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Competitive Diffusion Games on Graphs Made Temporal

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Abstract

Competitive diffusion games are a game-theoretic model of information spreading in a network, proposed by Alon et al. [Alo+10]. In such games, each player chooses an initial vertex in an undirected graph from which the information of the player diffuses across the edges through the network. The objective of every player is to maximize the number of vertices influenced by her first. We introduce a model for competitive diffusion games on temporal graphs. Temporal graphs are graphs that can change over time: Edges can be existent in one time step while they are absent in others. We investigate the existence of Nash equilibria in two-player diffusion games on temporal paths and temporal cycles. As a main result, we show that a Nash equilibrium always exists on temporal cycles where edges do not disappear over time, i.e. monotonically growing temporal cycles. We present an algorithm that finds such a Nash equilibrium in linear time. Furthermore, we show that a Nash equilibrium is guaranteed to exist on every temporal path where the last layer corresponds to the underlying graph. Furthermore, we show that there exist instances of temporal paths and cycles that do not admit a Nash equilibrium.

Zusammenfassung

Das von Alon u. a. [Alo+10] eingeführte Competitive Diffusion Game ist ein spieltheoretisches Modell über die Ausbreitung von Information in sozialen Netzwerken. In dem Spiel wählt jeder Spieler einen Startknoten in einem ungerichteten Graphen, von welchem sich die Information des Spielers über die Kanten durch den Graph ausbreitet. Das Ziel eines jeden Spielers ist es die Anzahl der von ihr zuerst beeinflussten Knoten zu maximieren. Unsere Arbeit erweitert Competitive Diffusion Games auf temporale Graphen. Temporale Graphen sind Graphen, die sich über die Zeit verändern können. Dadurch können Kanten in einem Zweitschritt existieren, während sie im nächsten Zeitschritt wieder vertschwinden. Wir untersuchen die Existenz von Nash-Gleichgewichten auf temporalen Pfaden und Kreisen für das Zwei-Spieler Competitive Diffusion Game. Als Hauptresultat zeigen wir, dass auf jedem temporalen Kreis, in dem Kanten über die Zeit nicht verschwinden dürfen, ein Nash-Gleichgewicht existiert. Wir beschreiben außerdem einen Algorithmus, mit dem sich ein solches Nash-Gleichgewicht in linearer Zeit finden lässt. Zusätzlich beweisen wir, dass auch auf jedem temporalen Pfad, in welchen die letzte Schicht dem zugrundeliegenden Graphen entspricht, immer ein Nash-Gleichgewicht existiert. Außerdem presentieren wir Instanzen von temporalen Pfaden und Kreisen für die das Competitive Diffusion Game kein Nash-Gleichgewicht besitzt.

Contents

Chapter 1

Introduction

Finding nodes of maximum influence in a social network is of high interest in various fields. Apart from applications in disease spreading and news propagation, it is important for advertising a company's product. In "viral marketing", companies try to influence as many costumers as possible by only selecting a small set of targets for promotion. Having convinced some users in a social network, the information about the company's product spreads naturally through the network by word of mouth recommendation (Domingos and Richardson [DR01]). In many settings, several alternative suppliers exist for one product, which imposes a competition among the companies. Alon et al. [Alo+10] introduced competitive diffusion games that model the diffusion of information in a game-theoretic setting, thereby focusing on the competitive aspect in many applications.

A social network is modeled by a graph consisting of a set of vertices, representing the users, and a set of edges, indicating the cooperation or friendships among the network participants. In competitive diffusion games, companies are represented by players, each trying to influence a maximum number of vertices in a graph. At the beginning of the game, each player chooses one initial vertex of the graph which is colored by the player's color (modeling an initially influenced user). Based on the chosen vertices, a propagation process takes place that models the spread of information through the network. In every time step, a colored vertex spreads her color to adjacent and so far uncolored vertices in the network. If an uncolored vertex is neighbored by two differently colored vertices, then the vertex is colored gray. The color gray does not further propagate through the network. The propagation process continues until no more vertices are colored. The utility of a player is defined by the number of vertices the player colored at the end of the game, that is, the number of users the company was able to successfully influence.

One natural approach to analyze game-theoretic models is to consider what happens if the players respond to each others strategies. In this case, there is either always a player that can derive benefit from changing her strategy or the game stabilizes in some strategy profile where each player plays the best response to the strategies of the other players. Such a stable strategy profile is called a Nash equilibrium. The Nash equilibrium is the most central and important solution concept for game-theoretic models (Maschler, Solan, and Zamir [MSZ13]). Playing a strategy that is part of a Nash equilibrium may reduce the costs occurring if company changes her strategy. The non-existence of a Nash equilibrium means that for every strategy profile, there exists a player that can improve by deviating from her strategy. Such a game is unstable in every case.

The existence or non-existence of Nash equilibria in competitive diffusion games has been shown for a variety of different graph classes (see Section 1.1). However, to the best of our knowledge, competitive diffusion games have never been investigated on temporal graphs. A variety of modern systems and applications can be naturally modeled as a temporal graph, since edges between vertices may exist in one time step while they may be absent in others. In social networks, for instance, relationships between individuals often change as individuals leave or enter a group (Michail [Mic16]). We introduce a model for competitive diffusion games on temporal graphs and investigate the existence of Nash equilibria for different temporal graph classes. In the following, we give an overview of our model.

We define a *temporal graph* $\mathcal{G} = (V, E, \tau)$ by a vertex set V, a maximal time label τ and an edge set E. The edges in E correspond to the interactions between the vertices in the different layers of the temporal graph. The *underlying* graph of a temporal graph is the graph that contains all edges that are present in at least one layer. A *temporal path* (*cycle*) is a temporal graph whose underlying graph is a path (cycle). The propagation process of a competitive diffusion game on a temporal graph is modeled as follows. We assume that a vertex propagates her color to a vertex in some time step t if the edge between the two vertices exists in the corresponding layer t of the temporal graph. The propagation process finishes as soon as no more vertices are colored. For the case that the propagation process takes more steps than layers exist, we assume that the propagation process continues with the edges of the last layer.

1.1 Related Work

Our game-theoretic model is based on the model of competitive diffusion games by Alon et al. [Alo+10]. A variety of research has focused on the existence of Nash equilibria in these games. After Alon et al. [Alo+10] and Takehara, Hachimori, and Shigeno [THS12] proved that even for graphs with small diameter a Nash equilibrium for two players is not guaranteed to exist, different graph classes have been analyzed. It has been shown that a Nash equilibrium for two players always exists on trees (Small and Mason [SM13]), paths, cycles, Cartesian grids (Roshanbin [Ros14]) and toroidal graphs (Sukenari et al. [Suk+16]). Recently, Fukuzono et al. [Fuk+20] analyzed chordal graphs, which are graphs where every induced cycles has exactly three vertices. They showed that a Nash equilibrium for two players is not guaranteed to exist on every chordal graph but on three subclasses of chordal graphs. Other work considers competitive diffusion games with three players. Bulteau, Froese, and Talmon [BFT16] showed that every path of size at least six does not admit a Nash equilibrium for three players, whereas for any number of players different from three, a Nash equilibrium can always be found. Apart from that, Bulteau, Froese, and Talmon [BFT16] proved that there is no Nash equilibrium for three players on any Cartesian grid. In contrast, every diffusion game on a cycle admits a Nash equilibrium, even if the number of players is three (Bulteau, Froese, and Talmon [BFT16]).

Other work on competitive diffusion games focuses on the complexity of finding Nash

	General	Superset	Monotonically Growing
Temporal Paths	✗ (Theorem 3.1)	✓ (Theorem 3.5)	✓ (Theorem 3.5)
Temporal Path Forests	✗ (Theorem 3.1)	✓ (Theorem 3.8)	✓ (Theorem 3.8)
Temporal Cycles	✗ (Theorem 4.1)	✗ (Theorem 4.1)	✓ (Theorem 4.27)

Table 1.1: Overview of our results. "✗" means that a Nash equilibrium is not guaranteed to exist. "✓" means that a Nash equilibrium always exists.

equilibria in general graphs for any number of k players. Etesami and Başar [EB16] showed that deciding the existence of a Nash equilibrium for general k is NP-complete. Furthermore, Ito et al. [Ito+15] showed that the existence problem is even W[1]-hard when parameterized by k.

The study of influence maximization in social networks was initiated by Kempe, Kleinberg, and Tardos [KKT03], who modeled the problem of choosing an influential set of users as an optimization problem. Since then, several game-theoretic alternatives have been proposed. Dürr and Thang [DT07] introduced Voronoi games which are similar to our model of reference, with the difference that in Voronoi games, the color of a vertex only depends on the distance to the vertices chosen by the players. Furthermore, Tzoumas, Amanatidis, and Markakis [TAM12] extended competitive diffusion games by Alon et al. [Alo+10], allowing a player to choose multiple initial vertices.

This is not the first work that analyzes temporal graphs in a game-theoretic setting. Bu et al. [Bu+19] integrated a game-theoretic approach for predicting edges in a temporal network. Erlebach and Spooner [ES20] analyzed the Cops and Roberts game on graphs with periodic edge-connectivity.

1.2 Our Contributions

We introduce a model for competitive diffusion games on temporal graphs and analyze the diffusion game with two players on temporal paths and temporal cycles. We give an overview of our results in Table 1.1. We start with investigating temporal paths in Chapter 3, giving an example of a temporal path that does not admit a Nash equilibrium in Section 3.1. As an additional specialization, we consider temporal graphs where edges that exist in some layer also have to exist in the last layer (superset temporal graphs). Superset temporal graphs can be found in social networks, where some particular event at the end of a considered time period reunifies all members of the social network, meaning that all interactions that happened over time reoccur, as, for instance, in a final meeting at the end of a conference. We show that a Nash equilibrium is guaranteed to exist on every superset temporal path in Section 3.2 and on every temporal graph whose components are superset temporal paths (superset temporal path forests) in Section 3.4. In Section 3.3, we outline that by relaxation of the conditions imposed by the superset property, a Nash equilibrium is no longer guaranteed to exist.

In Chapter 4, we consider temporal cycles. We show in Section 4.1 that a Nash equilibrium is not guaranteed to exist on every temporal cycle and neither on every superset temporal cycle. We continue by considering temporal cycles where edges do not disappear over time (monotonically growing temporal cycles). Monotonically growing temporal graphs can naturally be found in online social networks. While in real life humans are bounded by a maximal number of social contacts they can manage, in online social networks, relationships between people normally only appear instead of disappear. For instance, in facebook, friends are made but seldom deleted. As the main contribution of this work, we show in Section 4.2 that a Nash equilibrium is guaranteed to exist on every monotonically growing temporal cycle. We also state an algorithm that finds such a Nash equilibrium in linear time.

Chapter 2

Preliminaries

For two numbers $a, b \in \mathbb{N}$ with $a \leq b$ we denote the set $\{a, a+1, \ldots, b\}$ as $[a, b]$.

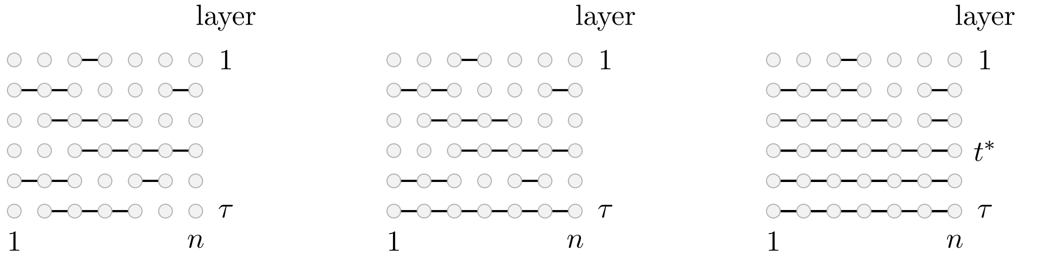
Graph. A *graph* is a tuple $G = (V, E)$ with a set V of vertices and a set $E \subseteq \binom{V}{2}$ of edges. Graph G is a *path* of size n if $V = [1, n]$ and $E = \{\{i, i+1\} \mid i \in [1, n-1]\}$. It is a *cycle* of size n if $V = [1, n]$ and $E = \{\{i, i+1\} \mid i \in [1, n-1]\} \cup \{\{n, 1\}\}$. Graph G is *connected* if there is a path from u to v for all $u, v \in V$. Vertex u is a *neighbor* of vertex v if $\{u, v\} \in E$.

Temporal Graph. Let $\mathcal{G} = (V, E, \tau)$ denote a *temporal graph*, where V is the set of vertices, $E \subseteq \binom{V}{2} \times [1, \tau]$ is the set of edges, and $\tau \in \mathbb{N}$ is the maximal time label. Let *layer* $t \in [1, \tau]$ of temporal graph $\mathcal{G}$ be the graph $G_t = (V, E_t)$, where E_t denotes the set of edges in $\mathcal{G}$ with time label t, i.e., $E_t := \{\{u, v\} \in \binom{V}{2} \mid (\{u, v\}, t) \in E\}$. Let the *underlying graph* of $\mathcal{G}$ be the graph $G^* = (V, E^*)$ with $E^* = \bigcup_{t \in [1,\tau]} E_t$. Temporal graph $\mathcal{G}$ is *connected* if the underlying graph of $\mathcal{G}$ is connected. Let a *component* of $\mathcal{G}$ be a maximal subgraph $\mathcal{G}' \subseteq \mathcal{G}$ that is connected.

A *temporal path* of size n (or a *temporal cycle* of size n) is a temporal graph $\mathcal{G} = (V, E, \tau)$ with $V = [1, n]$ such that the underlying graph of $\mathcal{G}$ is a path (or a cycle) of size n. We give an example of a temporal path in Figure 2.1a. A *temporal path forest* of size n is a temporal graph $\mathcal{G} = (V, E, \tau)$ with $V = [1, n]$ such that every component of $\mathcal{G}$ is isomorphic to a temporal path. An example of a temporal path forest is illustrated in Figure 2.2. The distance between two vertices v_1 and v_2 in a temporal path or temporal path forest is defined by $d(v_1, v_2) := |v_2 - v_1|$.

Superset Temporal Graph. Let $\mathcal{G} = (V, E, \tau)$ be a temporal graph. We say that $\mathcal{G}$ is a *superset temporal graph* if for all $t < \tau$ it holds that $E_t \subseteq E_\tau$. An example of a superset temporal path is illustrated in Figure 2.1b.

Monotonically Growing Temporal Graph. Let $\mathcal{G} = (V, E, \tau)$ be a temporal graph. We say that $\mathcal{G}$ is *monotonically growing* if for all layers $t, t' \in [1, \tau]$ with $t < t'$ it holds that $E_t \subseteq E_{t'}$. Let layer $t^* \in [1, \tau]$ of $\mathcal{G}$ be the first layer which is the underlying graph of $\mathcal{G}$, i.e., $G_{t^*} = G^*$ and $G_t \neq G^*$ for all $t < t^*$. An example of a monotonically growing temporal path is illustrated in Figure 2.1c.



(a) Example of a temporal path of size $n = 7$ with $\tau = 6$ layers. The underlying graph of a temporal path is a path connecting all vertices.

(b) Example of a superset temporal path. Every edge that exists in layer $t \in [1, \tau]$ also exists in layer τ.

(c) Example of a monotonically growing temporal path. Edges do not disappear over time. Layer t^* is the first layer that is the underlying graph.

Figure 2.1: Three examples of temporal paths.

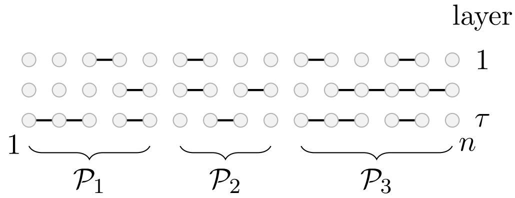


Figure 2.2: Example of a temporal path forest $\mathcal{P}$. There are three components in $\mathcal{P}$ (components $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$) which are isomorphic to temporal paths.

Central Vertex. Let $\mathcal{G} = (V, E, \tau)$ be a temporal path or temporal path forest and let $v_1, v_2 \in V$ with $v_1 \leq v_2$ be two vertices. Let L be the number of vertices in $[v_1, v_2]$. If L is odd, then we call $m = v_1 + \lfloor \frac{L}{2} \rfloor$ the *central vertex* of $[v_1, v_2]$. If L is even, then we call $m_l = v_1 + \frac{L}{2} - 1$ and $m_r = v_1 + \frac{L}{2}$ *central vertices* of $[v_1, v_2]$. Examples of central vertices in temporal path forests are illustrated in Figure 2.3.

Diffusion Game. A *diffusion game* $\Gamma = (\mathcal{G}, k)$, also called k-player diffusion game on $\mathcal{G}$, is defined by a temporal graph $\mathcal{G} = (V, E, \tau)$ and a number k of players, each having her distinct color in $[1, k]$. At the beginning of the game, in step 0, each Player $i \in [1, k]$ selects a single vertex $p_i \in V$, which is then colored by her color i. If two players choose the same vertex, then the game ends. The *strategy profile* of the game is a tuple $(p_1, \dots, p_k) \in V^k$ containing the initially chosen vertex of each player. We use the term position to refer to the chosen vertex p_i of Player i.

