertices in V_{edges} are drawn thickly in order to highlight the isomorphism between the induced graph $G_{(\text{fc})}[V_{\text{edges}}]$ and K_{n_G}.

5.1 General Hamiltonian graphs.

Karp [Kar72] showed that Independent Set is NP-complete on general graphs.

Theorem 5.1 (Karp [Kar72])**.** Independent Set *on general graphs is NP-complete.*

As a first result in this chapter we will give a simple polynomial-time many-one reduction from Independent Set on general graphs to Independent Set on Hamiltonian graphs, yielding the following.

Theorem 5.2. Independent Set *on Hamiltonian graphs with known Hamilton-cycle is NP-complete.*

In order to prove Theorem 5.2 we use a modified version of Definition 4.2 given in Chapter 4 that will be referred to as *fully-connected Hamiltonian closure* of a graph. One may think of the fully-connected Hamiltonian closure of G as the graph obtained by adding a disjoint K_{n_G}, and connecting every vertex in K_{n_G} to every vertex in G (see Figure 5.1).

Definition 5.3 (Fully-connected Hamiltonian closure of a graph.)**.** Let $G = (V, E)$ be a graph and let $\Phi := (v_0, \ldots, v_{n_G-1})$ with $v_i \in V$ for every $i \in \{0, \ldots, n_G-1\}$, where $v_i \neq v_j$ if $i \neq j$, be an enumeration of the vertices. The *fully-connected Hamiltonian closure of* G

(through Φ), denoted by $G_{(\mathrm{fc})} = (V_{(\mathrm{fc})}, E_{(\mathrm{fc})})$, is:

$$
\begin{aligned}
V_{(\mathrm{fc})} :=& V \uplus V_{\mathrm{edges}}, \text{ where} \\
& V_{\mathrm{edges}} := \{u_i \mid i \in \{0, \dots, n_G - 1\}\} \text{ and,} \\
E_{(\mathrm{fc})} :=& E \uplus E_{\mathrm{fc}},, \text{ where} \\
& E_{\mathrm{fc}} := \{\{u_i, w\} \mid i \in \{0, \dots, n-1\}, w \in V\} \cup \\
& \qquad\qquad \{\{v, w\} \mid v, w \in V_{\mathrm{edges}}, v \neq w\}.
\end{aligned}
$$

Note that $G_{(fc)}[V_{\mathrm{edges}}]$ is isomorphic to K_{n_G}. A schematic example of a fully-connected Hamiltonian closure of a graph G is given in Figure 5.1.

Analogously to our proof schemes and methods in the proofs of Theorem 3.1 and Theorem 4.1 in the previous chapters, we will prove that, given a graph G, the graph $G_{(\mathrm{fc})}$ is Hamiltonian, and that the size of its independent sets is polynomially dependent on G, as well as on the size of independent sets in G. This will then give rise to a polynomial-time many-one reduction from Independent Set on general graphs to Independent Set on Hamiltonian graphs.

Lemma 5.4. *The fully-connected Hamiltonian closure $G_{(\mathrm{fc})}$ of a graph G is Hamiltonian and can be constructed in polynomial time from G. Moreover, an explicit Hamilton-cycle can be given in $\mathcal{O}(|V(G_{(\mathrm{fc})})|)$ time after construction of $G_{(\mathrm{fc})}$.*

Proof. Denote by G' the Hamiltonian closure of G as defined in Definition 4.2. We know by construction that $V(G') = V(G_{(\mathrm{fc})})$ and $E(G') \subset E(G_{(\mathrm{fc})})$, and thus $G' \subset G_{(\mathrm{fc})}$. Since G' is known to be Hamiltonian (Lemma 4.3) and $G_{(\mathrm{fc})}[V(G')] = G_{(\mathrm{fc})}$, it follows that $G_{(\mathrm{fc})}$ is Hamiltonian. After reading G, $G_{(\mathrm{fc})}$ can be constructed in $\mathcal{O}(|V(G)|^2)$ time by adding $|V(G)|$ vertices—namely V_{edges})—and connecting them each to the altogether $2\,|V(G)|$ vertices in $G_{(\mathrm{fc})}$. Note that as $V(G') = V(G_{(\mathrm{fc})})$ a Hamilton-cycle in G' (see Lemma 4.3) is also a Hamilton-cycle in $G_{(\mathrm{fc})}$ that can be given in $\mathcal{O}(|V(G_{(\mathrm{fc})})|)$ time. □

Lemma 5.5. *Let $G = (V, E)$ be a graph and denote by $G_{(\mathrm{fc})} = (V_{(\mathrm{fc})}, E_{(\mathrm{fc})})$ the fully-connected Hamiltonian closure of G. Then G has an independent set of cardinality $k \in \mathbb{N}$ if and only if $G_{(\mathrm{fc})}$ has an independent set of cardinality k.*

Proof. Let $V_{(\mathrm{fc})} := V \cup V_{\mathrm{edges}}$ be as in Definition 5.3. For $k = 1$ the statement is trivially true, hence we assume that $k \geq 2$.

- $\Rightarrow$: Suppose G has an independent set $I \subset V$ of cardinality k. Since $E(G_{(\mathrm{fc})}[V]) = E(G)$, it follows that I is an independent set of cardinality k in $G_{(\mathrm{fc})}$.
- $\Leftarrow$: Suppose $I' \subset V_{(\mathrm{fc})}$ is an independent set of cardinality k in $G_{(\mathrm{fc})}$. Since $k \geq 2$, and as $N(v) \cup \{v\} = V_{(\mathrm{fc})}$ for every $v \in V_{\mathrm{edges}}$, it holds true that $I' \cap V_{\mathrm{edges}} = \emptyset$. Thus $I' \subset V = V(G)$ and therefore I' is an independent set of cardinality k in $G_{(\mathrm{fc})}[V] = G$. □

We are now ready to prove Theorem 5.2.

Proof of Theorem 5.2. The many-one polynomial-time reduction from Independent Set on general graphs to Independent Set on Hamiltonian graphs given by Lemma 5.4 together with Lemma 5.5, and the fact that Independent Set on general graphs is NP-hard (Theorem 5.1), yield that Independent Set is NP-hard on the class of Hamiltonian graphs with known Hamilton-cycle. Since Independent Set is trivially contained in NP, this concludes the proof. □

5.2 k-regular planar Hamiltonian graphs.

The computational complexity of Independent Set has already been studied on some subclasses of regular Hamiltonian graphs. Fleischner, Sabidussi, and Sarvanov [FSS10] showed that Independent Set remains NP-complete when restricted to planar 3-regular Hamiltonian graphs.

Theorem 5.6 (Fleischner, Sabidussi, and Sarvanov [FSS10, Proposition 2.1])**.** Independent Set *is NP-complete on planar* 3*-regular Hamiltonian graphs with known Hamilton-cycle.*

Fleischner, Sabidussi, and Sarvanov [FSS10] showed moreover that the same holds true for a specific subclass of planar 4-regular Hamiltonian graphs, yielding the following.

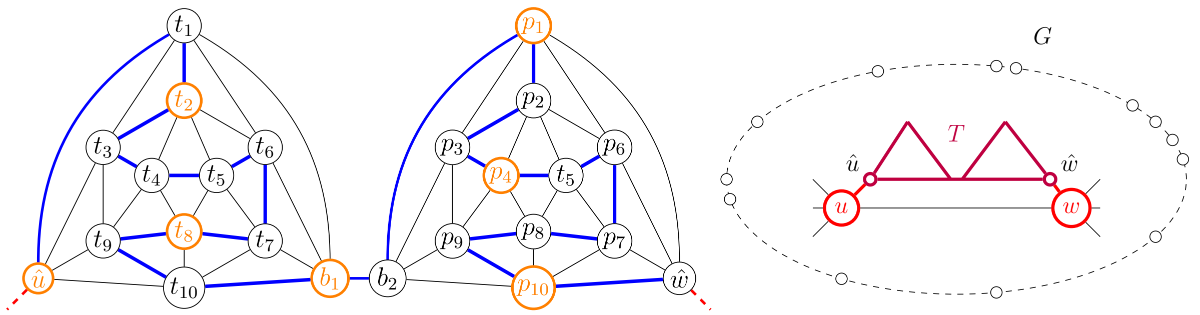
Theorem 5.7 (Fleischner, Sabidussi, and Sarvanov [FSS10, Proposition 4.3])**.** Independent Set *is NP-complete on planar* 4*-regular Hamiltonian graphs with known Hamilton-cycle.*

This raises the question whether Independent Set remains NP-complete on planar 5-regular Hamiltonian graphs. We answer this question by giving a polynomial-time many-one reduction proving that Independent Set remains indeed NP-complete on planar 5-regular Hamiltonian graphs.

Theorem 5.8. Independent Set *is NP-complete on* 5*-regular planar Hamiltonian graphs.*

The proof of Theorem 5.8 will be done via a polynomial-time many-one reduction from Independent Set on planar 3-regular Hamiltonian graphs. To this end, we introduce a „gadget"-graph that we will be referring to as the *T-graph.* We then define *T-insertions* analogously to the previously defined H-insertions and L-insertions, and, in the same vein, prove analogous properties adapted to the Independent Set problem (see Definition 3.10 and Definition 3.24 respectively).

Definition 5.9 (T-graph and T-insertion)**.** We define the T-graph to be a graph on 24 vertices with $V(T) := \{\hat{u}, \hat{w}, t_1, \ldots, t_{10}, p_1 \ldots, p_{10}, b_1, b_2\}$ as given by Figure 5.2a. The distinct vertices $\hat{u}$ and $\hat{w}$ will be referred to as the *apex-vertices* of T. Let G be a graph, and let $u, w \in V(G)$ be two distinct vertices in G. A *T-insertion on u, w* results in a new graph G' by a disjoint union of G and T (maybe after relabelling $V(T)$), and subsequently connecting u to $\hat{u}$ and w to $\hat{w}$ (see Figure 5.2b).



(a) Plane auxiliary graph T.

(b) T-insertion.

Figure 5.2: T-graph and a T-insertion. The apex-vertices in T are denoted by $\hat{u}, \hat{w}$. The thick dashed (red) edges in Figure 5.2a denote the edges that will connect the T-graph to G after performing a T-insertion as can be seen in Figure 5.2b (thick (red) edges connecting $u, \hat{u}$ and $w, \hat{w}$). The (orange) thickly drawn vertices in Figure 5.2a highlight an independent set of size seven in T, while the (blue) thick edges highlight a Hamilton-path in T. Figure 5.2b shows a schematically drawn detail of G after a T-insertion has been performed on two distinct connected degree-four vertices denoted by u and w. The two connected (purple) triangles schematically illustrate the T-graph. The dashed edges highlight some of the possible remaining edges in G.

Observation 7. It can be seen in Figure 5.2a that $\deg(v) = 5$ for every $v \in V(T) \setminus \{\hat{u}, \hat{w}\}$ and $\deg(\hat{u}) = \deg(\hat{w}) = 4$. Moreover $\mathcal{P} := (\hat{u}, t_1, \ldots, t_{10}, b_1, b_2, p_1, \ldots, p_{10}, \hat{w})$ is a Hamilton-path in T, meaning that it is a path with $V(P) = V(T)$ (see the (blue) thick highlighted path in Figure 5.2a).

The following two lemmata summarize some properties of the T-graph and T-insertions in a graph G that disclose their influence on the size of independent sets in G.

Lemma 5.10. *Let T be as in Definition 5.9. Then T has no independent set of size eight, and there is an independent set $I \subseteq V(T)$ with $|I| = 7$.*

Proof. An independent set as stated in the lemma is given by $I := \{\hat{u}, t_2, t_8, b_1, p_1, p_4, p_{10}\}$ as can be easily verified (see Figure 5.2a). To see this, let $T_1' := T[\{\hat{u}, t_1, \ldots, t_{10}, b_1\}]$ and $T_2' := T[\{\hat{w}, p_1, \ldots, p_{10}, b_2\}]$, and note that T can be constructed by a disjoint union of T_1' and T_2' and subsequently connecting b_1 and b_2. Moreover note that $T_1' \cong T_2'$. Let T_1 be the graph obtained from T_1' by connecting $\hat{u}$ with b_1. By a simple case distinction it follows that $\alpha(T_1) \leq 3$, and thus $\alpha(T_1') \leq 4$ where an independent set of cardinality four in T_1' must contain both $\hat{u}$ and b_1. As $T_1' \cong T_2'$, this yields that an independent set of cardinality four in T_2' must contain b_2 and $\hat{w}$, yielding the claim. Another way to see this would be via verification using a simple brute-force algorithm. $\square$

Lemma 5.11. *Let G be a graph and let $u, w \in V(G)$ be two distinct vertices in G with $\{u, w\} \in E(G)$. Let T be the graph as defined in Definition 5.9 with apex-vertices $\hat{u}, \hat{w}$. Let G' be the graph resulting from a T-insertion on u, w. Then,*

$$\alpha(G) \geq k \iff \alpha(G') \geq k + 7$$

for every $k \in \mathbb{N}$.

Proof. Recall that $\alpha(T) \leq 7$ by Lemma 5.10.

- $\Leftarrow$: Suppose that $\alpha(G') \geq k+7$ and let $I' \subset V(G')$ be an independent set of G' with $|I'| \geq k+7$ for some $k \in \mathbb{N}$. By Lemma 5.10 we know that $|I' \cap V(G'[V(T)])| \leq 7$. As by construction $V(G'[V(T)]) \cap V(G) = \emptyset$, it follows that $I := I' \setminus V(G'[V(T)])$ is an independent set in $G'[V(G)] = G$ with $|I| \geq |I'| - 7 \geq k$.

- $\Rightarrow$: Suppose that $\alpha(G) \geq k$ and let $I \subseteq V(G)$ be an independent set with $|I| \geq k$for some $k \in \mathbb{N}$. Note that $|I' \cap \{u, w\}| \leq 1$ as $\{u, w\} \in E(G)$. Without loss of generality assume that $u \notin I'$, else relabel u and w in G accordingly. We claim that $I' := I \cup \{\hat{u}, t_2, t_8, b_1, p_1, p_4, p_{10}\}$ is an independent set in G'. As $E(G') \cap \{v, w \mid v \in V(G), w \in \{\hat{u}, t_2, t_8, b_1, p_1, p_4, p_{10}\}\} = \{u, \hat{u}\}$, where $u \notin I$, and as $\{\hat{u}, t_2, t_8, b_1, p_1, p_4, p_{10}\}$ is an independent set in T (see Lemma 5.10), we conclude that I' is an independent set in G' with $|I'| \geq k+7$. □

An inductive use of Lemma 5.11 yields the following.

Corollary 5.12. *Let G be a graph and let $u_i, w_i \in V(G)$ be different vertices in G with $\{u_i, w_i\} \in E(G)$ for every $i \in \{1, \ldots, m\}$ and some $m \in \mathbb{N}$ with $m \leq \frac{n_G}{2}$. Let $T_i \cong T$ be graphs isomorphic to T as defined in Definition 5.9 with corresponding apex-vertices $\hat{u}_i, \hat{w}_i$ for every $i \in \{1, \ldots, m\}$. Let G' be the graph resulting from m consecutive T-insertion on u_i, w_i respectively. Then,*

$$\alpha(G) \geq k \iff \alpha(G') \geq k + 7m$$

for every $k \in \mathbb{N}$.