The temporal graph $\mathcal{G}$ is colored by the following propagation process over time. In step $t \in [1, \tau]$, we consider layer t of temporal graph $\mathcal{G}$. Every so far uncolored vertex v that has at least one neighbor in G_t that is colored in $i \in [1, k]$ and no neighbor colored in any other color $j \in [1, k] \setminus \{i\}$ is colored in i. Every uncolored vertex with at least two neighbors in G_t colored by two different colors $i, j \in [1, k]$, is colored by color 0, which

(a) Since the number of vertices in $[v_1, v_2]$ is even, there are two central vertices in $[v_1, v_2]$, denoted by m_l and m_r.

(b) Since the number of vertices in $[v_1, v_2]$ is odd, there is only one central vertex in $[v_1, v_2]$, denoted by m.

Figure 2.3: Illustrations for the definition of a central vertex.

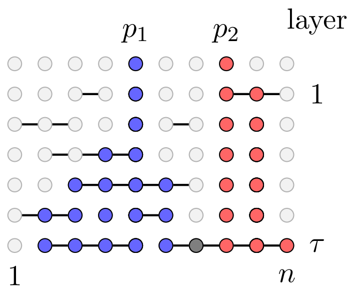


Figure 2.4: Example of a diffusion game on a temporal path. The strategy profile is $(p_1, p_2) = (5, 8)$. The outcome of the game is $U_1(p_1, p_2) = 5$ and $U_2(p_1, p_2) = 3$. The strategy profile $(5, 8)$ is not a Nash equilibrium, as, for instance, $(6, 8)$ yields a better outcome for Player 1, that is, $U_1(6, 8) = 6$.

we also call the color gray. In step $t > \tau$, the propagation process continues on G_τ until the coloring of the vertices does not change between consecutive steps.

In a diffusion game with strategy profile $(p_1, \ldots, p_k)$, we denote the number of vertices with color i in step t as $U_{t,i}(p_1, \ldots, p_k)$. Let the *pay-off* or *outcome* (of the game) of Player i, denoted by $U_i(p_1, \ldots, p_k)$, be the number of vertices with color i after the propagation process finished. A strategy profile $(p_1, \ldots, p_k)$ is a *Nash equilibrium* if for every Player $i \in [1, k]$ and every vertex $p' \in V$ it holds that $U_i(p_1, \ldots, p_{i-1}, p', p_{i+1}, \ldots, p_k) \leq U_i(p_1, \ldots, p_k)$. An example of a diffusion game is illustrated in Figure 2.4.

Our work only considers competitive diffusion games with two players. Consequently, speaking of diffusion games, we refer to two-player diffusion games.

Reachability. Let $\mathcal{G} = (V, E, \tau)$ be a temporal graph and let $v_1, v_2 \in V$ be two vertices. Assume that Player 1 plays on vertex v_1 in a one-player diffusion game on temporal graph $\mathcal{G}$, i.e., $p_1 = v_1$. If Player 1 colors vertex v_2 in step t, then we say that vertex v_1 *reaches* vertex v_2 in step t and write $\text{at}(v_1, v_2) = t$. Naturally, we say that v_1 reaches v_2 *until* or *latest* in step t for all $t \geq \text{at}(v_1, v_2)$. Additionally, vertex v_2 is reachable from vertex v_1 if v_1 reaches v_2 in some step $t \in \mathbb{N}$. A reachable vertex set Ω in $\mathcal{G}$ is a set of vertices such that every vertex $v \in \Omega$ is reachable from the same vertex in $\mathcal{G}$. In Figure 2.5, we give an example for our definition of reachability in a temporal path. Note that to better distinguish illustrations of reachability and diffusion games,

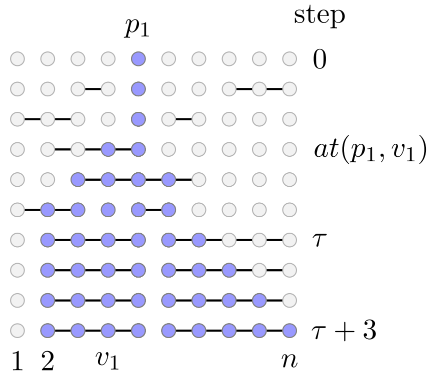


Figure 2.5: Given the temporal path and strategy profile (p_1, p_2) from Figure 2.4, this figure illustrates the reachability from position p_1. Position p_1 reaches vertex v_1 in step $\text{at}(p_1, v_1) = 3$ and until every step $t \geq 3$. From step $\tau + 1$ until step $\tau + 3$ the propagation process continues on layer τ. Thereby, all vertices in $[2, n]$ are reachable from position p_1. However, vertex 1 is not reachable from position p_1. Vertex set $[2, n]$ is a reachable vertex set in the given temporal graph.

we color vertices in illustrations about reachabiliy with a lighter color than they are colored in diffusion games.

Chapter 3

Temporal Paths

In this chapter, we investigate the existence of Nash equilibria in diffusion games on temporal paths. In Section 3.1, we show that there exists a class of temporal paths for which the corresponding diffusion game never admits a Nash equilibrium. Thereupon, we consider temporal paths where every edge existing in some layer also exists in the last layer, that are, superset temporal paths. We show in Section 3.2 that for superset temporal paths, a Nash equilibrium is always guaranteed to exist. In Section 3.3, we investigate the result of weakening the conditions imposed by the superset property, that is, we allow one edge to be absent in the last layer in a temporal path. We show that in this case the existence of Nash equilibria can no longer be guaranteed. In the last section of this chapter (Section 3.4) we consider temporal graphs, whose components are superset temporal paths, that are, superset temporal path forests. We show that also for superset temporal path forests the existence of a Nash equilibrium can always be guaranteed.

In the following sections, we refer to the two players as the left and the right player. The left player is the player that plays on a position with a lower number than the right player. Accordingly, vertices to the left of a player are vertices that have a lower number than the position of the player. Vertices to the right have a higher number than the player's position respectively.

3.1 Nash Equilibria on Temporal Paths

In this section, we present a non-existence result for Nash equilibria in diffusion games on temporal paths. In particular, we investigate a type of temporal path that has a characteristic "stair step" structure. An example is illustrated in Figure 3.1. In the following, we give an overview of how the players behave in a diffusion game on such a temporal graph.

Since in the temporal graph in Figure 3.1, edges build up from the left to the right an intuitively good strategy seems to be to play on the leftmost vertex, since thereby, the largest number of vertices is reached. However, in that case, the second player would play directly to the right of the first player and steal most of her vertices. This is illustrated in Figure 3.2a. Thus, playing very much to the left does not seem to be a good strategy anymore.

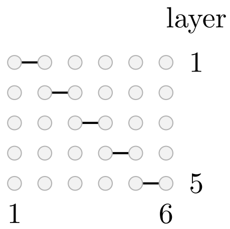
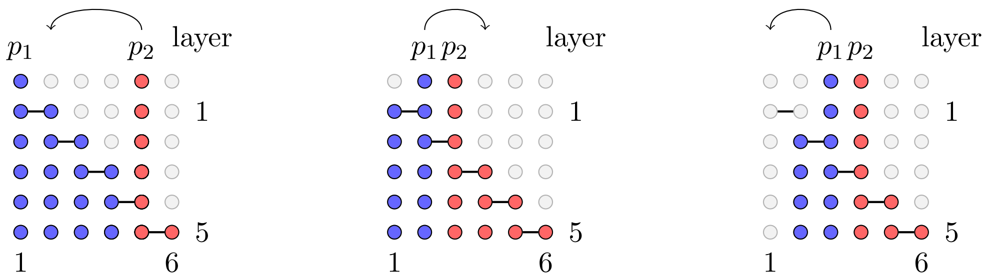


Figure 3.1: An example of the temporal paths considered in Section 3.1. There is no Nash equilibrium on this temporal path.



(a) Player 2 steals most of the vertices from Player 1 by moving next to her.

(b) Player 1 colors more vertices by moving to the right side of Player 2.

(c) Player 1 colors more vertices by playing on position 1.

Figure 3.2: Illustrations for the proof of Theorem 3.1.

Generally, we observe that the right player colors all vertices from her own position to the right and that the left player colors all vertices between the positions of the players. Consequently, if the players do not play next to each other, then the right player benefits from moving further to the left until she plays next to the left player. This can be observed in Figure 3.2a. However, if the players play on adjacent positions, then the left player colors at most two vertices. As a consequence, the left player either prefers moving further to the left in order to color more vertices to the right, or she prefers moving to the right of the other player's position in order to color all vertices to the right from there. These two cases are illustrated in Figure 3.2b and Figure 3.2c. We can see that for every strategy profile at least one player prefers a different position than the one she chose, which implies that no Nash equilibrium for the considered temporal path exists. In the following, we show that this applies to all temporal paths with such a "stair step" structure.

Theorem 3.1. *Let $\mathcal{P} = (V, E, \tau)$ be a temporal path of size $n \geq 6$. Let $\tau = n - 1$ and $E = \{(\{i, i+1\}, i) \mid i \in [1, n-1]\}$. There is no Nash equilibrium on $\mathcal{P}$.*

Proof. Let (p_1, p_2) be a strategy profile for $(\mathcal{P}, 2)$. Without loss of generality, assume that $p_1 < p_2$. Then, Player 1 colors all vertices in $[p_1, p_2 - 1]$ and Player 2 colors all vertices in $[p_2, n]$. We distinguish two different cases.

1. First, assume that $p_2 \neq p_1 + 1$. If Player 2 plays on position $p_1 + 1$, then Player 2

colors all vertices in $[p_1 + 1, n]$. Since $p_1 + 1 < p_2$, Player 2 colors more vertices than before.

2. Second, assume that $p_2 = p_1 + 1$. Then, Player 1 colors at most two vertices. We again consider two cases. Assume that $p_2 \leq 3$. If Player 1 moves to position 4, then she color all vertices in $[4, n]$. Since $n \geq 6$, these are at least three vertices. Thereby, she prefer moving to position 4. Otherwise, it holds that $p_2 \geq 4$. If Player 1 moves to position 1, then she color all vertices in $[1, p_2 - 1]$. Since $p_2 \geq 4$, these are at least three vertices. Thereby, there again exists a position for which Player 1 colors more vertices than for her current position.

In every case, there is a player that can change her position in order to color more vertices. We conclude that there is no strategy profile for $(\mathcal{P}, 2)$ that is a Nash equilibrium. □

From these counterexamples, the general non-existence of Nash equilibria on temporal paths directly follows.

Corollary 3.2. *A Nash equilibrium may fail to exist in a diffusion game on a temporal graph even if the underlying graph is a path.*

3.2 Superset Temporal Paths

Superset temporal graphs are temporal graphs where every edge existing in some layer also exists in the last layer. Consequently, in superset temporal paths, the last layer is a path connecting all vertices. In this section, we show that there exists a Nash equilibrium on every superset temporal path.

We start with an intuitive observation concerning diffusion games on all temporal graphs where the underlying graph is a path. We observe that in a temporal path, the left player cannot color any vertex to the right of the right player and vice versa. We also say that a player cannot "pass" a vertex colored by the other player. We show this in Observation 3.3.

Observation 3.3. *Let $(\mathcal{P}, 2)$ be a diffusion game on a temporal path $\mathcal{P}$ and let (p_1, p_2) be a strategy profile with $p_1 < p_2$. Player 2 cannot color any vertex $v \leq p_1$ and Player 1 cannot color any vertex $v \geq p_2$.*

Proof. Without loss of generality and for the sake of contradiction, assume that a vertex $v \leq p_1$ is colored by Player 2 in step t. Since the underlying graph of $\mathcal{P}$ is a path, vertex p_1 must have been colored by Player 2 in some step before t. This contradicts that Player 1 colors p_1 in step 0. □

In the following, we take a closer look at strategy profiles where both players play in the middle of a superset temporal path. We define the *middle* of a temporal path $\mathcal{P}$ of size n as the two central vertices of $[1, n]$ if n is even, and as the central vertex of $[1, n]$ and some vertex next to the central vertex if n is odd.

Since in the last layer of a superset temporal path all vertices are connected, every vertex is reachable from every position. Since additionally a player cannot "pass" a vertex colored by the other player, we conclude that a player playing in the middle

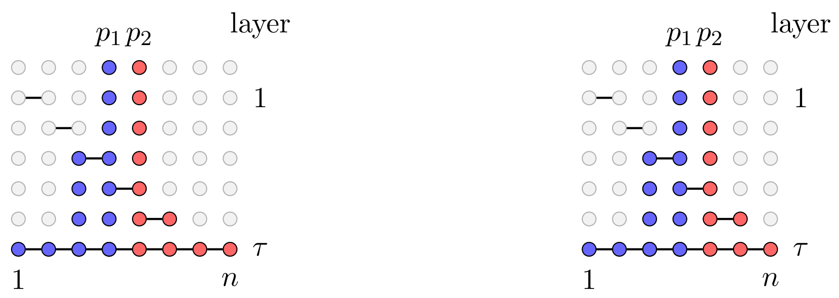


Figure 3.3: Two examples of diffusion games on superset temporal paths where both players plays in the middle of the temporal path. The left temporal path has even size and the right temporal path has odd size. It can be observed that in both diffusion games each player colors at least half the vertices of the temporal graph (rounded down).

colors at least half the vertices of the graph (rounded down). This is illustrated in Figure 3.3. We show that a strategy profile where both players play in the middle is Nash-stable in Lemma 3.4.

Lemma 3.4. *Let $(\mathcal{P}, 2)$ be a diffusion game on a superset temporal path $\mathcal{P}$ of size n. Let (p_1, p_2) be a strategy profile. Let $p_1 \in \{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor\}$ and $p_2 = p_1 + 1$. Then, strategy profile (p_1, p_2) is a Nash equilibrium*

Proof. Since the last layer of $\mathcal{P}$ is a path connecting all vertices, every position in $\mathcal{P}$ reaches every vertex in $\mathcal{P}$. Consequently, in particular p_1 reaches all vertices in $[1, p_1]$. Additionally, by Observation 3.3, Player 2 cannot color any vertex in $[1, p_1]$. We conclude that Player 1 colors all vertices in $[1, p_1]$. Symmetrically, Player 2 colors all vertices in $[p_2, n]$. Since the players play in the middle of $\mathcal{P}$, each player colors at least $\lfloor \frac{n}{2} \rfloor$ vertices. Since the players cannot "pass" a vertex colored by the other player, and since the players play on positions next to each other, there is no position a player could move to for which the player would color more than $\lfloor \frac{n}{2} \rfloor$ vertices. Thus, (p_1, p_2) is a Nash equilibrium. □

Apart from finding Nash equilibria in every superset temporal path, we can show that the Nash equilibria mentioned in Lemma 3.4 are the only Nash equilibria that exist. We show this in Theorem 3.5.

Theorem 3.5. *Let $(\mathcal{P}, 2)$ be a diffusion game on a superset temporal path $\mathcal{P}$ of size n. Let (p_1, p_2) be a strategy profile for $(\mathcal{P}, 2)$. Then, strategy profile (p_1, p_2) is a Nash equilibrium if and only if $p_1 \in \{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor\}$ and $p_2 = p_1 + 1$.*

Proof. Assume that strategy profile (p_1, p_2) is a Nash equilibrium for $(\mathcal{P}, 2)$. Without loss of generality, assume that $p_1 < p_2$. Since in the last layer of $\mathcal{P}$ all vertices are connected and since in a temporal path a player cannot "pass" a vertex colored by the other player, Player 1 colors all vertices in $[1, p_1]$ and Player 2 colors all vertices in $[p_2, n]$. For the sake of a contradiction, assume that $p_2 \neq p_1 + 1$. Then vertex $p_1 + 1$ is colored either by Player 1, by Player 2 or it is colored gray. In any case, there is at least one

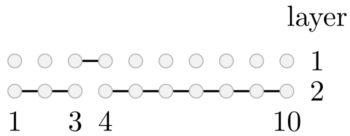


Figure 3.4: Illustrating an example of a temporal path where only one edge is absent in the last layer.

player that can increase her pay-off by moving to position $p_1 + 1$. Consequently, it must hold that $p_2 = p_1 + 1$. Now assume that $p_1 \notin \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$. If $p_1 < \lfloor \frac{n}{2} \rfloor$, then Player 1 can increase her pay-off by moving to position $p_2 + 1$. If $p_1 > \lceil \frac{n}{2} \rceil$, then Player 2 can increase her pay-off by moving to position $p_1 - 1$. We conclude that $p_1 \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$.

It has been shown in Lemma 3.4 that (p_1, p_2) is a Nash equilibrium if $p_1 \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$ and $p_2 = p_1 + 1$. This concludes the proof. □

We conclude that a Nash equilibrium is guaranteed to exist on every superset temporal path.