In a next step, we analyze the influence of T-insertions on the topology of G. We prove that T-insertions can be carried out in a way that, given a planar graph G, the resulting graph G' stays planar and can be constructed in polynomial time from G.

Lemma 5.13. *Let G be a plane Hamiltonian graph with some Hamilton-cycle $\mathcal{C} = \{v_0, \ldots, v_{n_G-1}, v_0\}$. Let $v_i, v_{i+1 \bmod n_G} \in V(G)$ be two distinct vertices that are neighbors with respect to $\mathcal{C}$ for some $i \in \{0, \ldots, n_G - 1\}$. Let T be as in Definition 5.9 with apex-vertices $\hat{u}, \hat{w}$. Then the graph G' resulting from a T-insertion on $v_i, v_{i+1 \bmod n_G}$ is a planar Hamiltonian graph that can be constructed in constant time after reading G.*

Proof. As $v_i, v_{i+1 \bmod n_G}$ are neighbors in $\mathcal{C}$, $\{v_i, v_{i+1 \bmod n_G}\} \in E(G)$. Thus $v_i, v_{i+1 \bmod n_G}$ lie on some common face F in a given planar embedding of G. Construct G' from G by a T-insertion on $v_i, v_{i+1 \bmod n_G}$ in a way that the T-graph is planar embedded „inside“ their common face F. Note that this is possible since T is a planar graph (see Figure 5.2a for a planar embedding of T). By Observation 7, we know that $\mathcal{P} := (\hat{u}, t_1, \ldots, t_{10}, b_1, b_2, p_1 \ldots, p_{10}, \hat{w})$ is a Hamilton-path in T and thus

$$\mathcal{C}' := v_0 \mathcal{P}^{\mathcal{C}} v_i \diamond (\hat{u}, t_1, \ldots, t_{10}, b_1, b_2, p_1 \ldots, p_{10}, \hat{w}) \diamond v_{i+1 \bmod n_G} \mathcal{P}^{\mathcal{C}} v_0$$

is by construction a Hamilton-cycle in G'. Note that the construction is possible as $\{v_i, \hat{u}\}, \{v_{i+1 \bmod n_G}, \hat{w}\} \in E(G')$. As T has a fixed number of vertices and edges, the construction of G' from G can be done by adding constant many vertices and edges to G, concluding the proof. □

We have now proven that, given a planar graph G, we can construct a planar graph G' by T-insertions such that the size of independent sets in G' is polynomially dependent on G, as well as on the size of independent sets in G. We will now use n_G many T-insertions in order to construct a planar 5-regular Hamiltonian graph G' from a given planar 3-regular Hamiltonian graph G, which will complete the reduction. To this end, let G be Hamiltonian graph, and let $\mathcal{C}$ be a Hamilton-cycle in G. Then the intuitive idea is to connect every two consecutive vertices $u, v \in V(G)$ along the cycle $\mathcal{C}$ in G by a T-insertion, while embedding the T-graph inside a face that contains the edge $\{u, v\}$. As every vertex in G has degree two, and the T-insertions give rise to two new neighbors for every vertex in G, the resulting graph is planar 5-regular Hamiltonian, proving Theorem 5.8.

Observation 8. Let G be a graph and let $u, w \in V(G)$ be two distinct vertices with $\deg(u) = \deg(w) = k$. Let G' be the graph resulting from a T-insertion on u, w, then $\deg(u) = (k+1) = \deg(w)$ in G' and $\deg(v) = 5$ for every $v \in V(G'[V(T)])$.

Observation 8 is obvious by construction and can easily be seen in Figure 5.2b. We are now ready to prove Theorem 5.8 by a polynomial-time many-one reduction from Independent Set on planar 3-regular Hamiltonian graphs.

Proof of Theorem 5.8. Let G be a planar 3-regular Hamiltonian graph with Hamilton-cycle $\mathcal{C} = (v_0 \dots, v_{n_G-1}, v_0)$. Construct G' from G by inductive, altogether n_G many, T-insertions on $v_i, v_{i+1 \bmod n_G}$ for every $i \in \{0, \dots, n_G - 1\}$. As every vertex $v \in V(\mathcal{C})$ will be part of two T-insertions, Observation 8 and an inductive use of Lemma 5.13 yield that G' is a planar 5-regular Hamiltonian graph that can be constructed in $\mathcal{O}(n_G)$ time from G. Applying Corollary 5.12 yields $\alpha(G) \geq k \iff \alpha(G') \geq k + 7n_G$. As by Theorem 5.6, Independent Set is known to be NP-complete on planar 3-regular Hamiltonian graphs with known Hamilton-cycle, and since Independent Set on 5-regular planar Hamiltonian graphs is contained in NP, this concludes the proof. □

Theorem 5.8 implies that Independent Set remains NP-complete on 5-regular Hamiltonian graphs. In line with the previous chapters, we will extend this statement by proving that Independent Set remains NP-complete on k-regular Hamiltonian graphs for every fixed $k \geq 3$.

Theorem 5.14. Independent Set *is NP-complete on k-regular Hamiltonian graphs with known Hamilton-cycle for every fixed $k \geq 3$.*

In order to prove Theorem 5.14, we will need the Definition 4.11 of *stacked graphs* (see Figure 4.2a) that have been introduced in Chapter 4. Recall that given a k-regular Hamiltonian graph G, the stacked graph G' of G is $(k+1)$-regular Hamiltonian. Thus, if we prove that the size of independent sets in G' is polynomially dependent on G

and the size of independent sets in G, we have proven that the construction of stacked graphs gives rise to a polynomial-time many-one reduction from INDEPENDENT SET on k-regular Hamiltonian graphs to INDEPENDENT SET on $(k+1)$-regular Hamiltonian graphs.

Lemma 5.15. *Let G be a k-regular Hamiltonian graph, and let $\mathcal{C}$ be a Hamilton-cycle in G. Let G' denote the stacked graph of G. Then it holds true that*

$$\alpha(G) \geq n \iff \alpha(G') \geq 2n$$

for every $n \in \mathbb{N}$.

Proof. Let $G = (V, E)$ and let $V(G') = V \cup V^*$ as given in Definition 4.11.

- $\Rightarrow$: Let $I \subseteq V(G)$ denote an independent set of G with $|I| \geq n$ for some $n \in \mathbb{N}$. Let $I^* := \{v^* \mid v \in I\} \subset V^*$. Then I^* is by construction an independent set in $G'[V^*] \cong G$. We claim that $I' := I \cup I^*$ is an independent set in G'. Note that $I \cap I^* = \emptyset$ and $|I'| \geq 2n$. By construction, I' is an independent set in $G'[V] = G$ and in $G'[V^*]$. Suppose that there are $v, w \in I'$ such that $\{v, w\} \in E(G')$, then neither $\{v, w\} \in E(G'[V])$ nor $\{v, w\} \in E(G'[V^*])$. Thus we may assume without loss of generality that $v \in I$ and $w = u^* \in I^*$ for some $u^* \in V^*$. By construction of G', it holds true that $\{v, u^*\} \in E(G') \Rightarrow \{v, u\} \in E(G)$, where u denotes the respective vertex in V (see Definition 4.11). But this is a contradiction to I being an independent set of G, since by construction of I' it holds true that $v, u \in I$.

- $\Leftarrow$: Let $I' \subset V(G')$ be an independent set in G' with $\alpha(G') \geq 2n$. Let $I := I' \cap V$ and $I^* := I' \cap V^*$, and note that $I \cap I^* = \emptyset$. By construction it holds true that $|I| \geq n$or $|I^*| \geq n$. As I is an independent set in G and I^* is an independent set in $G'[V^*] \cong G$ it follows that $\alpha(G) \geq \max\{|I|, |I^*|\}$, and thus $\alpha(G) \geq n$. □

As mentioned above, recall that by Lemma 4.12 from Chapter 4: given a k-regular Hamiltonian graph G, the stacked graph G' is $(k+1)$-regular Hamiltonian and can be constructed in polynomial time from G for any $k \in \mathbb{N}$ (see Figure 4.2b for a schematic illustration of a Hamilton-cycle in G'). The proof goes analogously to the proof of Theorem 3.29. For completion we will give an outline of the proof.

Proof of Theorem 5.14. Note that INDEPENDENT SET on k-regular Hamiltonian graphs is trivially contained in NP. The statement for $k = 4$ forms the base case of our induction, and follows by Theorem 5.7. Let $k \geq 4$ and define

$$\text{Red}: H_k^{\mathcal{C}} \to H_{k+1}^{\mathcal{C}}, \quad G = (V, E) \mapsto G' = (V', E'),$$

where G' is the stacked graph of G using similar notation as in the proof of Theorem 3.29. By Lemma 4.12, the function Red is well-defined and can be evaluated in time polynomial in $|V(G)|$. The proof now follows by Lemma 5.15 and induction on k. □

5.3 k-Hamiltonian-ordered graphs.

In this section, we will prove that Independent Set remains NP-complete when restricted to k-Hamiltonian ordered graphs for every fixed $k \geq 3$. We emphasize again that the property of being k-Hamiltonian ordered is way stronger than simple hamiltonicity; it requires graphs to be highly-connected and contain many different Hamilton-cycles.

Theorem 5.16. Independent Set *is NP-complete on k-Hamiltonian ordered graphs with known Hamilton-cycle for every $k \geq 3$.*

The main scheme of the proof of Theorem 5.16 will be analogous to the proof of Theorem 3.43: by induction on k, using polynomial-time many-one reductions from Independent Set on k-Hamiltonian ordered graphs to $(k+1)$-Hamiltonian ordered graphs. In this case, the general construction that gives rise to $(k+1)$-Hamiltonian ordered graphs turns out to be very simple. Given a k-Hamiltonian ordered graph G, we construct G' by adding a disjoint K_3 and connecting every vertex in G to the three vertices of K_3. This particular kind of construction will be needed in the chapters to come, which is why we will define it more generally.

Definition 5.17 (Product graph $G * H$)**.** Let G and H be two disjoint graphs. The *product graph* $G*H$ arises from a disjoint union of G and H and subsequently connecting every vertex in G to every vertex in H.

We will prove that constructing $G*K_3$ gives rise to a polynomial-time many-one reduction from Independent Set on k-Hamiltonian ordered graphs to $(k+1)$-Hamiltonian ordered graphs.

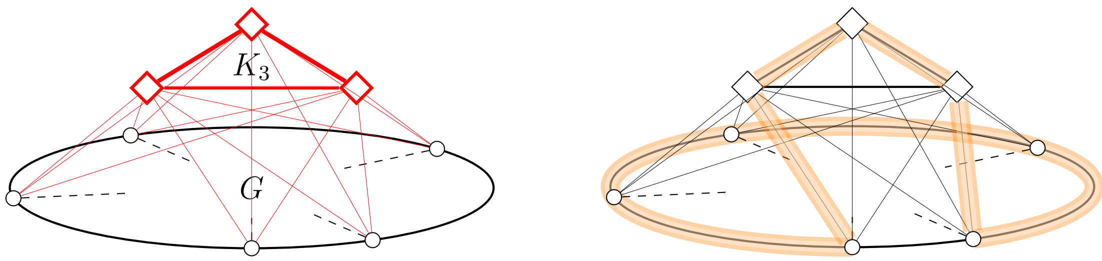
Lemma 5.18. *Let G be k-Hamiltonian-ordered for some $k \in \mathbb{N}$. Then $G*K_3$ is $(k+1)$-Hamiltonian-ordered and can be constructed in constant time after reading G.*

Proof. It is clear that $G * K_3$ is Hamiltonian if G is Hamiltonian (see Figure 5.3b), and that it can be constructed in constant time after reading G. Let $V(K_3) := \{x_1, x_2, x_3\} \subset V(G*K_3)$ be a labelling of the vertices in K_3. Let $\mathrm{Ord} = (u_0, \ldots, u_k) \in V(G*K_3)^{k+1}$ be an ordering of length $(k+1)$ in $G * K_3$. Note that $k+1 < n_G + 3$ as G is k-Hamiltonian ordered. Let $V(\mathrm{Ord}) := \{v \mid v \in \mathrm{Ord}\}$ denote the set of vertices that occur in Ord. There are two cases to consider:

Case 1: $V(\mathrm{Ord}) \cap V(K_3) = \emptyset$. Since G is k-Hamiltonian-ordered there is a cycle $\mathcal{C}$ in $(G * K_3)[V(G)]$ with respect to $(u_0, \ldots, u_{k-1})$. Suppose u_k is visited by $\mathcal{C}$ between u_0 and u_{k-1}, then we can write $\mathcal{C} = (u_0, \ldots, v, u_k, w, \ldots, u_{k-1}, \ldots, u, u_0)$ for some $u, v, w \in V(G)$. Then

$$\mathcal{C}' := u_0 \mathcal{P}^{\mathcal{C}} v \diamond (v, x_1, w) \diamond w \mathcal{P}^{\mathcal{C}} u \diamond (u, x_2, u_k, x_3, u_0)$$

is by construction a Hamilton-cycle in $G * K_3$ with respect to Ord.



(a) The graph $G * K_3$. (b) Hamilton-cycle in $G * K_3$.

Figure 5.3: $G * K_3$ and a Hamilton-cycle in $G * K_3$. The dashed edges represent some of the possible edges in G. In Figure 5.3a the (red) diamond-shaped vertices are the vertices of K_3 and the thick (red) edges are the edges of K_3 while the thin (red) edges represent the edges from G to K_3. The (orange) tube in Figure 5.3b highlights a Hamilton-cycle in $G * K_3$.

Case 2: $V(\mathrm{Ord}) \cap V(K_3) \neq \emptyset$. Let $\mathrm{Ord}' = (\widehat{u_0}, \ldots, \widehat{u_k})$ where $\widehat{u_i} = u_i$ if $u_i \notin V(K_3)$ and is left out if $u_i \in V(K_3)$. Thus $V(\mathrm{Ord}') := \{v \mid v \in \mathrm{Ord}'\} \subseteq V(G)$, and Ord' has length at most k. Now as G is k-Hamiltonian-ordered there is $\mathcal{C}$ in $(G * K_3)[V(G)]$ with respect to Ord'. As x_1, x_2, x_3 are connected to every vertex in $G * K_3$ we can extend $\mathcal{C}$ to a cycle visiting the x_i in any desired order and between any two desired consecutive vertices in $\mathcal{C}$; thus we can extend $\mathcal{C}$ to a Hamilton-cycle in $G * K_3$ with respect to Ord, concluding the proof. □

In a last step, we quantify the relation between the size of independent sets in G and $G * K_3$.

Lemma 5.19. *Let G be a graph. Then,*

$$\alpha(G) \geq k \iff \alpha(G * K_3) \geq k$$

for every $k \in \mathbb{N}$.

Proof. Since $G = (G * K_3)[V(G)]$, it holds true that any independent set $I \subset V(G)$ in G is an independent set in $G * K_3$. Note that for every graph H it holds true that $\alpha(H) \geq 1$, hence we may assume that $k > 1$. Let I' be an independent set in $G * K_3$ with $|I'| \geq 2$. Then $V(K_3) \cap I' = \emptyset$ as any vertex in $V(K_3)$ is connected to every other vertex in $G * K_3$. Hence $I' \subset V(G) = V(G * K_3) \setminus V(K_3)$ and as $(G * K_3)[V(G)] = G$ this concludes that I' is an independent set in G. □

We are now ready to prove Theorem 5.16. The proof goes analogously to the proof of Theorem 3.29 which is why we only give an outline.