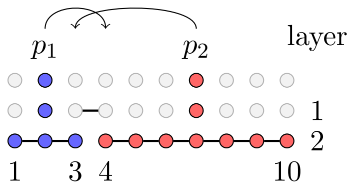
Corollary 3.6. *A Nash equilibrium is guaranteed to exist on every superset temporal path.*

3.3 Further Results on Temporal Paths

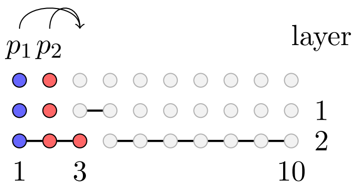
In the previous section, we showed that a Nash equilibrium is guaranteed to exist on every superset temporal path. In this section, we investigate what happens if we of weaken the conditions imposed by the superset property. We show that if we allow only a single edge to be absent in the last layer of a temporal path, then the existence of a Nash equilibrium can no longer be guaranteed.

An example of a temporal graph where all edges but at most one exist in the last layer is illustrated in Figure 3.4. The edge that is absent in the last layer is the one that connects vertices 3 and 4. Similarly as for the temporal path which we presented in Section 3.1, we observe that the players prefer playing on vertex 3 or 4, in order to reach all vertices in the graph. However, we can show that neither vertex 3 nor 4 can be part of a Nash equilibrium. In the following, we provide an intuition for these two behaviors.

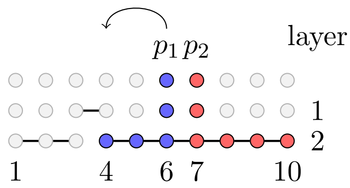
In Figure 3.5b to Figure 3.5c, we illustrate that there is always a player that benefits from moving to position 3 or 4 if no player plays on these positions. In Figure 3.5a, one player plays to the left and the other player to the right of vertices 3 and 4, which makes one player color the vertices in $[1, 3]$ and the other player color the vertices in $[4, 10]$. It can be observed that each player gains an additional vertex by moving to either position 3 or 4. In Figure 3.5b and Figure 3.5c, both players play on the same side of vertices 3 and 4, so that one side of the temporal graph is colored and the other side is not colored at all. Due to the proportion of the number of vertices on each side, there is always a player that prefers moving to a position for which the player also colors the vertices of the other side of the temporal path. In Figure 3.5d, we illustrate that vertices 3



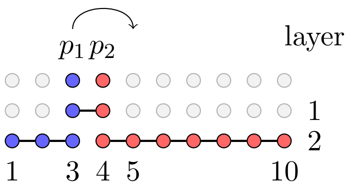
(a) Example of the case that Player 1 plays in $[1, 2]$ and that Player 2 plays in $[5, 10]$. Player 1 prefers position 4 and Player 2 prefers position 3.

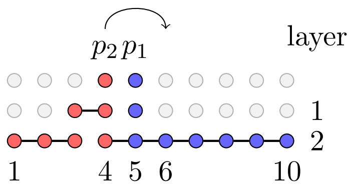


(b) Example of the case that both players play in $[1, 2]$. Both players prefer moving to position 3.



(c) Example of the case that both Players play in $[5, 10]$. There is always a player that colors at most three vertices (here Player 1) and that prefers moving to position 4.





(d) If Player 1 plays on position 3, then the best position for Player 2 is position 4. However, then, Player 1 rather moves to position 5, which makes Player 2 move to position 6. As a result, positions 3 and 4 cannot be part of a Nash equilibrium.

Figure 3.5: Illustrations for the proof that there is no Nash equilibrium on the temporal path of Figure 3.4. Figure 3.5b to Figure 3.5c illustrate that the players prefer playing on positions 3 and 4. Figure 3.5d illustrates that positions 3 and 4 cannot be part of a Nash equilibrium.

and 4 cannot be part of a Nash equilibrium. Since vertices 3 and 4 reach an unbalanced number of vertices to one as to the other side, a player playing on playing on position 3 or 4 is vulnerable to loosing many vertices to the other player, which makes her change her position after the other player played her best response.

We prove that no Nash equilibrium on the temporal path in Figure 3.4 exists in Theorem 3.7.

Theorem 3.7. *Let* $\mathcal{P} = (V, E, 2)$ *be a temporal path of size* 10 *with edges* $E = \{(\{i, i+1\}, 2) \mid i \in [1, 9]\} \setminus \{(\{3, 4\}, 2)\} \cup \{(\{3, 4\}, 1)\}$. *There is no Nash equilibrium on* $\mathcal{P}$.

Proof. For the sake of contradiction assume that strategy profile (p_1, p_2) is a Nash equilibrium.

We first show that vertices 3 and 4 cannot be part of a Nash equilibrium. Without loss of generality, assume that $p_1 = 3$. Since (p_1, p_2) is a Nash equilibrium, Player 2 plays the best response to the strategy of Player 1, so that $p_2 = 4$. Thereby, Player 1 colors three vertices and Player 2 colors seven vertices. Since Player 1 would color six vertices by moving to position 5, strategy profile $(3, 4)$ is not stable. The proof that a strategy profile containing position 4 is not stable works analogously. We conclude that positions 3 and 4 cannot be part of a Nash equilibrium.

There are three remaining cases for strategy profiles that could be Nash equilibria.

1. If $p_1 \in [1,2]$ and $p_2 \in [1,2]$, then it is clear that position 3 is better for both players.

2. Without loss of generality assume that $p_1 \in [1,2]$ and that $p_2 \in [5,10]$. Then, Player 1 colors the vertices in $[1,3]$ and Player 2 colors the vertices in $[4,10]$. However, if Player 1 plays on position 4, then she color at least the vertices in $[1,4]$, and if Player 2 plays on position 3, then she color at least the vertices in $[3,10]$. Thus, both players prefer to move to either position 3 or position 4.

3. Finally, assume that both players play on a vertex in $[5,10]$. Then, both players can only color vertices in $[4,10]$. Since there are seven vertices in $[4,10]$, for every strategy profile, there is a player that colors at most three vertices. However, a player playing on position 4 colors at least the vertices in $[1,4]$, which are more than three vertices.

For every strategy profile, there is a player that would change her position in order to color more vertices. We conclude that there is no Nash equilibrium on $\mathcal{P}$. □

Throughout this section, we showed a non-existence result for Nash equilibria in diffusion games on temporal paths where only one edge is absent in the last layer. We conclude that after relaxing the conditions imposed for superset temporal paths, the existence of a Nash equilibrium can no longer be guaranteed.

3.4 Temporal Path Forests

In this section, we investigate the existence of Nash equilibria on temporal graphs whose components are isomorphic to temporal paths, namely, temporal path forests. We show that our results for temporal paths from Section 3.1 and Section 3.2 also extend to temporal path forests.

In Corollary 3.2 in Section 3.1, we proved that a Nash equilibrium may fail to exist on a temporal path. Clearly, temporal path forests are a generalization of temporal paths. Consequently, neither for temporal path forests, a Nash equilibrium is guaranteed to exist.

We further showed that on a superset temporal path, a Nash equilibrium can always be found. In the following, we prove that this also applies to superset temporal path forests. An example of a superset temporal path forest is illustrated in Figure 3.6. We start with giving an intuitive description of a Nash equilibrium on such a graph.

It is clear that in every diffusion game on a superset temporal path forest $\mathcal{F}$, a player can only color vertices in the component that the player plays in. Additionally, since every component of $\mathcal{F}$ is isomorphic to a superset temporal path, a player colors all vertices of a component if no other player plays in her component. As a result, in order to color a large number of vertices, a player prefers playing on a vertex of a very large component and preferably by herself. We denote the two largest components of a temporal path forest by $\mathcal{P}_1$ and $\mathcal{P}_2$ and their size by n_1 and n_2.

In the following, we consider strategy profiles where both players play on vertices of the largest component of $\mathcal{F}$, that is on a vertex in $\mathcal{P}_1$. We showed in Lemma 3.4 in Section 3.2 that in a diffusion game on a superset temporal path, two positions in

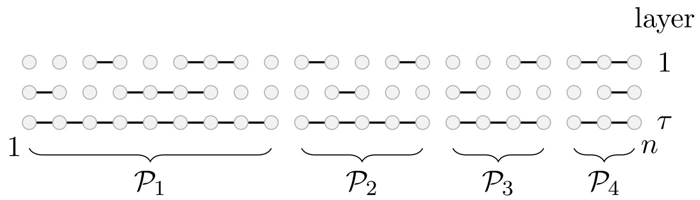


Figure 3.6: Example of a superset temporal path forest with four components $\mathcal{P}_1, \ldots, \mathcal{P}_4$.

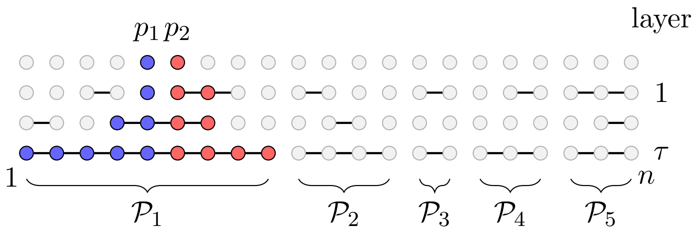
the middle of a superset temporal path form a Nash equilibrium. Consequently, if the players play on two positions in the middle of component $\mathcal{P}_1$, then no player prefers moving to a different vertex in $\mathcal{P}_1$. Thereby, each player colors roughly half the vertices of $\mathcal{P}_1$ which results in a pay-off of at least $\lfloor \frac{n_1}{2} \rfloor$. If it holds that $n_2 \leq \lfloor \frac{n_1}{2} \rfloor$, then no player prefers moving to a vertex of some other component of $\mathcal{F}$, so that we have a Nash equilibrium. An example of this situation is illustrated in Figure 3.7a. Otherwise, at least on player prefers moving to a vertex in $\mathcal{P}_2$. We get a new strategy profile, where one player plays on a vertex in $\mathcal{P}_2$ and the other player plays on a vertex in $\mathcal{P}_1$. This is exemplified in Figure 3.7b. Clearly, the player that just changed her position to the best remaining option in the temporal path forest would not move again. The other player now colors all vertices of the largest component of $\mathcal{F}$, so that that player also does not want to move to any other vertex. It follows that the new strategy profile is a Nash equilibrium. In Theorem 3.8, we formally prove that the mentioned strategy profiles for the two different cases are Nash equilibria.

Theorem 3.8. *There is a Nash equilibrium on every superset temporal path forest.*

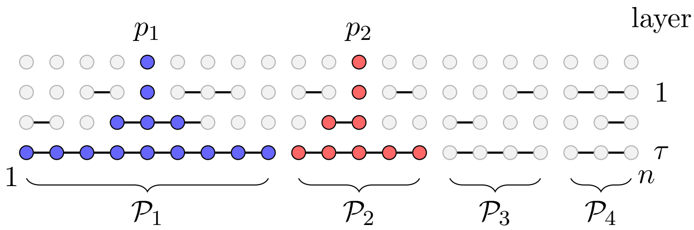
Proof. Let $(\mathcal{F}, 2)$ be a diffusion game on a superset temporal path forest $\mathcal{F}$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_m$ be the components of $\mathcal{F}$ of size $n_1, \ldots, n_m$. Without loss of generality, assume that $n_1 \geq n_2 \geq n_j$ for all $j \in [3, m]$. We now describe how to construct a strategy profile (p_1, p_2) for $(\mathcal{F}, 2)$ that is a Nash equilibrium. Let p_1 be a central vertex of $\mathcal{P}_1$. For position p_2 we consider two different cases.

1. First, assume that $\lfloor \frac{n_1}{2} \rfloor \geq n_2$. If n_1 is even, then let p_2 be the other central vertex of $\mathcal{P}_1$. Otherwise, let $p_2 = p_1 + 1$. Since positions p_1 and p_2 are in the middle of component $\mathcal{P}_1$ which is isomorphic to a superset temporal path, we conclude by Theorem 3.5 that no player prefers moving to a different vertex in $\mathcal{P}_1$. Additionally, both players color at least $\lfloor \frac{n_1}{2} \rfloor$ vertices. A player that would move to a vertex of some other component of $\mathcal{F}$ would color at most n_2 vertices. Since we assumed that $\lfloor \frac{n_1}{2} \rfloor \geq n_2$, no player prefers moving to a vertex of some other component of $\mathcal{F}$. Thus, strategy profile (p_1, p_2) is a Nash equilibrium.

2. Otherwise, it holds that $\lfloor \frac{n_1}{2} \rfloor < n_2$. Since Player 1 plays on a central vertex of $\mathcal{P}_1$, Player 2 can color at most $\lfloor \frac{n_1}{2} \rfloor$ vertices by playing on any vertex in $\mathcal{P}_1$. If Player 2 plays on a vertex of some other component, then she colors all vertices of that component. Since $\lfloor \frac{n_1}{2} \rfloor \leq n_2$ and $n_2 \geq n_j$ for all $j \in [3, m]$, the best option for Player 2 is playing on a vertex in $\mathcal{P}_2$. Consequently, let p_2 be a vertex in $\mathcal{P}_2$. Since thereby Player 1 colors all vertices in the largest component of $\mathcal{F}$, she clearly does



(a) If $\lfloor \frac{n_1}{2} \rfloor \geq n_2$, then a Nash equilibrium consists of two vertices in the middle of $\mathcal{P}_1$.



(b) If $\lfloor \frac{n_1}{2} \rfloor < n_2$, then a Nash equilibrium consists of one player playing on a central vertex of $\mathcal{P}_1$ and the other player playing on a vertex in $\mathcal{P}_2$.

Figure 3.7: Illustrations for the proof that there exists a Nash equilibrium on every superset temporal path forest. We distinguish two cases dependent on the ratios of numbers of vertices in the two largest components of a superset temporal path forest, i.e., n_1 compared to n_2.

not prefer moving to any other vertex in $\mathcal{F}$. As a result, strategy profile (p_1, p_2) is a Nash equilibrium.

We conclude that there is a Nash equilibrium on every superset temporal path forest. □

Throughout this chapter, we showed that a Nash equilibrium may fail to exist on some temporal paths and temporal path forests. However, we could prove a Nash equilibrium is guaranteed to exist on all superset temporal paths and superset temporal path forests. In the next chapter, we analyze another class of temporal graphs, that is, we study temporal cycles.

Chapter 4

Temporal Cycles

In this chapter, we analyze the existence of Nash equilibria in diffusion games on temporal cycles. We showed in Chapter 3 that there always exists a Nash equilibrium on superset temporal paths and superset temporal path forests. Therefore, we start with investigating the existence of Nash equilibria in diffusion games on superset temporal cycles. After showing that a Nash equilibrium may fail to exist on a superset temporal cycle (Section 4.1), we consider a more restricted subclass of temporal cycles, that are, monotonically growing temporal cycles. As the main contribution of this thesis, we prove that for every diffusion game on a monotonically growing temporal cycle a Nash equilibrium is guaranteed to exist (Section 4.2). We also give an algorithm that finds a Nash equilibrium on every monotonically growing temporal cycle in linear time.

4.1 Superset Temporal Cycles

Recall that in Section 3.1 we presented a non-existence result for Nash equilibria in diffusion games on temporal paths. An example of the temporal paths we considered is illustrated in Figure 4.1. In this section, we show that a Nash equilibrium is not guaranteed to exist on every superset temporal cycle. For this, we investigate superset temporal cycles, which are the result of adding an additional layer to the end of the temporal paths considered in Section 3.1. An example of such a superset temporal cycle is presented in Figure 4.2. We show that no Nash equilibrium exists on all superset temporal cycles with the same structure.

In the illustration of the temporal cycle in Figure 4.2, vertices are not presented on top of each other. Thereby, it is quite hard to follow a propagation process over time. Thus, instead of using the illustration in Figure 4.2, we rather use an illustration that resembles the presentation of the temporal path, illustrated in Figure 4.3. We split the temporal cycle between vertices 1 and 6, so that we can refer to a left and a right player similarly as we did on temporal paths. Furthermore, vertices to the left are vertices until vertex 1 and vertices to the right are vertices until vertex 6 respectively.

Since the temporal cycle in Figure 4.3 is very similar to the temporal path in Figure 4.1, we get a similar coloring of the two temporal graphs for the same strategy profiles. The only difference in the coloring is that all vertices that stay uncolored in the temporal path are distributed evenly between the players in the temporal cycle. Con-

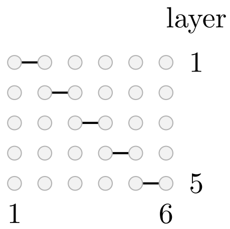


Figure 4.1: Illustration of a temporal path for which we showed in Section 3.1 that no Nash equilibrium exists.

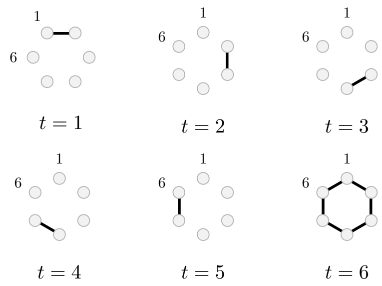


Figure 4.2: Example of a superset temporal cycle with layers $t \in [1, 6]$ for which no Nash equilibrium exists. It corresponds to the temporal path in Figure 4.1, extended by an additional layer that connects all vertices to a cycle.