Proof of Theorem 5.16. The proof follows by induction on $k \geq 3$. To see this, note that if Independent Set on k-Hamiltonian-ordered graphs is known to be NP-hard, then Lemma 5.18 and Lemma 5.19 give rise to a polynomial-time many-one reduction from Independent Set on k-Hamiltonian ordered graphs to Independent Set on $(k +$

1)-Hamiltonian ordered graphs, concluding that the latter problem is NP-hard. As Independent Set on k-Hamiltonian ordered graphs is trivially contained in NP, the induction start for $k = 3$, proven as Theorem 5.2, together with Observation 5 conclude the proof. □

A direct corollary to Theorem 5.16, due to Theorem 3.42, reads as follows.

Corollary 5.20. Independent Set *is NP-complete on k-connected Hamiltonian graphs with known Hamilton-cycle for every $k \in \mathbb{N}$.*

6 The Clique Problem on Hamiltonian Graphs

The problem of deciding whether a graph contains a clique of size k for some integer k can be stated as follows.

CLIQUE
Input: An undirected graph G, an integer $k \in \mathbb{N}$.
Question: Is there a set $K \subseteq V(G)$ with $|K| = k$ such that for all $v, w \in I$, it holds true that $\{v, w\} \in E(G)$?

CLIQUE is part of the famous NP-complete problems that have been studied by Karp [Kar72] in the very beginnings of computational complexity theory. There have been countless results concerning cliques in graphs, such as upper-bounds for the maximal cliques in several graph-classes, as well as characterisations of graph-classes by excluded clique-minors [Kur30; MM65]. Pardalos and Xue [PX94] gathered many different results on cliques in graphs, and highlighted many applications for some variations of the CLIQUE problem. Abu-Khzam et al. [Abu+05] point out applications in computational biology, while Bag, Ruj, and Sakurai [BRS15] give applications for „proof-of-work“ in cryptocurrencies. As an NP-complete problem, CLIQUE is widely believed to be intractable. This has lead to the development of many polynomial-time approximation algorithms, as well as exact exponential-time algorithms [CP90; Öst02; PX94]. The computational complexity of CLIQUE, and variations thereof, restricted to numerous graph-classes, have been extensively studied in the past few decades [Che+06; ES11; Fei+91].

In this chapter we will analyze the computational complexity of CLIQUE on Hamiltonian graphs and restricted subclasses thereof—namely k-regular Hamiltonian graphs and k-ordered Hamiltonian graphs. We will prove that CLIQUE remains NP-complete when restricted to k-Hamiltonian ordered graphs for every fixed $k \geq 3$. CLIQUE is trivially polynomial-time solvable on k-regular Hamiltonian graphs.

Throughout this chapter we will write $\omega(G) \geq k$ if G has a clique of size at least $k \in \mathbb{N}$. Without loss of generality we assume the input graphs to have $\omega(G) > 3$, as it can be verified in polynomial time (by a simple brute-force algorithm) whether a graph contains a clique of order at least three.

6.1 General Hamiltonian graphs.

Karp [Kar72] showed that CLIQUE is NP-complete on general graphs.

Theorem 6.1 (Karp [Kar72]). CLIQUE *on general graphs is* NP*-complete.*

As a first result in this chapter we will give a simple polynomial-time many-one reduction from CLIQUE on general graphs to CLIQUE on Hamiltonian graphs, yielding the following result.

Theorem 6.2. CLIQUE *on Hamiltonian graphs with known Hamilton-cycle is* NP*-complete.*

In order to prove Theorem 6.2, we make use of Definition 4.2 from Chapter 4; that is the Hamiltonian closure of a graph. The idea is that, given a graph G, the Hamiltonian closure G' of G is known to be Hamiltonian and the relation between the size of cliques in G and G' is by construction easily understood.

Lemma 6.3. *Let G be a graph, and let G' denote the Hamiltonian closure of G. Then,*

$$\omega(G) \geq k \iff \omega(G') \geq k$$

for every $k \geq 3$.

Proof. Since we have assumed that $\omega(G) > 3$, and since by construction $G \subset G'$, we only need to prove the claim for $k \geq 4$. Let G be a graph with $\omega(G) \geq k \geq 4$. As $G \subset G'$, it follows immediately that $\omega(G') \geq k$. Note that for any vertex $v \in V(G') \setminus V(G)$ it holds by construction true that $\deg(v) = 2$, and thus v can only be part of cliques of size at most 3 in G'. Therefore, for any clique $K \subset G'$ with $|V(K)| \geq 4$ it follows that $V(K) \subset V(G)$. As $G'[V(G)] = G$, this concludes the proof. □

Recall that by Lemma 4.3, for any graph G, the Hamiltonian closure is Hamiltonian and can be constructed in polynomial time from G.

Proof of Theorem 6.2. Lemma 4.3 together with Lemma 6.3, and the fact that CLIQUE is known to be NP-hard on general graphs by Theorem 6.1, prove that CLIQUE is NP-hard on Hamiltonian graphs. Since CLIQUE on Hamiltonian graphs is trivially contained in NP, it follows that CLIQUE is NP-complete on Hamiltonian graphs. □

6.2 Planar and k-regular Hamiltonian graphs.

In this section we argue that CLIQUE can be solved in polynomial time when restricted to k-regular Hamiltonian graphs as well as when restricted to planar graphs.

Theorem 6.4. CLIQUE *is polynomial-time solvable on any subclass of planar Hamiltonian graphs.*

Proof. A well-known characterisation of planar graphs, known as the Theorem of Kuratowski (Wagner), states that planar graphs are exactly the graphs that do not contain K_5 or $K_{3,3}$ as a (topological) minor (see for example [Die12, Theorem 4.4.6] for a reference). This yields that, given a planar graph G, there is no subgraph $H \subset G$ with $H \cong K_5$.

Note that K_5 is a minor of K_t for every $t \geq 5$, concluding that there is no $H \subset G$ with $H \cong K_t$ for every $t \geq 5$. Thus, we conclude that $\omega(G) \leq 4$. A naive brute-force algorithm testing whether G contains a clique of size k for every $k \in \{2, 3, 4\}$ runs in $\mathcal{O}(n_G^4)$ time. □

Theorem 6.5. Clique *is polynomial-time solvable when restricted to k-regular Hamiltonian graphs for any fixed $k \in \mathbb{N}$.*

Proof. Note that for any complete graph K_t with $t \in \mathbb{N}$ it holds true that $\deg(v) = (t-1)$ for every $v \in V(K_t)$. If G is a k-regular graph for some fixed $k \in \mathbb{N}$, this yields that G cannot contain cliques of order $k+2$ or higher as subgraphs. This means that there is no $H \subseteq G$ such that $H \cong K_t$ for any $t \geq k+2$. Hence G may only contain cliques of order at most $k+1$. A naive brute-force algorithm testing whether G contains a clique of size p for every $p \in \{2, \ldots, k+1\}$ runs in $\mathcal{O}(n_G^{(k+1)})$ time. As k was arbitrary but fixed (and not part of the input), this concludes the proof. □

6.3 k-Hamiltonian-ordered graphs.

This section is dedicated to the proof of the following.

Theorem 6.6. Clique *on k-Hamiltonian ordered graphs is NP-complete for every $k \geq 3$.*

We will prove Theorem 6.6 by induction on k using polynomial-time many-one reductions from Clique on k-Hamiltonian ordered graphs to Clique on $(k+1)$-Hamiltonian ordered graphs. The proof will be completely analogous to the proof of Theorem 5.16, by using the same construction. Given a k-Hamiltonian ordered graph G, we will construct $G * K_3$ which is known to be $(k+1)$-Hamiltonian ordered by Lemma 5.18. Note that the relation between the sizes of cliques in G and $G * K_3$ is obvious by construction. We will however give a proof for completion.

Lemma 6.7. *Let G be a graph. Then,*

$$\omega(G) \geq k \iff \omega(G * K_3) \geq k+3$$

for every $k \geq 3$.

Proof. Recall that $V(G) \cap V(K_3) = \emptyset$ by construction of $G * K_3$.

- $\Rightarrow$: Let $K \subseteq G$ be a clique in G with $|V(K)| \geq k$. As $G \subset G * K_3$ and $V(K) \cap V(K_3) = \emptyset$, it follows that $K * K_3 \subseteq G * K_3$ since every vertex in K_3 is connected to ever other vertex in $G * K_3$. Note that $K * K_3$ is by construction a complete subgraph of $G * K_3$ with $|V(K * K_3)| = |V(K)| + |V(K_3)| \geq k+3$.

- $\Leftarrow$: Let $K' \subseteq G * K_3$ be a clique with $|V(K')| \geq k+3$. Let $K := (G * K_3)[V(K')] - K_3$. Then K is by construction a clique, as every induced subgraph of a complete graph is again complete. By construction, $V(K) \subseteq V(G)$ and thus K is a clique in G with $|V(K)| \geq |V(K')| - |V(K_3)| \geq k$. □

The inductive proof of Theorem 6.6 goes analogously to the proof of Theorem 3.43.

Proof of Theorem 6.6. The proof follows by induction on $k \geq 3$. To see this, note that if CLIQUE on k-Hamiltonian-ordered graphs is known to be NP-hard, then Lemma 6.7 and Lemma 5.18 give rise to a polynomial-time many-one reduction from CLIQUE on k-Hamiltonian ordered graphs to CLIQUE on $(k+1)$-Hamiltonian ordered graphs, concluding that the latter problem is NP-hard. As CLIQUE on k-Hamiltonian ordered graphs is trivially contained in NP, the induction start for $k = 3$, proven as Theorem 6.2, together with Observation 5 concludes the proof. □

A direct corollary to Theorem 6.6, due to Theorem 3.42, is given by the following.

Corollary 6.8. CLIQUE *is NP-complete on k-connected Hamiltonian graphs with known Hamilton-cycle for every $k \in \mathbb{N}$.*

7 The Treewidth Problem on Hamiltonian Graphs

In recent years, the notion of *treewidth* has gained a lot of interest, as it turned out to play a fundamental role in several important and very deep graph-theoretic results [RS86; RS90]. A formal definition of the *treewidth* of a graph uses the notion of a so-called *tree decomposition* of a graph. Intuitively, a tree decomposition partitions a graph into a „tree-like" structure. Formal definitions for *treewidth* and *treedecomposition* read as follows.

Definition 7.1 (Tree decomposition Diestel [Die12])**.** Let G be a graph, $\mathcal{T}$ a tree and $V_{\mathcal{T}} : V(T) \to 2^{V(G)}$, where $2^{V(G)}$ denotes the powerset of $V(G)$. The pair $(\mathcal{T}, V_{\mathcal{T}})$ is called a *tree decomposition* of G if it satisfies the following three conditions:

(TD1) $\bigcup_{t \in V(\mathcal{T})} V_{\mathcal{T}}(t) = V(G)$.

(TD2) For every edge $\{v, w\} \in E(G)$, there is $t \in V(\mathcal{T})$ such that $v, w \in V_{\mathcal{T}}(t)$.

(TD3) Let $t_1, t_3 \in V(\mathcal{T})$ and $\mathcal{P}$ a path in $\mathcal{T}$ connecting t_1 and t_3, then $V_{\mathcal{T}}(t_1) \cap V_{\mathcal{T}}(t_3) \subset V_{\mathcal{T}}(t_2)$ for all $t_2 \in V(\mathcal{P})$.

The images of $V_{\mathcal{T}}$ will be referred to as *bags*.

Definition 7.2 (Width and treewidth Diestel [Die12])**.** Let G be a graph and $(\mathcal{T}, V_{\mathcal{T}})$ be a tree decomposition. Then the *width* of $(\mathcal{T}, V_{\mathcal{T}})$ is defined as

$$\operatorname{width}(\mathcal{T}, V_{\mathcal{T}}) := \max_{t \in \mathcal{T}} (|V_{\mathcal{T}}(t)| - 1).$$

The *treewidth of* G is defined as

$$tw(G) := \min_{(\mathcal{T}, V_{\mathcal{T}})} \operatorname{width}(\mathcal{T}, V_{\mathcal{T}});$$

the minimal width over every possible tree decomposition of G.

Determining the treewidth of a graph has many different practical applications beside its numerous applications in graph-theory. Bodlaender [Bod94] gathered countless results regarding the treewidth of graphs, and highlighted many practical applications, such as in evolution theory and natural language processing. It turns out that many intractable problems become polynomial-time solvable when restricted to graph classes of bounded treewidth [AP89; BK08; Bod97]. Unfortunately, the problem of deciding whether a graph G has treewidth k for some integer k is intractable. This decision problem, formally known as Treewidth, can be stated as follows.

TREEWIDTH
Input: An undirected graph G, an integer $k \in \mathbb{N}$.
Question: Is there a tree decomposition $(\mathcal{T}, V_\mathcal{T})$ of G such that $\text{width}(\mathcal{T}, V_\mathcal{T}) = k$?

Arnborg, Corneil, and Proskurowski [ACP87] have shown that TREEWIDTH is NP-complete on general graphs. However, when k is a fixed constant, TREEWIDTH can be solved in linear time [Bod96]. The computational complexity of TREEWIDTH has been studied restricted to numerous graph classes [BT97; KK95; Klo96].

In this chapter we will prove that TREEWIDTH remains NP-complete on Hamiltonian graphs, as well as on k-Hamiltonian ordered graphs for every fixed $k \geq 3$.

Throughout this chapter we will write $tw(G) \leq k$ if G has tree-width at most k for some $k \in \mathbb{N}$.

7.1 General Hamiltonian graphs.

As mentioned above, TREEWIDTH is known to be NP-complete on general graphs [ACP87]. We will use this result to prove that TREEWIDTH remains NP-complete on Hamiltonian graphs.

Theorem 7.3. TREEWIDTH *is NP-complete on Hamiltonian graphs with known Hamilton-cycle.*

The proof of Theorem 7.3 will be done via a polynomial-time many-one reduction from TREEWIDTH on general graphs. Before we embark on the proof of Theorem 7.3, we cover some helpful definitions and lemmata regarding tree decompositions and treewidth that were inspired by Diestel [Die12]. A notion that we will need later on are the so-called *adhesion sets*. An adhesion set refers to the intersection of two bags in a tree decomposition.

Definition 7.4 (Adhesion set)**.** Let G be a graph. Let $(\mathcal{T}, V_\mathcal{T})$ be a tree decomposition of G and let $\{t_1, t_2\} \in E(T)$. Then $V_\mathcal{T}(t_1) \cap V_\mathcal{T}(t_2)$ is called the *adhesion set* of $V_\mathcal{T}(t_1)$ and $V_\mathcal{T}(t_2)$.

One notable property of adhesion sets is that they function as separators in the underlying graph.

Remark. The name adhesion set comes from the fact that $V_\mathcal{T}(t_1) \cap V_\mathcal{T}(t_2)$ separates $\bigcup_{t \in V(\mathcal{T}_1)} V_\mathcal{T}(t)$ from $\bigcup_{t \in V(\mathcal{T}_2)} V_\mathcal{T}(t)$ in G, where $\mathcal{T}_1, \mathcal{T}_2$ are the two components of $\mathcal{T} - \{t_1, t_2\}$.