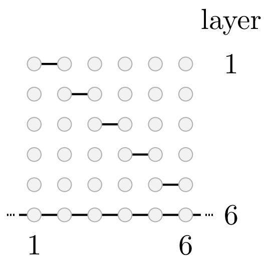


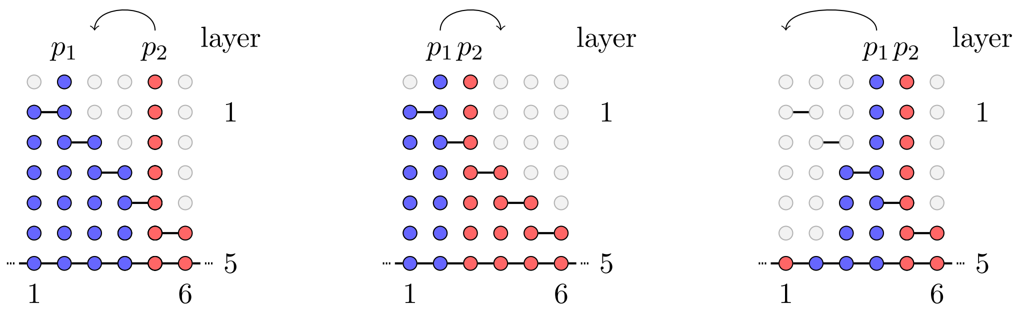
Figure 4.3: A different illustration of the temporal cycle from Figure 4.2.

sequently, also in the temporal cycle, the left player colors all vertices in between the two positions of the players and the right player colors all vertices from her own position to the right. We conclude that if the players do not play next to each other, then the right player prefers moving further to the left until she is next to the left player. This is illustrated in Figure 4.4a.

Furthermore, we observe that the left player can color at most one vertex to the left before the graph becomes a cycle and that all other vertices to the left are distributed evenly between the players. This results in two different behaviors of the left player for the case that the players play on adjacent positions. In the first case, exemplified in Figure 4.4b, the left player prefers moving to the right of the right player in order to color all vertices to the right from there. In the other case, exemplified in Figure 4.4c, the left player prefers moving further to the left in order to not have to share the vertices to the left with the right player.

In Theorem 4.1, we show that no Nash equilibrium exists in all diffusion games on superset temporal cycles with the same structure as the one in Figure 4.3.

Theorem 4.1. *Let $\mathcal{C} = (V, E, \tau)$ be a temporal cycle of size $n \geq 6$. Let $\tau = n$ and $E = \{(\{i, i+1\}, i) \mid i \in [1, n-1]\} \cup \{(\{n, 1\}, \tau), (\{i, i+1\}, \tau) \mid i \in [1, n-1]\}$. There is no Nash equilibrium on $\mathcal{C}$.*



(a) Player 2 colors more vertices by moving next to Player 1.

(b) Player 1 colors more vertices by moving to the other side of Player 2.

(c) Player 1 colors more vertices by playing on position 1.

Figure 4.4: Illustrations for the proof that no Nash equilibrium exists on the temporal cycle from Figure 4.3.

Proof. Let (p_1, p_2) be a strategy profile for $(\mathcal{C}, 2)$. Without loss of generality, assume that $p_1 < p_2$. Then, Player 1 colors all vertices in $[p_1, p_2 - 1]$ and Player 2 colors all vertices in $[p_2, n]$. If $p_1 > 1$, then Player 1 colors vertex $p_1 - 1$. If $p_1 > 2$, then the vertices $[1, p_1 - 2]$ are distributed evenly between the players. Thus, if the number of vertices in $[1, p_1 - 2]$ odd, then there is one gray colored vertex in $[1, p_1 - 2]$. We consider two different cases.

1. First, assume that $p_2 \neq p_1 + 1$. If Player 2 plays on position $p_1 + 1$, then Player 2 colors all vertices in $[p_1 + 1, n]$ and the same number of vertices in $[1, p_1 - 2]$. Since $p_1 + 1 < p_2$, these are more vertices for Player 2 than before.

2. Second, assume that $p_2 = p_1 + 1$. If $p_1 \geq 3$, then at least one vertex in $[1, p_1]$ is colored gray or by Player 2. If Player 1 moves to vertex 1, then she colors all vertices $[1, p_1]$ by herself. Consequently, vertex 1 is a better position for Player 1 than her current position. Otherwise, it holds that $p_1 \leq 2$. In this case, Player 1 colors at most two vertices. Since we assumed that $p_2 = p_1 + 1$, it follows that $p_2 \leq 3$. If Player 1 moves to position 4, then she colors all vertices in $[4, n]$. Since $n \geq 6$, these are at least three vertices, so that position 4 is better for Player 1 than her current position.

In any case, there is a player that would change her position in order to color more vertices. We conclude that there is no strategy profile for $(\mathcal{C}, 2)$ that is a Nash equilibrium. □

We conclude that a Nash equilibrium may fail to exist in a diffusion game on a superset temporal cycle.

Corollary 4.2. *A Nash equilibrium may fail to exist in a diffusion game on a temporal cycle, even if the last layer of the temporal cycle corresponds to the underlying graph.*

In the next section, we analyze a subclass of superset temporal cycles, that is, we study, monotonically growing temporal cycles.

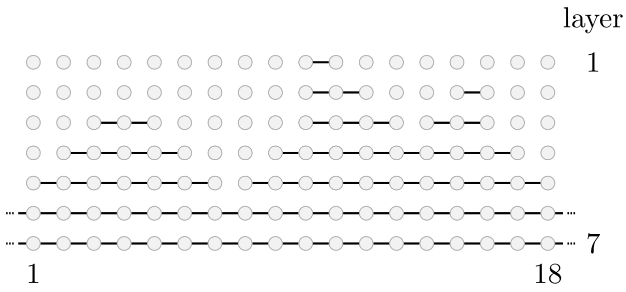


Figure 4.5: Example of a monotonically growing temporal cycle.

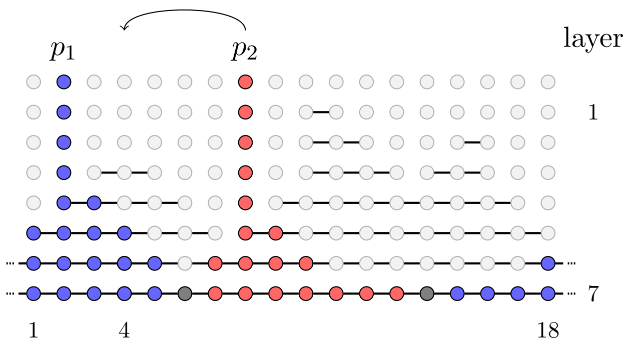
4.2 Monotonically Growing Temporal Cycles

In this section, we analyze a subclass of superset temporal cycles, namely, monotonically growing temporal cycles. In monotonically growing temporal cycles, edges do not disappear over time, meaning that an edge existing in a layer $t \in [1, \tau]$ also exists in every layer $t' > t$. An example of such a monotonically growing temporal cycle is illustrated in Figure 4.5. In this section, we show that a Nash equilibrium exists on every monotonically growing temporal cycle.

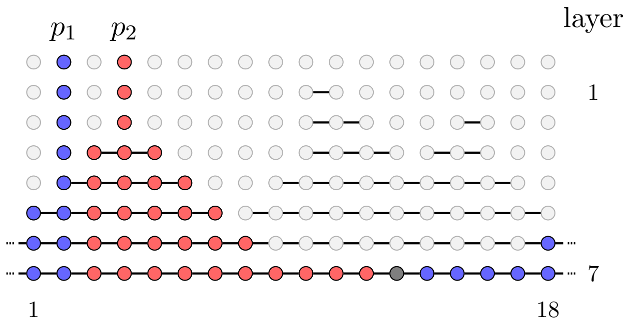
In Figure 4.6a, we illustrate a diffusion game on the monotonically growing temporal cycle from Figure 4.5. We observe that the strategy profile is not a Nash equilibrium since, for instance, Player 2 benefits from moving to position 4. A diffusion game with the new strategy profile is illustrated in Figure 4.6b. While Player 2 gains three additional points by changing her position, Player 1 suffers from her change by coloring two vertices less than before. This is not a surprising observation, considering the number of gray and uncolored vertices in the graph. Since in the diffusion game in Figure 4.6a, only two vertices are not colored by some player, it is most likely that Player 2 "steals" a point from Player 1 by moving to a better position. We can show that every diffusion game on a monotonically growing temporal cycle results in no uncolored and at most two gray vertices. We conclude that if for some strategy profile, there is no position for which Player 2 can "steal" a point from Player 1, then it is most likely that there is neither a position for which Player 2 can color more vertices. Consequently, strategy profiles where no player can "steal" a point from the other player are promising candidates for being a Nash equilibrium. We call such strategy profiles *balanced* strategy profiles.

Definition 4.3. Let $(\mathcal{G}, 2)$ be a diffusion game on a temporal graph $\mathcal{G}$ and let (p_1, p_2) be a strategy profile. Then, $\Delta_t(p_1, p_2) := U_{t,1}(p_1, p_2) - U_{t,2}(p_1, p_2)$ is the *ratio of pay-offs at time* t. Moreover, $\Delta_t(p_1, p_2)$ is *not improvable by Player* 1 if there is no position p_1' such that $\Delta_t(p_1', p_2) > \Delta_t(p_1, p_2)$. It is *not improvable by Player* 2 if there is no position p_2' such that $\Delta_t(p_1, p_2') < \Delta_t(p_1, p_2)$. $\Delta(p_1, p_2)$ denotes the ratio of pay-offs (of the game), i.e., $\Delta(p_1, p_2) = U_1(p_1, p_2) - U_2(p_1, p_2)$. A *balanced* strategy profile is a strategy profile such that the ratio of pay-offs is not improvable by any player.

For the case that a balanced strategy profile results in at most one gray vertex, we can easily show that the profile is always a Nash equilibrium. The case that a balanced



(a) Player 1 colors nine vertices and Player 2 colors seven vertices. Two vertices are colored gray. The strategy profile is not a Nash equilibrium since, for instance, Player 2 benefits from moving to position 4.



(b) Illustration of the diffusion game from Figure 4.6a after Player 2 has moved to position 4. Now Player 1 colors seven and Player 2 colors ten vertices.

Figure 4.6: Two diffusion games on the monotonically growing temporal cycle from Figure 4.5

strategy profile results in two gray vertices is more complex. In that case, a player may still be able to benefit from changing her position if both players gain one additional point. However, we will later see that on every monotonically growing temporal cycle, we can find at least one balanced strategy profile that is a Nash equilibrium. Therefore, a main part of our proof for the existence of Nash equilibria on monotonically growing temporal cycles focuses on finding balanced strategy profiles. Note that balanced strategy profiles are Nash equilibria in a variation of diffusion games where the outcome corresponds to the ratio of pay-offs. This variation is a zero-sum game. Since in a zero sum game the outcome of one player is the negative outcome of the other player, they are easier to analyze than other games (Maschler, Solan, and Zamir [MSZ13]).

We divide the proof into several subsections. In Section 4.2.1, we show how to find balanced strategy profiles on monotonically growing temporal cycles. We leave out the proof of several sublemmas for sake of readability. We show these sublemmas in the last subsection of this section (Section 4.2.4). In Section 4.2.2, we formally analyze the relationship between balanced strategy profiles and Nash equilibria and derive which of the balanced strategy profiles found in Section 4.2.1 are also a Nash equilibrium.

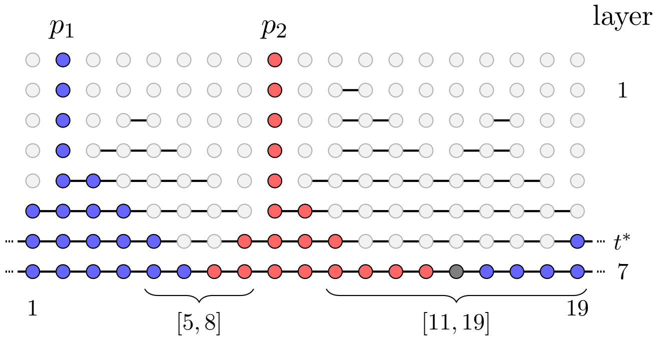


Figure 4.7: A diffusion game on a monotonically growing temporal cycle. The vertices in $[5, 8]$ and $[11, 19]$ are uncolored in step $t^* - 1$. These uncolored vertices are distributed evenly between the players in the following propagation process. Thereby, the ratio of pay-offs in step $t^* - 1$ equals the ratio of pay-offs after the propagation process finished. In particular, it holds that $\Delta_{t^*-1}(p_1, p_2) = 4 - 2 = 2$ and $\Delta(p_1, p_2) = 10 - 8 = 2$.

In Section 4.2.3, we summarize our result, stating an algorithm that returns a Nash equilibrium for every monotonically growing temporal cycle in linear time.

4.2.1 Balanced Strategy Profiles

In this subsection, we derive how to find balanced strategy profiles for diffusion games on monotonically growing temporal cycles. We can simplify this problem by considering diffusion games on monotonically growing temporal path forests. In the following, we explain the background of this simplification.

Recall that layer t^* in a monotonically growing temporal cycle is the first layer that is a cycle connecting all vertices. Consequently, every layer before layer t^* is a path forest and every layer following layer t^* is a cycle. Since we analyze temporal cycles instead of non-temporal cycles, we assume that $t^* > 0$. We can show that from the property that, from layer t^* on, every layer is a cycle, it follows that the players color the same number of vertices from step t^* on. This is illustrated by an example in Figure 4.7. As a consequence, the ratio of pay-offs in step $t^* - 1$ equals the ratio of pay-offs after the propagation process finished. We conclude that if for a strategy profile the ratio of pay-offs in step $t^* - 1$ is not improvable by any player, then also the ratio of pay-offs after the propagation process finished is not improvable by any player and vice versa. We show this in Lemma 4.4.

Lemma 4.4. *Let $(\mathcal{C}, 2)$ be a diffusion game on a monotonically growing temporal cycle $\mathcal{C}$ with strategy profile (p_1, p_2). Then, $\Delta(p_1, p_2)$ is not improvable by any player if and only if $\Delta_{t^*-1}(p_1, p_2)$ is not improvable by any player.*

Proof. Since each player starts spreading her color from one position and since $\mathcal{C}$ is a cycle from layer t^* on, the players color the same number of vertices from step t^* on. It follows that $\Delta_{t^*-1}(p_1, p_2) = \Delta(p_1, p_2)$ for all $p_1, p_2 \in V$. Consequently, $\Delta(p_1, p_2)$ is not improvable by any player if and only if $\Delta_{t^*-1}(p_1, p_2)$ is not improvable by any player. □

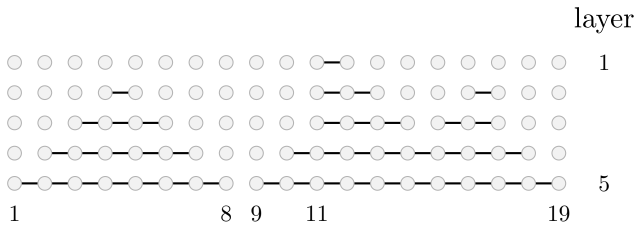


Figure 4.8a: The monotonically growing temporal path forest that consists of layers 1 to $t^* - 1$ of the temporal cycle in Figure 4.7.

We conclude that in order to find balanced strategy profiles on a monotonically growing temporal cycle, it is enough to find balanced strategy profiles until step $t^* - 1$. Layers 1 to $t^* - 1$ of a monotonically growing temporal cycle form a monotonically growing temporal path forest. By that, we can focus on finding balanced strategy profiles in diffusion games on monotonically growing temporal path forests where we assume that the coloring process on the last layer is happening only once. In the following, we define the modified diffusion game which we apply to monotonically growing temporal path forests in this section.

Definition 4.5. A *reduced* diffusion game on a temporal graph $\mathcal{G} = (V, E, \tau)$ is a diffusion game $(\mathcal{G}, 2)$ where the propagation process ends in step τ.

Summarizing, the simplified goal of this subsection is to find balanced strategy profiles for reduced diffusion games on monotonically growing temporal path forests. As argued above, using this, it is possible to find balanced strategy profiles in diffusion games on monotonically growing temporal cycles. Thus, at any time we speak of diffusion games on monotonically growing temporal path forests in this subsection, we refer to reduced diffusion games. Accordingly, we adjust the definition of reachability of a vertex. For a temporal graph $\mathcal{G} = (V, E, \tau)$, we assume that a vertex $v_1 \in V$ reaches another vertex $v_2 \in V$ if vertex v_1 reaches vertex v_2 until step τ.

In order to intuitively derive balanced strategy profiles, we discuss the outcome of various reduced diffusion games on the monotonically growing temporal path forest depicted in Figure 4.8a. We will identify an *optimal* strategy of a player, i.e., a position for which a player gets a high pay-off compared to the other player, independently of where the other player plays. Having identified such a strategy in every monotonically growing temporal path forest, we will be able to find balanced strategy profiles.