A particular result regarding tree decomposition and adhesion sets that will be of special interest in proving that TREEWIDTH is NP-complete on Hamiltonian graphs, can be stated as follows.

Lemma 7.5 (Diestel [Die12, Lemma 12.3.4])**.** *Let G denote a graph and let $(\mathcal{T}, V_\mathcal{T})$ be a tree decomposition of G. Any set of vertices $I \subset V(G)$ that is not contained in a bag of $(\mathcal{T}, V_\mathcal{T})$ contains two vertices that are separated by an adhesion set of $(\mathcal{T}, V_\mathcal{T})$ in G.*

Remark. It is worth noting that we call two vertices u, v *separated* in G by some set $X \subset V(G)$, if it holds true that $u, v \notin X$, and every path connecting u and v contains a vertex in X.

For the mentioned reduction we will make use of the product graph $G * K_{n_G}$, which turns out to be Hamiltonian by construction. To this end, recall Definition 5.17 of the product graphs.

Lemma 7.6. *Let G be a graph of order n_G. Then, the product graph $G * K_{n_G}$ is Hamiltonian.*

Proof. We prove this lemma by explicitly giving a Hamilton-cycle in $G * K_{n_G}$. To this end, let $\{u_0, \ldots, u_{n_G-1}\}$ be an enumeration of the vertices in K_{n_G}. A Hamilton-cycle in $G * K_{n_G}$ is then given by

$$\mathcal{C} = (v_0, u_0, v_1, u_1, \ldots, v_{n_G-1}, u_{n_G-1}, v_0),$$

as can be easily verified. □

In a last step towards the proof of Theorem 7.3, we quantify the relation between the treewidth of $G * K_{n_G}$ and zhe treewidth of G.

Lemma 7.7. *Let G denote a graph. Then,*

$$tw(G) \leq k \Leftrightarrow tw(G * K_{n_G}) \leq n_G + k$$

for every $k \in \mathbb{N}$.

Proof. $\Rightarrow$: Let $(\mathcal{T}, V_{\mathcal{T}})$ be a tree decomposition of G, such that width$((\mathcal{T}, V_{\mathcal{T}})) = tw(G) \leq k$ for some $k \in \mathbb{N}$. Now construct $(\mathcal{T}', V'_{\mathcal{T}})$ with $\mathcal{T}' := \mathcal{T}$ and $V'_{\mathcal{T}}(t) := V_{\mathcal{T}}(t) \cup V(K_{n_G})$ for every $t \in V(\mathcal{T}')$. Then $(\mathcal{T}', V'_{\mathcal{T}})$ is a tree decomposition of $G * K_{n_G}$. To see this it suffices to verify the tree decomposition axioms from Definition 7.1:

- $\bigcup_{t \in V(\mathcal{T}')} V'_{\mathcal{T}}(t) = V(G * K_{n_G})$ holds by construction.

- Let $\{u, v\} \in E(G * K_{n_G})$ then $u, v \in V'_{\mathcal{T}}(t)$ for some $t \in V(\mathcal{T}')$. This holds obviously true for $\{u, v\} \in E((G * K_{n_G})[V(G)])$ and $\{u, v\} \in E((G * K_{n_G})[V(K_{n_G})])$, since $V_{\mathcal{T}}(t) \subset V'_{\mathcal{T}}(t)$ and $V(K_{n_G}) \subset V'_{\mathcal{T}}(t)$ for every $t \in V(\mathcal{T}')$. Therefore, let $\{u, v\} \in E(G * K_{n_G})$ with $u \in V(G)$ and $w \in V(K_{n_G})$. Then there is $t \in V(\mathcal{T}')$ such that $u \in V_{\mathcal{T}}(t) \subset V'_{\mathcal{T}}(t)$ and hence by construction also $w \in V'_{\mathcal{T}}(t)$.

- Let $\mathcal{P}$ be a path in $\mathcal{T}'$ connecting $t_1, t_2 \in V(\mathcal{T}')$, and recall that $\mathcal{T}' = \mathcal{T}$. Then, since $(\mathcal{T}, V_{\mathcal{T}})$ is a tree decomposition of G, $(V_{\mathcal{T}}(t_1) \cap V_{\mathcal{T}}(t_2)) \subset V_{\mathcal{T}}(t)$ for every $t \in V(\mathcal{P})$. Hence it holds true that for all $t \in V(\mathcal{P})$, $V'_{\mathcal{T}}(t_1) \cap V'_{\mathcal{T}}(t_2) = (V_{\mathcal{T}}(t_1) \cap V_{\mathcal{T}}(t_2)) \cup V(K_{n_G}) \subset V_{\mathcal{T}}(t) \cup V(K_{n_G}) = V'_{\mathcal{T}}(t)$.

As $(\mathcal{T}', V'_{\mathcal{T}})$ is a tree decomposition of $G * K_{n_G}$, it holds true that $tw(G * K_{n_G}) \leq \text{width}((\mathcal{T}', V'_{\mathcal{T}}))$, where $\text{width}((\mathcal{T}', V'_{\mathcal{T}})) = \max_{t \in V(\mathcal{T}')} (|V'_{\mathcal{T}}(t)| - 1) = \max_{t \in V(\mathcal{T})} (|V_{\mathcal{T}}(t)| - 1) + n_G \leq k + n_G$.

$\Leftarrow$: Let $(\mathcal{T}', V'_{\mathcal{T}})$ be a tree decomposition of $G * K_{n_G}$ such that $\text{width}((\mathcal{T}', V'_{\mathcal{T}})) = tw(G * K_{n_G}) \leq k + n_G$, and define $(\mathcal{T}, V_{\mathcal{T}}) := (\mathcal{T}', V'_{\mathcal{T}} \cap V(G))$, where $V_{\mathcal{T}}(t) = (V'_{\mathcal{T}} \cap V(G))(t) := V'_{\mathcal{T}}(t) \cap V(G)$ for every $t \in V(\mathcal{T}')$. Then $(\mathcal{T}, V_{\mathcal{T}})$ is by construction a tree decomposition of G. Suppose for the sake of contradiction that $\text{width}((\mathcal{T}, V_{\mathcal{T}})) > k$. Then there is $\hat{t} \in V(\mathcal{T})$ with $\left|V_{\mathcal{T}}(\hat{t})\right| \geq k + 2$. Let $I := V_{\mathcal{T}}(\hat{t}) \cup V(K_{n_G})$ and note that $|I| \geq n_G + k + 2$. Since $\text{width}((\mathcal{T}', V'_{\mathcal{T}})) \leq k + n_G$, there cannot be a $t' \in \mathcal{T}'$ such that $I \subset V'_{\mathcal{T}}(t')$. By Lemma 7.5, there must be two distinct vertices $v, w \in I$ and two vertices $t'_v, t'_w \in V(\mathcal{T}')$ with $\{t'_v, t'_w\} \in E(\mathcal{T}')$, such that v and w are separated by the adhesion set $A := V'_{\mathcal{T}}(t'_v) \cap V'_{\mathcal{T}}(t'_w)$ in $G * K_{n_G}$. Let $\mathcal{T}'_1$ and $\mathcal{T}'_2$ be the two components of $\mathcal{T}' - \{t'_v, t'_w\}$, and without loss of generality we may assume that $t'_v \in V(\mathcal{T}'_1)$ and $t'_w \in V(\mathcal{T}'_2)$. Now there may hold exactly one of the following three cases for the dependencies of v and w:

(i) $v, w \in V(K_{n_G})$. This cannot be true as there is $t'_0 \in V(\mathcal{T}')$ such that $V(K_{n_G}) \subset V'_{\mathcal{T}}(t'_0)$ (this follows as a special case from Lemma 7.5). But as $v, w \notin A$ since they are separated by A, it follows that $V(K_{n_G}) \not\subset A$ and thus v, w are not separated by A in $G * K_{n_G}$.