Since a player wants to maximize the number of vertices the player colors herself, intuitively, it seems to be a good strategy to play on a vertex that reaches a large number of vertices. We say that for a monotonically growing temporal path forest $\mathcal{F} = (V, E, \tau)$, a vertex $v \in V$ reaches the *maximum number of reachable vertices* in $\mathcal{F}$, if there is no other vertex $v' \in V$ that reaches more vertices than v. In the example in Figure 4.8a, at most eight vertices can be reached from any vertex. Among four other vertices, vertex 11 reaches eight vertices. Thus, vertex 11 reaches the maximum number of reachable vertices in the graph. The vertices that are reachable from that position are illustrated in Figure 4.8b. For illustrative purposes, we assume in the following that Player 1 plays on vertex 11.

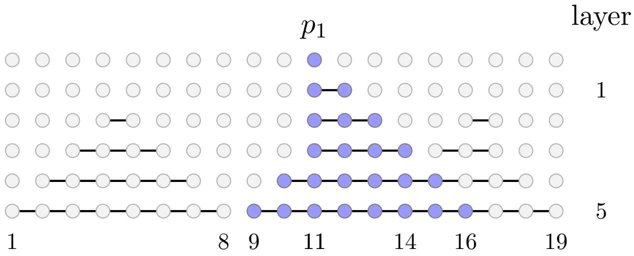


Figure 4.8b: This figure illustrates the vertices that are reachable by Player 1 if she plays on position 11. Player 1 reaches the vertices in $[9, 16]$ which are eight vertices. Eight is the maximum number of vertices reachable from any vertex in the temporal graph.

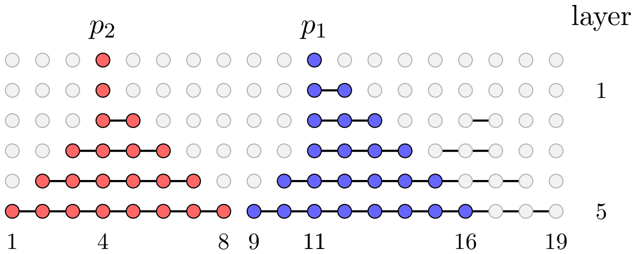


Figure 4.8c: A diffusion game where Player 2 plays on a position reaching a completely different vertex set than Player 1. Each player colors exactly the vertices the player reaches.

In the following, we discuss the ratio of pay-offs of the game for different positions of Player 2, assuming that Player 1 plays on vertex 11. We identify three types of positions Player 2 can play on. First, Player 2 can play on a position that reaches a completely different vertex set than Player 1. This happens if she plays on a position in $[1, 8]$ or on position 19. Second, Player 2 can reach a (not necessarily proper) subset of the vertices that Player 2 reaches, which applies to positions in $[9, 14]$. Third, Player 2 can play on a position reaching some vertices that Player 2 reaches and some vertices that Player 2 does not reach. This holds for positions in $[15, 18]$.

We first consider the case that Player 2 reaches completely different vertices than Player 1. An example of such a diffusion game is illustrated in Figure 4.8c. Since the players reach completely different vertices, each player colors exactly the vertices the player reaches. Consequently, the ratio of pay-offs only depends on the number of vertices the players reach.

Observation 4.6. *Let $(\mathcal{F}, 2)$ be a reduced diffusion game on a monotonically growing temporal path forest $\mathcal{F}$. Let (p_1, p_2) be a strategy profile and let Ω_1 and Ω_2 be the reachable vertex sets from p_1 and p_2. If $\Omega_1 \cap \Omega_2 = \emptyset$, then $\Delta(p_1, p_2) = |\Omega_1| - |\Omega_2|$.*

Second, we consider the case that Player 2 reaches some vertices that Player 1 reaches and some vertices that Player 1 does not reach. In the considered example, this happens if Player 2 plays on a vertex in $[15, 18]$. Assuming that Player 2 plays on vertex 16, we illustrate the vertices that are reachable from the two positions of the players in Figure 4.8d. It is clear that in any case, each player colors the vertices that only the

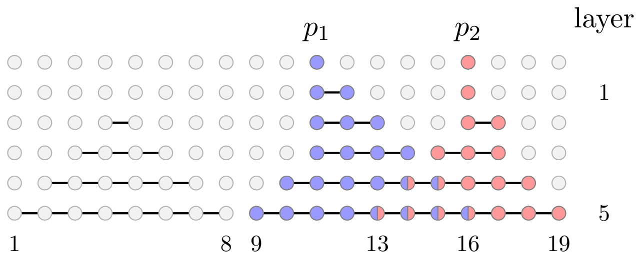


Figure 4.8d: Illustration of the vertices that are reachable from the two positions of the players for the case that Player 1 plays on vertex 11 and Player 2 plays on vertex 16. The vertices that both players reach are colored in both colors, blue and red. The players have some reachable vertices in common, that are, the vertices in [13, 16]. At the same time, each player reaches some vertices the other player does not reach.

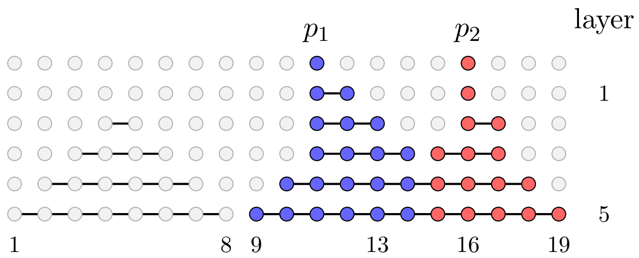


Figure 4.8e: Illustration of a diffusion game with the strategy profile from Figure 4.8d. The players color the same number of vertices in the vertex set that both players reach, that is, each player colors two vertices in [13, 16]. Additionally, each player colors the vertices that only the player reaches herself. Thereby, the ratio of pay-offs can be computed by the number of vertices the players reach, that is, $\Delta(p_1, p_2) = 8 - 7 = 1$

player herself reaches. Apart from that, it can be observed that each player colors exactly half of the vertices that both players reach. The result of the diffusion game is illustrated in Figure 4.8e. In fact, we can show that for all strategy profiles where the players have some reachable vertices in common and at the same time each player reaches some vertices the other player does not reach, the players always color the same number of vertices in the vertex set that both players reach. We show this in Lemma 4.7 in Section 4.2.4. We state Lemma 4.7 here for the sake of readability.

Lemma 4.7. *Let $(\mathcal{F}, 2)$ be a reduced diffusion game on a monotonically growing temporal path forest $\mathcal{F}$. Let (p_1, p_2) be a strategy profile and let Ω_1 and Ω_2 be the reachable vertex sets from p_1 and p_2. Assume that $\Omega_1 \cap \Omega_2 \neq \emptyset$ and that $\Omega_1 \not\subseteq \Omega_2$ and $\Omega_2 \not\subseteq \Omega_1$. Then, the players color the same number of vertices in $\Omega_1 \cap \Omega_2$.*

By Lemma 4.7, the players color the same number of vertices in the vertex set that both players reach. In addition, each player colors the vertices that only the player reaches herself. We conclude that the ratio of pay-offs only depends on the number of vertices the players reach. This is the same result that we got for the previous case where the players reach completely different vertex sets. Thus, we combine the two cases into

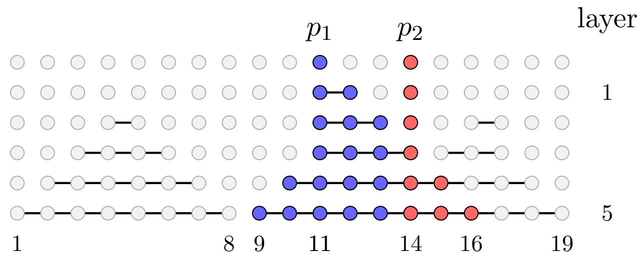


Figure 4.8f: A diffusion game where Player 2 plays on a vertex reaching a subset of the vertices Player 1 reaches. Player 1 reaches the vertices in $[9, 16]$ and Player 2 reaches the vertices in $[11, 16] \subseteq [9, 16]$.

one lemma:

Lemma 4.8. *Let $(\mathcal{F}, 2)$ be a diffusion game on a monotonically growing temporal path forest $\mathcal{F}$. Let (p_1, p_2) be a strategy profile and let Ω_1 and Ω_2 be the reachable vertex sets from p_1 and p_2. Assume that $\Omega_1 \not\subseteq \Omega_2$ and $\Omega_1 \not\subseteq \Omega_2$. Then, $\Delta(p_1, p_2) = |\Omega_1| - |\Omega_2|$.*

Proof. If $\Omega_1 \cap \Omega_2 = \emptyset$, then the statement follows by Observation 4.6. Otherwise, it holds that $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $\Omega_1 \not\subseteq \Omega_2$ and $\Omega_2 \not\subseteq \Omega_1$. By Lemma 4.7, the players color the same number of vertices in $\Omega_1 \cap \Omega_2$. Clearly, Player 1 colors all vertices in $\Omega_1 \setminus (\Omega_1 \cap \Omega_2)$ and Player 2 colors all vertices in $\Omega_2 \setminus (\Omega_1 \cap \Omega_2)$. We conclude that also in this case, it holds that $\Delta(p_1, p_2) = U_1(p_1, p_2) - U_2(p_1, p_2) = |\Omega_1| - |\Omega_2|$. □

It remains to consider the case that Player 2 reaches a (not necessarily proper) subset of the vertices that Player 1 reaches. An example of such a diffusion game, is illustrated in Figure 4.8f. Note that it is not possible that Player 1 reaches a subset of the vertices Player 2 reaches, since Player 1 reaches the maximum number of reachable vertices in the graph. In Figure 4.8f, Player 1 colors all vertices in between the positions of the players. However, generally, it hard to say how the vertices in between the positions of the players are colored, since it depends on the temporal graph. Nevertheless, it is possible to conclude how the other vertices in the temporal graph are colored. We already showed in Observation 3.3 in Section 3.2 that a player cannot "pass" a vertex colored by the other player in diffusion games on temporal paths. Clearly, this also applies to temporal path forests. For the example in Figure 4.8f this implies that Player 1 cannot color any vertex to the right of Player 2, that is, any vertex to the right of vertex 14. Clearly, Player 2 reaches all vertices to the right of her which also Player 1 reaches. As a result, in the example in Figure 4.8f, Player 2 colors at least the vertices in $[14, 16]$. Analogously, Player 1 colors at least the vertices in $[9, 11]$. We show that this generally holds if one player reaches a subset of the vertices the other player reaches in Lemma 4.9.

Lemma 4.9. *Let $(\mathcal{F}, 2)$ be a reduced diffusion game on a a monotonically growing temporal path forest $\mathcal{F}$. Let (p_1, p_2) be a strategy profile with $p_1 < p_2$. Let $\Omega_1 = [\alpha_1, \beta_1]$ and $\Omega_2 = [\alpha_2, \beta_2]$ be the reachable vertex sets from p_1 and p_2. Assume that $\Omega_2 \subseteq \Omega_1$. Then, Player 1 colors the vertices in $[\alpha_1, p_1]$ and Player 2 colors the vertices in $[p_2, \beta_1]$.*

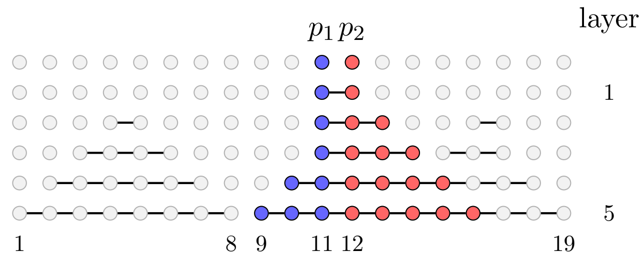


Figure 4.8g: Illustration of a diffusion game with the best response of Player 2 given that Player 1 plays on position 11. By playing directly next to Player 1 on vertex 12, Player 2 colors two vertices more than Player 1 so that $\Delta(p_1, p_2) = -2$.

Proof. We first show that $\beta_1 = \beta_2$. Since $p_1 < p_2$ and $\Omega_2 \subseteq \Omega_1$, it is clear that $\beta_1 \geq \beta_2$. Since p_1 reaches β_1 and since p_1 has to pass p_2 in order to reach β_1, it follows that p_2 also reaches β_1. We conclude that $\beta_1 = \beta_2$.

By Observation 3.3 from Section 3.2, a player cannot "pass" a vertex colored by the other player on temporal paths. This observation also applies to temporal path forests. Consequently, Player 1 cannot color any vertex $v > p_2$ and Player 2 cannot color any vertex $v < p_1$. As a result, Player 1 surely colors the vertices in $[\alpha_1, p_1]$ and Player 2 surely colors the vertices in $[p_2, \beta_1]$. □

Finally, we have considered all cases of strategy profiles in diffusion games on monotonically growing temporal path forests. For the first two considered cases of strategy profiles, i.e., none of the players reaches a subset of the vertices the other players reaches, we showed that the ratio of pay-offs only depends on the number of vertices the players reach (Lemma 4.8). Consequently, regarding these strategy profiles, the best strategy of Player 1 is to play on a position reaching the maximum number of reachable vertices in the graph. However, considering our results for strategy profiles where one player reaches a subset of the vertices the other players reaches (Lemma 4.9), we observe that such a position is not always optimal. In Figure 4.8g, we illustrate a diffusion game where Player 1 plays on position 11 (reaching the maximum number of reachable vertices) and where Player 2 plays directly next to her on vertex 12 (reaching a subset of the vertices that Player 1 reaches). In this diffusion game, Player 2 colors two vertices more than Player 1. We observe that Player 1 would have played better if she would have started with position 12. Vertex 12 reaches the same number of vertices as vertex 11 but is a central vertex of the vertex set it reaches. Thereby, Player 2 can steal at most half the vertices from Player 1.As a result, if Player 1 plays on vertex 12, then Player 2 can color at most the same number of vertices as Player 1 in any case. A diffusion game where Player 1 plays on vertex 12 and with a best response of Player 2 is illustrated in Figure 4.8h.

We conclude that in general, the best strategy is to play on a vertex that not only reaches the maximum number of reachable vertices but that additionally is a central vertex of the vertex set it reaches. In the following, we call such a position an *optimal position*.

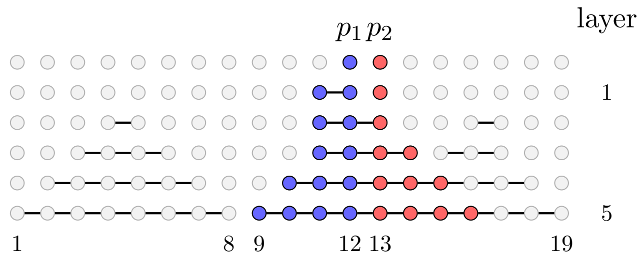


Figure 4.8h: A diffusion game where Player 1 initially chooses vertex 12. Since Player 1 plays on a central vertex of its reachable vertex set, Player 2 can color at most the same number of vertices as Player 1 by playing on a vertex that reaches a subset of the vertices Player 1 reaches.

Definition 4.10. Let $\mathcal{F}$ be a monotonically growing temporal path forest. Let L^* be the maximum number of vertices that are reachable from any vertex in $\mathcal{F}$. If a vertex $v \in V$ reaches L^* vertices and is a central vertex of its reachable vertex set, then v is an *optimal* position (in $\mathcal{F}$) .

In the following, we analyze how optimal positions are related to balanced strategy profiles. We show in Lemma 4.11 that if a player plays on an optimal position, then the other player can color at most the same number of vertices as the player playing on the optimal position.

Lemma 4.11. *Let $(\mathcal{F}, 2)$ be a reduced diffusion game on a monotonically growing temporal path forest $\mathcal{F}$ with strategy profile (p_1, p_2). If p_1 is an optimal position, then $\Delta(p_1, p_2) \geq 0$.*

Proof. Let Ω_1 and Ω_2 be the reachable vertex sets from p_1 and p_2 respectively. Let L^* be the maximum number of vertices that are reachable from any vertex in $\mathcal{F}$. Since p_1 reaches L^* vertices, it cannot hold that $\Omega_1 \subset \Omega_2$. We consider two cases. First assume that $\Omega_2 \subseteq \Omega_1$. Since p_1 is a central vertex of Ω_1, we conclude by Lemma 4.9 that Player 1 colors at least half the vertices of Ω_1. Thereby, Player 2 cannot color more vertices than Player 1, so that $\Delta(p_1, p_2) \geq 0$. Otherwise, it holds that $\Omega_1 \not\subseteq \Omega_2$ and $\Omega_2 \not\subseteq \Omega_1$. In this case, the ratio of pay-offs can be computed by $\Delta(p_1, p_2) = |\Omega_1| - |\Omega_2|$ (Lemma 4.8). Since p_1 reaches L^* vertices, p_2 cannot reach more vertices than p_1, so that $\Delta(p_1, p_2) \geq 0$. We conclude that $\Delta(p_1, p_2) \geq 0$ holds in both cases. □

If there is more than one optimal position in a monotonically growing temporal path forest and if both players play on an optimal position, then each player colors at least as many vertices as the other player. As a result, the players color the same number of vertices and we found a balanced strategy profile. We show this in Lemma 4.12.