(ii) $v \in V(K_{n_G}), w \in V_{\mathcal{T}}(\hat{t})$. This cannot be true as $\{v, w\} \in E(G*K_{n_G})$ and by $(TD2)$ from Definition 7.1 there is $t'_0 \in V(\mathcal{T}')$ such that $v, w \in V'_{\mathcal{T}}(t'_0)$ which is again a contradiction to v, w being separated by A in $G * K_{n_G}$.

(iii) $v, w \in V_{\mathcal{T}}(\hat{t})$. Now this means that v, w are contained in the same bag $V_{\mathcal{T}}(\hat{t}) \subseteq V'_{\mathcal{T}}(\hat{t}) \subset I$. Without loss of generality assume that $\hat{t} \in V(\mathcal{T}'_1)$. As v, w are separated by A in $G * K_{n_G}$ it must hold true that $V(K_{n_G}) \subseteq A \subseteq V'_{\mathcal{T}}(t'_v)$. Hence we can conclude that $V'_{\mathcal{T}}(\hat{t}) \not\subset V'_{\mathcal{T}}(t'_v)$, as otherwise $|V'_{\mathcal{T}}(t'_v)| \geq k + 2 + n_G$ which would be a contradiction to $tw((\mathcal{T}', V'_{\mathcal{T}})) \leq k + n_G$ (note that this implies that $\hat{t} \neq t'_v$). Thus there is a vertex $x \in V'_{\mathcal{T}}(\hat{t}) \setminus V'_{\mathcal{T}}(t'_v) \subset V_{\mathcal{T}}(\hat{t})$. Now since A separates v, w in $G * K_{n_G}$ it must hold that $V(K_{n_G}) \cap \bigcup_{t \in V(\mathcal{T}'_1)} V'_{\mathcal{T}}(t) = \emptyset$. But since there is an edge $\{x, u\} \in E(G * K_{n_G})$ for every $u \in V(K_{n_G})$, there must be a $t'_u \in V(\mathcal{T}'_2)$ with $u, x \in V'_{\mathcal{T}}(t'_u)$ due to $(TD2)$ from Definition 7.1. Finally this is a contradiction to $(TD3)$ from Definition 7.1 as $x \notin V'_{\mathcal{T}}(t'_v)$, hence $x \notin A$ but $x \in V'_{\mathcal{T}}(\hat{t}) \cap V'_{\mathcal{T}}(t'_u)$.

As none of the three cases are possible, the assumption that $\text{width}(\mathcal{T}, V_{\mathcal{T}}) > k$ must have been false, concluding the proof. $\square$

We are now ready to prove Theorem 7.3 via a polynomial-time many-one reduction from TREEWIDTH on general graphs.

Proof of Theorem 7.3. Since TREEWIDTH on Hamiltonian graphs is trivially contained in NP, and since TREEWIDTH on general graphs is known to be NP-complete [ACP87], the polynomial-time many-one reduction due to Lemma Lemma 7.7 and Lemma 7.6

reducing TREEWIDTH on general graphs to TREEWIDTH on Hamiltonian graphs with known Hamilton-cycle concludes the proof. □

7.2 k-Hamiltonian-ordered graphs.

In this section, we prove that TREEWIDTH remains NP-complete when restricted to k-Hamiltonian ordered graphs.

Theorem 7.8. TREEWIDTH *is NP-complete on k-Hamiltonian-ordered graphs for every $k \geq 3$.*

The proof is completely analogous to the proof of Theorem 5.16 and uses the exact same construction as needed in the polynomial-time many-one reductions for the inductive proof of Theorem 5.16. Given a k-Hamiltonian ordered graph we will construct $G * K_3$ for which we have already proven that it is $(k+1)$-Hamiltonian ordered (see Lemma 5.18). A slight adaption to Lemma 7.7 then yields the following.

Corollary 7.9. *Let G denote a graph. Then,*

$$tw(G) \leq k \iff tw(G * K_3) \leq k + 3$$

for every $k \in \mathbb{N}$.

Thus, given a k-Hamiltonian ordered graph G, $G * K_3$ is $(k+1)$-Hamiltonian-ordered and can be constructed in polynomial time from G, where $tw(G * K_{n_G})$ is linearly dependent on $tw(G)$. This gives rise to a polynomial-time many-one reduction proving Theorem 7.8. The proof goes analogously to Theorem 3.43, which is why we only provide a general outline.

Proof of Theorem 7.8. The proof follows by induction on $k \geq 3$. To see this, note that if TREEWIDTH on k-Hamiltonian-ordered graphs is known to be NP-hard, then Lemma 5.18 and Corollary 7.9 give rise to a polynomial-time many-one reduction from TREEWIDTH on k-Hamiltonian ordered graphs to TREEWIDTH on $(k+1)$-Hamiltonian ordered graphs, concluding that the latter problem is NP-hard. As TREEWIDTH on k-Hamiltonian ordered graphs is trivially contained in NP, the induction start for $k = 3$, proven as Theorem 7.3, together with Observation 5 conclude the proof. □

A direct corollary to Theorem 7.8, due to Theorem 3.42, is given by the following.

Corollary 7.10. TREEWIDTH *is NP-complete on k-connected Hamiltonian graphs with known Hamilton-cycle for every $k \in \mathbb{N}$.*

8 Conclusion

We have shown, that the NP-complete problems Feedback Vertex Set, 3-Coloring, Independent Set, Clique and Treewidth remain NP-complete when restricted to Hamiltonian graphs. Even for finer restrictions, as to k-Hamiltonian ordered graphs, all of the studied problems remain intractable. As the property of being k-Hamiltonian ordered can be seen as a strong Hamiltonian property, this indicates that, from a computational complexity point of view, hamiltonicity may not be as restrictive as one might think. Except for Treewidth, we proved that further restrictions on the vertex degrees do not affect the tractability of the studied problems either.

The reductions we gave proving that Independent Set and Clique remain NP-complete on Hamiltonian graphs were fpt-reductions, yielding that Independent Set and Clique remain $W[1]$-hard on Hamiltonian graphs (see [Pap03] for definitions of FPT and $W[1]$). The same holds true for Independent Set and Clique on k-Hamiltonian ordered graphs. In the proofs of other results, such as that Feedback Vertex Set remains NP-complete on planar 4-regular Hamiltonian graphs, we were not able to give parametrized reductions, raising the question whether Feedback Vertex Set may be fixed parameter tractable on planar 4-regular Hamiltonian graphs. Similar questions arise for some of the other studied problems, motivating further study of their computational complexity on a finer scale.

We have shown that 3-Coloring remains NP-complete on k-ordered Hamiltonian graphs, as well as on k-connected graphs. In contrast to the other studied problems, we were not able to give a reduction proving that 3-Coloring remains NP-complete when restricted to the finer class of k-Hamiltonian ordered graphs. This raises the question whether 3-Coloring becomes be polynomial-time solvable on k-Hamiltonian ordered graphs, or whether it remains intractable.

At this point we want to conjecture that similar results hold true for Dominating Set, which is the problem of deciding whether a graph contains k vertices whose neighborhoods cover the whole graph. We think that by similar techniques as the ones used in this thesis, namely by the inductive use of polynomial-time many-one reductions relying on similar constructions as the ones we have given, Dominating Set remains NP-complete when restricted to k-Hamiltonian ordered, as well as to k-regular graphs for every $k \geq 3$. Some first results of our research in that direction lead moreover to the conjecture that this might even hold true for planar 3-regular Hamiltonian graphs.

Finally, given that all of the studied problems remain NP-complete on Hamiltonian graphs, a natural question to ask is: What are classical problems that become non-trivially tractable on Hamiltonian graphs?

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