Lemma 4.12. *Let $(\mathcal{F}, 2)$ be a reduced diffusion game on a monotonically growing temporal path forest $\mathcal{F}$ with strategy profile (p_1, p_2). Assume that there are at least two optimal positions in $\mathcal{F}$. If p_1 and p_2 are optimal positions, then (p_1, p_2) is a balanced strategy profile.*

Proof. Since p_1 and p_2 are both optimal positions, we conclude by Lemma 4.11 that $\Delta(p_1, p_2) \geq 0$ and $\Delta(p_1, p_2) \leq 0$, which implies that $\Delta(p_1, p_2) = 0$. Additionally, by Lemma 4.11, no player can improve the ratio of pay-offs by moving to a different position. As a result, (p_1, p_2) is a balanced strategy profile. □

In order to show that at least one optimal position exists in every monotonically growing temporal path forest, we use the following Lemma 4.13 which we prove in Section 4.2.4.

Lemma 4.13. *Let $\mathcal{F} = (V, E, \tau)$ be a monotonically growing temporal path forest and let Ω be a set of vertices that is reachable from a vertex $v \in V$. Let m be a central vertex of Ω. Then, vertex m reaches all vertices in Ω.*

By Lemma 4.13, every central vertex of an arbitrary reachable vertex set Ω reaches at least the vertices in Ω. Thereby, it is not hard to conclude that in every monotonically growing temporal path forest there exists a position that reaches the maximum number of reachable vertices and that is a central vertex of its reachable vertex set. We show this in Lemma 4.14.

Lemma 4.14. *There exists at least one optimal position in every monotonically growing temporal path forest.*

Proof. Let $\mathcal{F} = (V, E, \tau)$ be a monotonically growing temporal path forest. We construct an optimal position in $\mathcal{F}$ as follows. Let L^* be the maximum number of vertices reachable from any vertex in $\mathcal{F}$. Let $v \in V$ be a vertex that reaches L^* vertices. Assume that v reaches vertex set $\Omega \subseteq V$ and let m be a central vertex of Ω. By Lemma 4.13, m reaches all vertices in Ω. Consequently, m reaches L^* vertices and is a central vertex of its reachable vertex set. As a result, m is an optimal position in $\mathcal{F}$. □

For the case that the maximum number of vertices reachable from any vertex (denoted by L^*) is even, we will later show that always at least two optimal positions exist. However, if L^* is odd, then it is possible that only one optimal position exists. We can show that in this case the optimal position and any vertex adjacent to the optimal position form a balanced strategy profile. An example of a diffusion game where Player 1 plays on the optimal position and where Player 2 next to her is illustrated in Figure 4.9. In order to prove that such a strategy profile is always stable, we need the following Lemma 4.15 which we show in Section 4.2.4.

Lemma 4.15. *Let $\mathcal{F} = (V, E, \tau)$ be a monotonically growing temporal path forest. Assume that vertex $m \in V$ reaches the vertices in $[\alpha, \beta]$ and that m is a central vertex of $[\alpha, \beta]$. Assume that the number of vertices in $[\alpha, \beta]$ is odd. Then vertex $m+1$ reaches at least the vertices in $[\alpha + 1, \beta]$.*

Using Lemma 4.15, we can show a strategy profile where Player 1 plays on the only optimal position and where Player 2 plays next to Player 1 results in a ratio of pay-offs of 1. Furthermore, we can prove that Player 1 always colors at least one vertex more than Player 2 if Player 1 plays on the only optimal position. Consequently, if Player 2 plays next to Player 1, then Player 2 plays a best response to Player 1 (regarding the ratio of pay-offs). By additionally showing that also Player 1 plays a best response to Player 2, we can conclude that the players play a balanced strategy profile. We show this in Lemma 4.16.

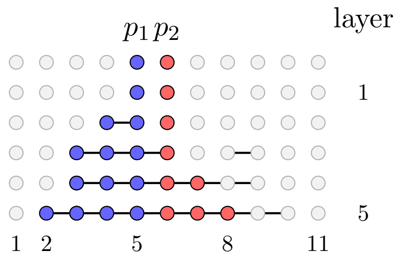


Figure 4.9: A diffusion game on a monotonically growing temporal path forest where only one optimal position exists. The optimal position is vertex 5 which is the position of Player 1. Player 2 colors only one vertex less than Player 1 by playing directly next to her on vertex 6. The illustrated strategy profile is balanced.

Lemma 4.16. *Let $(\mathcal{F}, 2)$ be a reduced diffusion game on a monotonically growing temporal path forest $\mathcal{F}$ with strategy profile (p_1, p_2). Assume that there is only one optimal position in $\mathcal{F}$. If p_1 is an optimal position and if $p_2 = p_1 + 1$, then (p_1, p_2) is a balanced strategy profile.*

Proof. Assume that p_1 and p_2 reach the vertices in Ω_1 and Ω_2. Assume that $\Omega_1 = [\alpha_1, \beta_1]$. Let L^* be the maximum number of vertices reachable from any vertex in $\mathcal{C}$.

We first compute the ratio of pay-offs for strategy profile (p_1, p_2). By Lemma 4.15, p_2 reaches at least the vertices in $[\alpha_1 + 1, \beta_1]$ which are at least $L^* - 1$ vertices. If p_2 would additionally reach $\beta_1 + 1$, then p_2 would be a central vertex of a reachable vertex set of size L^*. This contradicts that p_2 is the only optimal position in $\mathcal{C}$. Additionally, p_2 cannot reach $\alpha_1 - 1$, since otherwise p_2 would reach more than L^* vertices. We conclude that p_2 reaches at most the vertices in $[\alpha_1, \beta_1]$ and at least the vertices in $[\alpha_1 + 1, \beta_1]$. As a result, $\Omega_2 \subseteq \Omega_1$. Since there is only one optimal position in $\mathcal{F}$, the number of vertices in Ω_1 must be odd. Consequently, there is only one central vertex of Ω_1, which is p_1. Since p_1 is the only central vertex of Ω_1 and since $\Omega_2 \subseteq \Omega_1$, we conclude by Lemma 4.9 that Player 1 colors one vertex more than Player 2, so that $\Delta(p_1, p_2) \geq 1$. Since $p_2 = p_1 + 1$, we get $\Delta(p_1, p_2) = 1$.

It remains to show that $\Delta(p_1, p_2)$ is not improvable by any player. We first consider possibly better positions for Player 2, then for Player 1. For the sake of contradiction, assume that Player 2 prefers moving to a different position p_2' reaching the vertices in Ω_2', so that $\Delta(p_1, p_2') < 1$. We consider two cases.

1. First assume that $\Omega_1 \not\subseteq \Omega_2'$ and $\Omega_2' \not\subseteq \Omega_1$. By Lemma 4.8, the ratio of pay-offs can be computed by $\Delta(p_1, p_2') = |\Omega_1| - |\Omega_2'|$. Since there is only one optimal position in $\mathcal{F}$, there is only one reachable vertex set of size L^*, which is Ω_1. Since $\Omega_1 \neq \Omega_2'$ this implies that $|\Omega_2'| < L^*$. It follows that $\Delta(p_1, p_2') = |\Omega_1| - |\Omega_2'| \geq 1$. This contradicts that p_2' is a better position for Player 2 than p_2.

2. Otherwise, it holds that $\Omega_1 \subseteq \Omega_2'$ or $\Omega_2' \subseteq \Omega_1$. Since $|\Omega_1| = L^*$, it cannot hold that $\Omega_1 \subset \Omega_2'$. Consequently, $\Omega_2' \subseteq \Omega_1$. Since Player 1 plays on a central vertex of Ω_1 and since the number of vertices in Ω_1 is odd, we conclude by Lemma 4.9 that

Player 2 colors less vertices than Player 1, so that $\Delta(p_1, p_2') \geq 1$. This contradicts that p_2' is a better position for Player 2 than p_2.

We conclude that $\Delta(p_1, p_2)$ is not improvable by Player 2. For the sake of contradiction, assume that Player 1 prefers moving to a different position p_1' reaching the vertices in Ω_1', so that $\Delta(p_1', p_2) > 1$. We consider two cases.

1. First assume that $\Omega_1' \not\subseteq \Omega_2$ and $\Omega_2 \not\subseteq \Omega_1'$. By Lemma 4.8, the ratio of pay-offs can be computed by $\Delta(p_1', p_2) = |\Omega_1'| - |\Omega_2|$. We showed before that $|\Omega_2| = L^* - 1$ or $|\Omega_2| = L^*$. Since L^* is the maximum number of reachable vertices in $\mathcal{F}$, it holds that $|\Omega_1'| \leq L^*$. Hence, $\Delta(p_1', p_2) = |\Omega_1'| - |\Omega_2| \leq 1$. This contradicts that p_1' is a better position for Player 1 than p_1.

2. Otherwise, it holds that $\Omega_1' \subseteq \Omega_2$ or $\Omega_2 \subseteq \Omega_1'$. First assume that $\Omega_1' \subseteq \Omega_2$. Since Player 2 plays on a central vertex of Ω_2, it follows by Lemma 4.9 that $\Delta(p_1', p_2) \leq 0$. Otherwise, $\Omega_2 \subset \Omega_1'$, which implies that $|\Omega_2| = L^* - 1$. Consequently, $|\Omega_2| = |\Omega_1'| + 1$. Again, we consider two cases. First assume that the number of vertices in Ω_2 is odd. Since Player 2 plays on the only central vertex of Ω_2, she also plays on one of two central vertices of Ω_1. Consequently, by Lemma 4.9, Player 2 colors at least as many vertices as Player 1, that is, $\Delta(p_1', p_2) \leq 0$. Otherwise, the number of vertices in Ω_2 is even, so that the number of vertices in Ω_1 is odd. Thus, Player 2 plays either on the central vertex of Ω_1 or directly next to the central vertex of Ω_1. By Lemma 4.9, in any case, Player 1 colors at most one vertex more than Player 2, that is, $\Delta(p_1', p_2) \leq 1$.

   All cases contradict that p_1' is a better position for Player 1 than p_1.

As a result, $\Delta(p_1, p_2)$ is also not improvable by Player 1. We conclude that (p_1, p_2) is a balanced strategy profile. □

Using Lemma 4.12 and Lemma 4.16, we can find balanced strategy profiles for reduced diffusion games on all monotonically growing temporal path forests. We summarize our results in the following corollary:

Corollary 4.17. *Let $(\mathcal{F}, 2)$ be a reduced diffusion game on a monotonically growing temporal path forest $\mathcal{F}$ with strategy profile (p_1, p_2).*

1. *Assume that there are at least two optimal positions in $\mathcal{F}$. If p_1 and p_2 are both optimal positions in $\mathcal{F}$, then strategy profile (p_1, p_2) is balanced.*

2. *Otherwise, let p_1 be an optimal position in $\mathcal{F}$ and $p_2 = p_1 + 1$. Then, strategy profile (p_1, p_2) is balanced.*

Finally, we apply our results for reduced diffusion games on monotonically growing temporal path forests to diffusion games on monotonically growing temporal cycles. Since layers 1 to $t^* - 1$ of a monotonically growing temporal cycle constitute a monotonically growing temporal path forest, we define an optimal position in a monotonically growing temporal cycle as a position reaching the maximum number of reachable vertices until step $t^* - 1$ and being a central vertex of its reachable vertex set until step $t^* - 1$.

Definition 4.18. Let $\mathcal{C}$ be a monotonically growing temporal cycle. Let L^* be the maximum number of vertices that are reachable from any vertex in $\mathcal{C}$ until step t^*-1. If a vertex $v \in V$ reaches L^* vertices until step t^*-1 and is a central vertex of the vertex set vertex v reaches until step t^*-1, then v is an *optimal* position (in $\mathcal{C}$).

In the following, we assume that edge $\{n, 1\}$ is always one of the last edges that appears in a monotonically growing temporal cycle, i.e., $(\{n, 1\}, t)$ does not exist in every step $t < t^*$. Thereby, we can directly apply our results for reduced diffusion games on monotonically growing temporal path forests to diffusion games on monotonically growing temporal cycles.

Lemma 4.19. *Let $(\mathcal{C}, 2)$ be a diffusion game on a monotonically growing temporal cycle $\mathcal{C}$ with strategy profile (p_1, p_2).*

1. *Assume that there are at least two optimal positions in $\mathcal{C}$. If p_1 and p_2 are both optimal positions in $\mathcal{C}$, then strategy profile (p_1, p_2) is balanced.*
2. *Otherwise, let p_1 be an optimal position in $\mathcal{C}$ and let $p_1 = p_1 + 1$. Then, a strategy profile (p_1, p_2) is balanced.*

Proof. Let $\mathcal{C} = (V, E, \tau)$. Let $\mathcal{F}$ be the monotonically growing temporal path forest that corresponds to layers 1 to t^*-1 of $\mathcal{C}$, that is, $\mathcal{F} = (V, E', t^*-1)$ with $E' = \{(e, t) \in E \mid t \in [1, t^*-1]\}$. By the definition of $\mathcal{F}$ and by the definition of a reduced diffusion game, it follows that strategy profile (p_1, p_2) is balanced in a reduced diffusion game on $\mathcal{F}$ if and only if $\Delta_{t^*-1}(p_1, p_2)$ is not improvable by any player in a diffusion game on $\mathcal{C}$. By Lemma 4.4, it holds that $\Delta_{t^*-1}(p_1, p_2) = \Delta(p_1, p_2)$. We conclude that strategy profile (p_1, p_2) is balanced in a reduced diffusion game on $\mathcal{F}$ if and only if strategy profile (p_1, p_2) is balanced in a diffusion game on $\mathcal{C}$. In Corollary 4.17, we state in which case (p_1, p_2) is balanced in a reduced diffusion game on $\mathcal{F}$. As we defined optimal positions in $\mathcal{F}$ and $\mathcal{C}$ analogously, we can directly apply these results to diffusion games on monotonically growing temporal cycles. □

4.2.2 From Balanced Strategy Profiles to Nash Equilibria

In the previous subsection, we found balanced strategy profiles on all monotonically growing temporal cycles. In this section, we derive which of these balanced strategy profiles are Nash equilibria.

If we get at most one gray or uncolored vertex in a diffusion game on a temporal graph with some balanced strategy profile, then the profile is always a Nash equilibrium.

Lemma 4.20. *Let $(\mathcal{G}, 2)$ be a diffusion game on a temporal graph $\mathcal{G}$. Let (p_1, p_2) be a balanced strategy profile. If there is at most one gray or uncolored vertex after the propagation process finished for strategy profile (p_1, p_2), then (p_1, p_2) is a Nash equilibrium.*

Proof. Without loss of generality and for the sake of contradiction, assume that Player 1 gets a higher pay-off by moving to position p_1'. Since (p_1, p_2) is a balanced strategy profile, the outcome of Player 2 increases by at least the same number. Since all but at most one vertex is colored by one of the players for (p_1, p_2), it is not possible that both players get an additional point. Therefore, Player 1 cannot get a higher pay-off by moving to position p_1'. Thus, (p_1, p_2) is a Nash equilibrium. □

If a balanced strategy profile results in more than one gray or uncolored vertex, then we cannot directly conclude that the profile is a Nash equilibrium. To deal with this case, we start by showing that at most two vertices can be colored gray in every diffusion game on some monotonically growing temporal cycle. Additionally, there will never be any uncolored vertex in the graph.

Lemma 4.21. *Let $(\mathcal{C}, 2)$ be a diffusion game on a monotonically growing temporal cycle $\mathcal{C}$. Then, there are at most two gray and no uncolored vertices in the graph for any strategy profile.*

Proof. Since $\mathcal{C}$ is monotonically growing, the last layer of $\mathcal{C}$ is a cycle. Consequently, the players continue spreading their color on the last layer of $\mathcal{C}$ until they are stopped by a vertex that is already colored. Since the underlying graph is a cycle and since each player starts from one position, the players meet at exactly two places. At each place, at most one vertex can be colored gray. Clearly, no uncolored vertices remain. □

Since we cannot directly conclude that a balanced strategy profile resulting in two gray vertices is a Nash equilibrium, we preferably use balanced strategy profiles that result in at most one gray vertex in order to find a Nash equilibrium. For the sake of readability, we recall our partial characterization of balanced strategy profiles from Section 4.2.1.

Lemma 4.19. *Let $(\mathcal{C}, 2)$ be a diffusion game on a monotonically growing temporal cycle $\mathcal{C}$ with strategy profile (p_1, p_2).*

1. *Assume that there are at least two optimal positions in $\mathcal{C}$. If p_1 and p_2 are both optimal positions in $\mathcal{C}$, then strategy profile (p_1, p_2) is balanced.*

2. *Otherwise, let p_1 be an optimal position in $\mathcal{C}$ and let $p_1 = p_1 + 1$. Then, a strategy profile (p_1, p_2) is balanced.*

If the players play on adjacent positions in a monotonically growing temporal cycle, then it is clear that the strategy profile results in at most one gray vertex. This is the case for the balanced strategy profiles mentioned in Case 2 in Lemma 4.19. Consequently, strategy profiles of this type are always Nash equilibria. It remains to consider Case 1 from Lemma 4.19, i.e. more than one optimal position exists. In that case, every pair of optimal positions is a balanced strategy profile. In order to derive for which monotonically growing temporal cycles there exist two adjacent optimal positions, we recall the definition of an optimal position.

Definition 4.18. Let $\mathcal{C}$ be a monotonically growing temporal cycle. Let L^* be the maximum number of vertices that are reachable from any vertex in $\mathcal{C}$ until step $t^* - 1$. If a vertex $v \in V$ reaches L^* vertices until step $t^* - 1$ and is a central vertex of the vertex set vertex v reaches until step $t^* - 1$, then v is an *optimal* position (in $\mathcal{C}$).

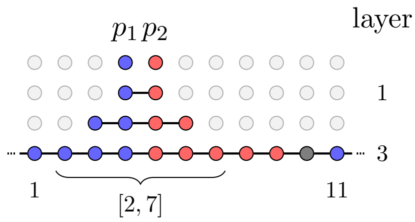


Figure 4.10: Diffusion game on a monotonically growing temporal cycle where $L^* = 6$ is even. Vertex set $[2, 7]$ is a reachable vertex set until step $t^* - 1$ of size L^*. Since the number of vertices in $[2, 7]$ is even, both players can play on a central vertex of $[2, 7]$. Thereby, the players play on adjacent optimal positions.

For the case that L^* is even, we can easily find two adjacent optimal positions. An example of this case is illustrated in Figure 4.10. It is easy to see that every pair of central vertices of a reachable vertex until step $t^* - 1$ of size L^* is a pair of adjacent optimal positions.

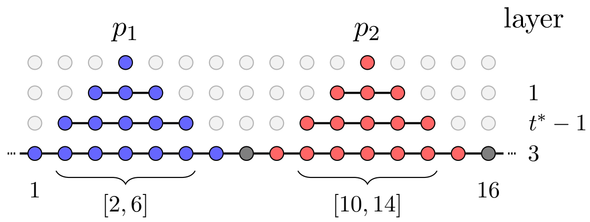
It remains to consider the case that L^* is odd. In that case, there do not always exist two adjacent optimal positions. However, we can derive conditions for the existence of gray vertices dependent on the number of vertices between the positions of the players. In order to refer to the number of vertices between the two positions, we define the distances between two vertices in a temporal cycle.

Definition 4.22. Let $\mathcal{C} = (V, E, \tau)$ be a monotonically growing temporal cycle. Let $u, v \in V$ with $u < v$. The distances between u and v are defined by $d_1(u, v) := v - u$ and $d_2(u, v) := u + (|V| - v)$.

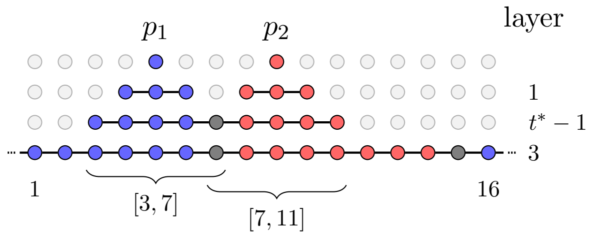
We first investigate in which diffusion games we get gray vertices until step $t^* - 1$. In Figure 4.11a, we illustrate a diffusion game where the players do not have reachable vertices in common until step $t^* - 1$. It is clear that in this case, we do not get any gray vertices until step $t^* - 1$. Otherwise, the players have reachable vertices in common until step $t^* - 1$ as exemplified in Figure 4.11b. We can show that in this case we get a gray vertex until step $t^* - 1$ if and only if $d_1(p_1, p_2)$ is even. The proof is based on the following lemma which we show in Section 4.2.3.

Lemma 4.23. *Let $(\mathcal{F}, 2)$ be a reduced diffusion game on a monotonically growing temporal path forest $\mathcal{F}$. Let p_1, p_2 with $p_1 < p_2$ be two optimal positions in $\mathcal{F}$, reaching the vertex sets Ω_1 and Ω_2. Assume that the number of vertices in Ω_1 and Ω_2 is odd. Assume that $\Omega_1 \cap \Omega_2 \neq \emptyset$. Then, a vertex in $[p_1, p_2]$ is colored gray if and only if $d(p_1, p_2)$ is even.*

It remains to consider in which case we get gray vertices after step $t^* - 1$. Since a monotonically growing temporal cycle is a cycle from layer $t^* - 1$ and since the players play on optimal positions, we can also here reduce the appearance of gray vertices to the distance of the optimal positions. All in all, we can show that we get two gray vertices in a monotonically growing temporal cycle where L^* is odd if and only if the distances between the positions of the players are even in both directions. We show this in Lemma 4.24.



(a) Player 1 reaches the vertices in $[2, 6]$ and Player 2 reaches the vertices in $[10, 14]$ until step $t^* - 1$. Consequently, there are no vertices that both players reach before the graph becomes a cycle. As a result, we do not get any gray vertex until step $t^* - 1$.



(b) Player 1 reaches the vertices in $[3, 7]$ and Player 2 reaches the vertices in $[7, 11]$ until step $t^* - 1$. Thus, both players reach vertex 7 before the graph becomes a cycle. Consequently, we can get a gray vertex before step $t^* - 1$, here, for instance, vertex 7.

Figure 4.11: Two examples of diffusion games on a monotonically growing temporal cycles where L^* is odd. In both examples there do not exist two adjacent optimal positions.

Lemma 4.24. *Let $(\mathcal{C}, 2)$ be a diffusion game on a monotonically growing temporal cycle $\mathcal{C}$. Let L^* be the maximum number of vertices that are reachable from any position in $\mathcal{C}$ until step $t^* - 1$. Assume that L^* is odd. Let $p_1, p_2 \in V$ be two optimal positions in $\mathcal{C}$ with $p_1 < p_2$. Then, there are two gray vertices in the graph for strategy profile (p_2, p_2) if and only if both $d_1(p_1, p_2)$ and $d_2(p_1, p_2)$ are even.*

Proof. Assume that p_1 and p_2 reach the vertex sets Ω_1 and Ω_2 until step $t^* - 1$. We consider two cases. If $\Omega_1 \cap \Omega_2 = \emptyset$, then each player colors the vertex set the player reaches until step $t^* - 1$. Since both players play on an optimal position, they reach the same number of vertices until step $t^* - 1$ and each Player i plays on the only central vertex of Ω_i. Since additionally from step t^* on the temporal graph is a cycle, we get two gray vertices for (p_1, p_2) if and only if $d_1(p_1, p_2)$ is even and $d_2(p_1, p_2)$ is even.

Otherwise, it holds that $\Omega_1 \cap \Omega_2 \neq \emptyset$. Without loss of generality assume that the vertices in $[p_1, p_2]$ are colored in step $t^* - 1$. By Lemma 4.23, a vertex in $[p_1, p_2]$ is colored gray if and only if $d_1(p_1, p_2)$ is even. With the same arguments as in the previous case we conclude that we get another gray vertex after step $t^* - 1$ if and only if $d_2(p_1, p_2)$ is even. We conclude that we get two gray vertices if and only if $d_1(p_1, p_2)$ and $d_2(p_1, p_2)$ are both even. □

We conclude by Lemma 4.24 that if L^* is odd and if the players play on optimal positions that have odd distance in at last one direction, then we get at most one gray vertex in the graph. Thereby, also in this case, we found a Nash equilibrium. It remains to consider the case that L^* is odd and that all optimal positions have even distance in both directions. We show in Lemma 4.25 that in this case, in fact, every pair of optimal positions is a Nash equilibrium.

Lemma 4.25. *Let $(\mathcal{C}, 2)$ be a diffusion game on a monotonically growing temporal cycle $\mathcal{C}$. Let L^* be the largest number of vertices that are reachable from any position in $\mathcal{C}$ until step $t^* - 1$. Assume that L^* is odd. Assume that at least two optimal position exist in $\mathcal{C}$ and that for all optimal positions $v_1, \ldots, v_m$, distances $d_1(v_i, v_j)$ and $d_2(v_i, v_j)$ with $i, j \in [1, m]$ are even. Let $p_1 = v_1$ and $p_2 = v_2$. Then, strategy profile (p_1, p_2) is a Nash equilibrium.*

Proof. For the sake of contradiction, assume that (p_1, p_2) is not a Nash equilibrium. Without loss of generality, assume that Player 1 gets a higher outcome by moving to position p_1'.

We first show that $\Delta(p_1', p_2) = 0$ and that (p_1', p_2) results in no gray vertices. Since p_1 and p_2 are both optimal positions, we conclude by Lemma 4.19 that (p_1, p_2) is a balanced strategy profile. Consequently, Player 2 gets at least the same number of additional points for (p_1', p_2) as Player 1. Since all optimal positions have even distance in both directions, we conclude by Lemma 4.24 that all pairs of optimal positions result in two gray vertices. Consequently, also (p_1, p_2) results in two gray vertices. By Lemma 4.21 there are no uncolored vertices in the graph. Hence, only two vertices are not colored by one of the players for (p_1, p_2). As a result, each player gets exactly one additional point for (p_1', p_2), so that $\Delta(p_1, p_2) = \Delta(p_1', p_2)$. Additionally, there are no gray and uncolored vertices left for (p_1', p_2). Since p_1 and p_2 are both optimal positions, we conclude by Lemma 4.11 that $\Delta(p_1, p_2) = 0$, so that also $\Delta(p_1', p_2) = 0$. We summarize that $\Delta(p_1', p_2) = 0$ and that (p_1', p_2) results in no gray vertices.

Let Ω_1' and Ω_2 be the reachable vertex sets from p_1' and p_2 until step $t^* - 1$. We consider two cases.

1. First assume that $\Omega_1' \subseteq \Omega_2$ or $\Omega_2 \subseteq \Omega_1'$. Since $|\Omega_2| = L^*$, it follows that $\Omega_1' \subseteq \Omega_2$. Since Player 2 plays on the only central vertex of Ω_2, we conclude by Lemma 4.9 that Player 2 colors more vertices than Player 1 for (p_1', p_2). This contradicts that $\Delta(p_1', p_2) = 0$.

2. Otherwise, it holds that $\Omega_1' \not\subseteq \Omega_2$ and $\Omega_2 \not\subseteq \Omega_1'$. By Lemma 4.8, the ratio of payoffs for (p_1', p_2) can be computed by $\Delta(p_1', p_2) = |\Omega_1'| - |\Omega_2|$. Since $\Delta(p_1', p_2) = 0$ and $|\Omega_2| = L^*$, it follows that $|\Omega_1'| = L^*$. Let m be the central vertex of Ω_1'. By Lemma 4.13, m and p_1' reach the same vertex set Ω_1 until step $t^* - 1$. Consequently, we get the same coloring of the temporal graph for strategy profile (m, p_2) and (p_1', p_2). Since (p_1', p_2) results in no gray vertices, also (m, p_2) results in no gray vertices. However, m and p_2 are both optimal positions. This contradicts that all pairs of optimal positions result in two gray vertices.

Thus, (p_1, p_2) is a Nash equilibrium. □

Finally, we have found a Nash equilibrium for all cases of monotonically growing temporal cycles. We summarize our results in an algorithm in the next subsection.

4.2.3 The Algorithm

The following algorithm returns a Nash equilibrium for every monotonically growing temporal cycle in linear time.

Algorithm 4.26.

Input: Monotonically growing temporal cycle $\mathcal{C} = (V, E, \tau)$ with $t^* > 0$
Output: Nash equilibrium (p_1, p_2)

Description: Let L^* be the maximum number of vertices that are reachable from any vertex in $\mathcal{C}$ until step $t^* - 1$.

1. If L^* is even, then let p_1 and p_2 be the two central vertices of a reachable vertex set of size L^* until step $t^* - 1$.

2. Otherwise, L^* is odd. Let $\Omega_1, \ldots, \Omega_z$ be the reachable vertex sets in $\mathcal{C}$ until step $t^* - 1$ of size L^*. Let $v_1, v_2, \ldots, v_z$ be the central vertices of these vertex sets.

   (a) If $z \geq 2$ we consider two cases.

      i. If there is a pair of vertices v_i, v_j with $i \neq j \in [1, z]$ such that $d_1(v_i, v_j)$ or $d_2(v_i, v_j)$ is odd, then let $p_1 = v_i$ and $p_2 = v_j$

      ii. Otherwise, let $p_1 = v_1$ and $p_2 = v_2$.

   (b) Otherwise, let $p_1 = v_1$ and $p_2 = v_1 + 1$

We show the correctness of Algorithm 4.26 in Theorem 4.27 and prove that it runs in linear time in Theorem 4.28.

Theorem 4.27. *Algorithm 4.26 returns a Nash equilibrium for every monotonically growing temporal cycle $\mathcal{C}$ with $t^* > 0$.*

Proof. First of all, note that by Lemma 4.19, every strategy profile returned by Algorithm 4.26 is a balanced strategy profile. In the following, we iterate over all cases of Algorithm 4.26 and show that Algorithm 4.26 always returns a Nash equilibrium. In Case 1 and Case 2b, the players play next to each other. Consequently, for these strategy profiles, we get at most one gray vertex. By Lemma 4.20, it follows that these strategy profiles are Nash equilibria.

Otherwise a strategy profile returned by Algorithm 4.26 falls under Case 2a. Consequently, at least two optimal positions exist in $\mathcal{C}$. In Case 2(a)i, the players play on optimal positions with odd distance in at least one direction. We conclude by Lemma 4.24 that we get at most one gray vertex. By Lemma 4.20, it follows that the strategy profile is a Nash equilibrium. Considering Case 2(a)ii, all optimal positions in $\mathcal{C}$ have even distance in both directions. By Lemma 4.25, we conclude that (v_1, v_2) is a Nash equilibrium. Thus, also Case 2(a)ii returns a Nash equilibrium. □

Theorem 4.28. *Let $\mathcal{C} = (V, E, \tau)$ be a monotonically growing temporal cycle of size n. Algorithm 4.26 on $\mathcal{C}$ runs in $O(n \cdot \tau)$-time.*

Proof. First, we need to compute step t^*, i.e. the first layer in which $\mathcal{C}$ is a complete cycle. For this, we check for every step $t \in [1, \tau]$ if the number of edges in E_t is n. This can be done in $\mathcal{O}(\tau \cdot n)$-time.

The next step is to compute L^*. To do so, we create for every vertex $v \in V$ a one-player diffusion game on $\mathcal{C}$ with strategy profile (v). In every step $t \in [1, \tau]$ of a diffusion game on $\mathcal{C}$, a vertex colors at most two other vertices. Performing n diffusion games on $\mathcal{C}$ until step $t^* - 1$ runs in $\mathcal{O}(n \cdot t^*)$-time. Finding the maximal number of reachable vertices until step $t^* - 1$ out of all reachable vertex sets until step $t^* - 1$ can be done in $\mathcal{O}(n)$-time. Altogether, L^* can be computed in $\mathcal{O}(n \cdot \tau)$-time.

In the following, we consider the different cases of Algorithm 4.26. Note that we have computed all reachable vertex sets of size L^* before. Computing the central vertices of these vertex sets can be done in constant time. Consequently, in Case 1 and Case 2b, we get a solution in constant time.

In the following, we consider Case 2a. We assume that L^* is odd and that $z \geq 2$. If the number n of vertices in $\mathcal{C}$ is odd, then either $d_1(v_1, v_2)$ or $d_2(v_1, v_2)$ is odd. Consequently, if n is odd, then (v_1, v_2) is a solution for Case 2(a)i. Otherwise, n is even. It follows that for all v_i, v_j with $i \neq j \in [1, z]$, $d_1(v_i, v_j)$ is even if and only if $d_2(v_i, v_j)$ is even. Thus, it is enough to consider the distances between two optimal positions in one direction. Without loss of generality, assume that $v_1 < v_i$ with $i \in [2, z]$. We compute all distances $d_1(v_1, v_i)$. If $d_1(v_1, v_i)$ is odd, then we found a solution for Case 2(a)i. Otherwise, $d_1(v_1, v_i)$ is even for all $i \in [2, z]$. This implies that also $d_1(v_j, v_i)$ is even for all $j \in [2, z]$. Thus, all optimal positions have even distance in both directions, so that Case 2(a)ii holds. We summarize that for Case 2(a)i and Case 2(a)ii, we only have to compute the distances between v_1 and all other optimal positions v_i with $i \in [2, z]$. This can be done in $\mathcal{O}(z)$-time and thus, in $\mathcal{O}(n)$-time.

Altogether, Algorithm 4.26 runs in $\mathcal{O}(\tau \cdot n)$-time. □

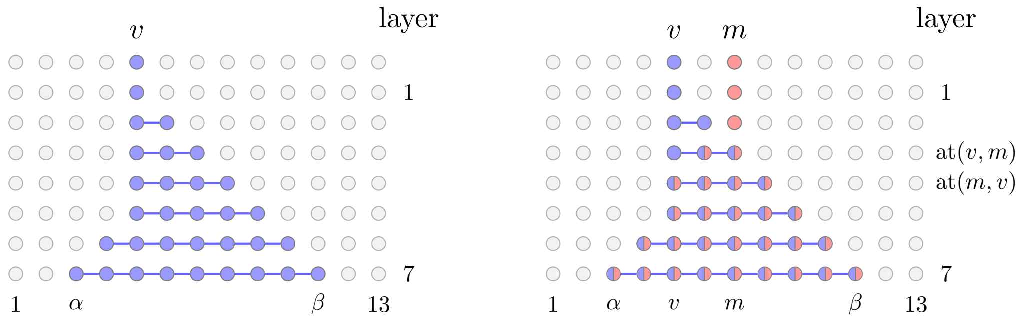
We summarize our results in Corollary 4.29.

Corollary 4.29. *A Nash equilibrium is guaranteed to exist on every monotonically growing temporal cycle $\mathcal{C} = (V, E, \tau)$ and can be found in $\mathcal{O}(\tau \cdot n)$-time.*

4.2.4 Filling the Gaps: Reachability

In this subsection, we show four lemmas: Lemma 4.13, Lemma 4.7, Lemma 4.15 and Lemma 4.23. The first three lemmas were used in Section 4.2.1 to find balanced strategy profiles on monotonically growing temporal path forests. The last lemmas has been used in Section 4.2.2.

All four lemmas are concerned with the reachability of vertices in reduced diffusion games on monotonically growing temporal path forests. Therefore, we use the adjusted definition of reachability for reduced diffusion games: For a temporal graph $\mathcal{G} = (V, E, \tau)$, we say that a vertex $v_1 \in V$ *reaches* another vertex $v_2 \in V$ if vertex v_1 reaches vertex v_2



(a) Assuming that v reaches the vertices in $[\alpha, \beta]$, v must have reached certain vertices in previous steps. These vertices are colored in blue in the respective layers. By the assumption that v reaches the blue colored vertices, we conclude that the blue colored edges must exist in the temporal graph.

(b) This figure illustrates that central vertex m of $[\alpha, \beta]$ reaches all vertices in $[\alpha, \beta]$. The vertices reachable from m are colored in red. Step $\text{at}(v, m)$ is the step in which v reaches m and step $\text{at}(m, v)$ the step in which m reaches v.

Figure 4.12: Illustration for the proof of Lemma 4.13.

until step τ. With $\text{at}(v_1, v_2)$, we refer to the step in which v_1 reaches v_2. Analogously, $\text{at}(v_2, v_1)$ is the step in which v_2 reaches v_1. Note that it does not necessarily hold that $\text{at}(v_1, v_2) = \text{at}(v_2, v_1)$.

We start with the proof of Lemma 4.13. In Lemma 4.13, we state that if a vertex v reaches a vertex set $[\alpha, \beta]$ in a monotonically growing temporal path forest $\mathcal{F}$, then also every central vertex m of $[\alpha, \beta]$ reaches the vertices in $[\alpha, \beta]$. In order to give an intuition why Lemma 4.13 holds, we construct an example that illustrates the situation. Since $\mathcal{F}$ is monotonically growing and since v reaches the vertices in $[\alpha, \beta]$, certain edges must exist in the temporal graph $\mathcal{F}$. We mark these edges in the illustration in Figure 4.12a. Considering this illustration, it is easy to see that, by the marked edges, central vertex m of $[\alpha, \beta]$ is also able to reach all vertices in $[\alpha, \beta]$. The vertices that are reachable from m are illustrated in Figure 4.12b.

In order to prove that m reaches all vertices in $[\alpha, \beta]$, we assume that v lies to the left of m and consider the vertices in $[\alpha, v]$, $[v, m]$ and $[m, \beta]$ independently. Since v reaches the vertices in $[m, \beta]$ and since m lies on the way from v to β, it is clear that also m reaches all vertices in $[m, \beta]$. Additionally, if m reaches α, then m also reaches the vertices between α and m. We show in Observation 4.30 that m reaches vertex α if m reaches v until step $\tau - d(\alpha, v)$.

Observation 4.30. *Let $\mathcal{F} = (V, E, \tau)$ be a monotonically growing temporal path forest. Let $v_1, v_2 \in V$ with $v_1 < v_2$. Assume that v_1 reaches the vertices in $[\alpha, \beta]$. If v_2 reaches v_1 until step $\tau - d(\alpha, v_1)$, then v_2 reaches vertex α.*

Proof. Vertex v_1 reaches α and edges do not disappear by the monotonicity of the temporal graph. At least $d(\alpha, v_1)$ steps are necessary in order to reach α from v_1. Consequently, every vertex reaching v_1 until step $\tau - d(\alpha, v_1)$ also reaches α. □

We conclude that in order to show that m reaches all vertices in $[\alpha, m]$, we only have to prove that m reaches v until step $\tau - d(\alpha, v)$, that is, $\text{at}(m, v) \leq \tau - d(\alpha, v)$. In Figure 4.12b, it can be observed that all edges between v and m exist as soon as v reaches m, that is, from step $\text{at}(v, m)$ on. We conclude that latest in step $\text{at}(v, m) - 1$, m begins reaching one vertex in $[v, m]$ every step. We state this in Observation 4.31.

Observation 4.31. *Let $\mathcal{F} = (V, E, \tau)$ be a monotonically growing temporal path forest and let $v_1, v_2 \in V$. Then, latest in step $at(v_1, v_2) - 1$, v_1 begins reaching one vertex in $[v_1, v_2]$ every step.*

By Observation 4.31, we are able to compute an upper bound for the step in which m reaches v depending on $\text{at}(v, m) - 1$. After some computations, we are able to conclude that $\text{at}(m, v) \leq \tau - d(\alpha, v)$. We mentioned in Observation 4.30 that in this way m reaches all vertices in $[\alpha, m]$ and consequently all vertices in $[\alpha, \beta]$.

Lemma 4.13. *Let $\mathcal{F} = (V, E, \tau)$ be a monotonically growing temporal path forest and let Ω be a set of vertices that is reachable from a vertex $v \in V$. Let m be a central vertex of Ω. Then, vertex m reaches all vertices in Ω.*

Proof. Let $\Omega = [\alpha, \beta]$. We can assume that $v \leq m$ because of the following symmetry. If the number of vertices in $[\alpha, \beta]$ is odd, then there is only one central vertex in $[\alpha, \beta]$. Consequently, the case $v \leq m$ is analogous to the case $v \geq m$. If the number of vertices in $[\alpha, \beta]$ is even, then let m_l and m_r be the central vertices of $[\alpha, \beta]$ such that $m_l < m_r$. We can divide the proof into two cases, that is, either $v \leq m_l$ or $v \geq m_r$. For both cases we have to show that both central vertices m_r and m_l reach all vertices in $[\alpha, \beta]$. Because of symmetry, it is enough to prove this for $v \leq m_l$.

We first show that $\text{at}(m, v) \leq \tau - d(\alpha, v)$. By Observation 4.31, latest in step $\text{at}(v, m) - 1$, m begins reaching one vertex in $[v, m]$ every step. At least $d(v, m)$ steps are necessary in order to reach v from m. Consequently, an upper bound for the step in which m reaches v can be computed as follows:

$$\text{at}(m, v) \leq \text{at}(v, m) - 1 + d(v, m). \tag{4.1}$$

In the following, we compute $\text{at}(v, m)$. Since v reaches β and since v has to pass m in order to reach β, we conclude that v reaches m no later than in step $\tau - d(m, \beta)$, so that $\text{at}(v, m) \leq \tau - d(m, \beta)$. Applying this to Inequation (4.1), we get

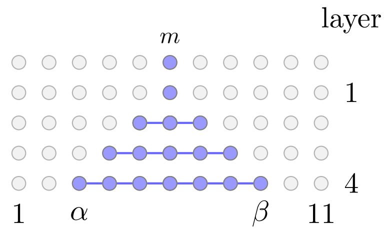
$$\text{at}(m, v) \leq \tau - d(m, \beta) - 1 + d(v, m). \tag{4.2}$$

Since m is a central vertex of $[\alpha, \beta]$, it holds that $d(m, \beta) \geq d(\alpha, m) - 1$. Applying Inequation (4.2), we get

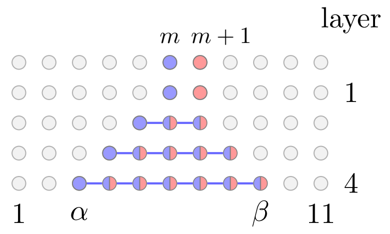
$$\text{at}(m, v) \leq \tau - (d(\alpha, m) - 1) - 1 + d(v, m) = \tau - d(\alpha, m) + d(v, m). \tag{4.3}$$

Since $\alpha \leq v \leq m$, it holds that $d(\alpha, m) - d(v, m) = d(\alpha, v)$. With Inequation (4.3), we conclude that $\text{at}(m, v) \leq \tau - d(\alpha, v)$.

Since m reaches v no later than in step $\tau - d(\alpha, v)$, we conclude by Observation 4.30 that m reaches all vertices in $[\alpha, m]$. Since v reaches the vertices in $[m, \beta]$ and since $v \leq m$, it is clear that also m reaches the vertices in $[m, \beta]$. We conclude that m reaches all vertices in $[\alpha, \beta]$. □



(a) Assuming that m is a central vertex of the reachable vertex set $[\alpha, \beta]$, m reaches all blue colored vertices. The blue colored edges are the edges that must exist by the assumption that m reaches the vertices in $[\alpha, \beta]$.



(b) Illustrates that vertex $m+1$ reaches all vertices in $[\alpha+1, \beta]$. The vertices reachable from $m+1$ are colored in red.

Figure 4.13: Illustration for the proof of Lemma 4.15.

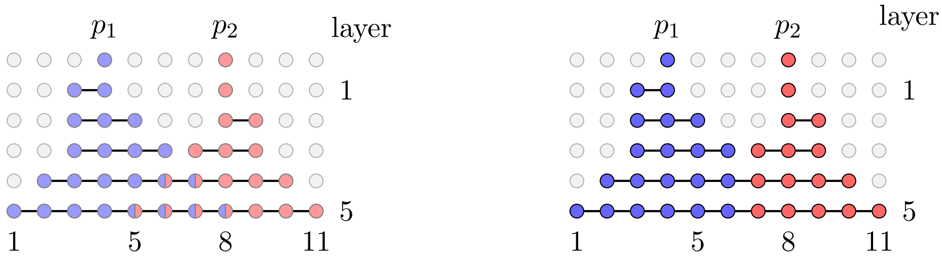
In the following, we show Lemma 4.15. In Lemma 4.15, we assume that $[\alpha, \beta]$ is a reachable vertex set of odd size in a monotonically growing temporal path forest and that m is the central vertex of this vertex set. We state that the vertex next to vertex m, that is, vertex $m+1$, reaches at least the vertices in $[\alpha+1, \beta]$. We illustrate the situation in Figure 4.13a. As in the illustration for the proof of Lemma 4.13, we highlight all edges that must exist by the assumption that m reaches the vertices in $[\alpha, \beta]$. It can be observed in Figure 4.13b that by the existence of these edges, vertex $m+1$ is able to reach all vertices in $[\alpha+1, \beta]$. The proof of Lemma 4.15 is very similar to the proof of Lemma 4.13.

Lemma 4.15. *Let $\mathcal{F} = (V, E, \tau)$ be a monotonically growing temporal path forest. Assume that vertex $m \in V$ reaches the vertices in $[\alpha, \beta]$ and that m is a central vertex of $[\alpha, \beta]$. Assume that the number of vertices in $[\alpha, \beta]$ is odd. Then vertex $m+1$ reaches at least the vertices in $[\alpha+1, \beta]$.*

Proof. Since m reaches all vertices in $[m+1, \beta]$ and since $m < m+1$, clearly also $m+1$ reaches the vertices in $[m+1, \beta]$. Since m reaches α and since edges do not disappear by the monotonicity of the graph, it follows that a vertex reaching m until step $\tau - d(\alpha + 1, m)$ also reaches vertex $\alpha + 1$. In the following, we prove that $m+1$ reaches m until step $\tau - d(\alpha+1, m)$.

Since m reaches β, it follows that m reaches $m+1$ latest in step $\tau - d(m+1, \beta)$. Consequently, edge $\{m, m+1\}$ exists in step $\tau - d(m+1, \beta)$. Thus, also $m+1$ reaches m latest in step $\tau - d(m+1, \beta)$. Since m is the only central vertex of $[\alpha, \beta]$, it holds that $d(m+1, \beta) = d(\alpha+1, m)$. As a result, vertex $m+1$ reaches m until step $\tau - d(\alpha + 1, m)$. We conclude that m reaches $\alpha+1$ and thereby all vertices in $[\alpha+1, m]$. Thus, vertex $m+1$ reaches all vertices in $[\alpha+1, \beta]$. □

The last remaining open lemma from Section 4.2.1 is Lemma 4.7, where we consider strategy profiles with $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $\Omega_1 \not\subseteq \Omega_2$ and $\Omega_2 \not\subseteq \Omega_1$. We state that for all diffusion games on monotonically growing temporal path forests with these strategy profiles, the players color the same number of vertices in $\Omega_1 \cap \Omega_2$. We give an example



(a) Both players reach the vertices in $[5, 8]$, but only Player 1 reaches the vertices in $[1, 4]$ and only Player 2 reaches the vertices in $[9, 11]$.

(b) The players color the same number of vertices in the vertex set that both players reach, that is, each player colors two vertices in $[5, 8]$.

Figure 4.14: Figure 4.14a illustrates an example of the type of strategy profiles considered in Lemma 4.15. Figure 4.14b illustrates the diffusion game with that strategy profile.

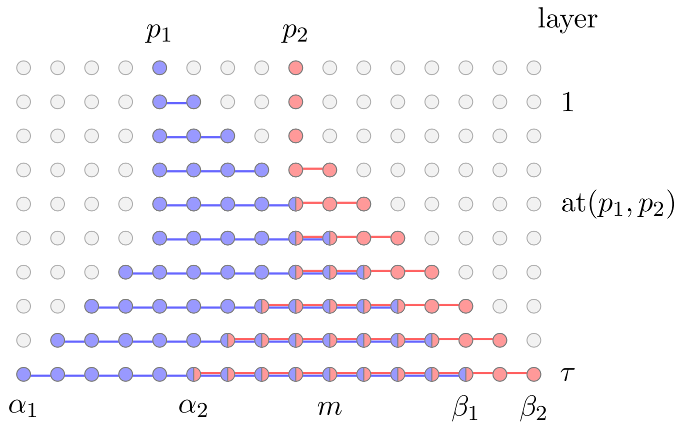
of these strategy profiles and of a reduced diffusion game with such a strategy profile in Figure 4.14.

In order to show Lemma 4.7, we prove two lemmas that are associated with the central vertex (vertices) of $\Omega_1 \cap \Omega_2$. In Lemma 4.35, we show that if the number of vertices in $\Omega_1 \cap \Omega_2$ is even, then each player reaches the central vertex of $\Omega_1 \cap \Omega_2$ that is closer to her before the other player. For the odd case, we prove that the central vertex of $\Omega_1 \cap \Omega_2$ is reached from both players at the same time. In Lemma 4.33, we show that in any case the central vertex (vertices) of $\Omega_1 \cap \Omega_2$ are in between the positions of the players. By Lemma 4.33 and Lemma 4.35, we can conclude that the players color the same number of vertices in $\Omega_1 \cap \Omega_2$.

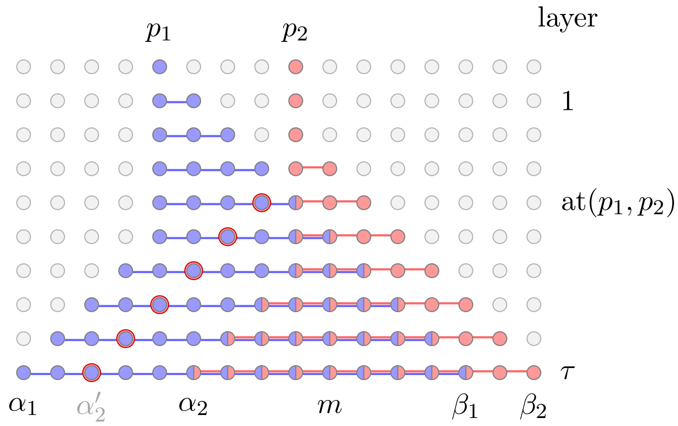
We start with the proof of Lemma 4.33. For illustration, we assume in the following that the number of vertices in $\Omega_1 \cap \Omega_2$ is odd. In Lemma 4.33, we state that central vertex m of $\Omega_1 \cap \Omega_2$ lies in between the positions of the players. In order to give an intuition for that, we construct an contradictory example where m does not lie in between the positions of the players.

The example is illustrated in Figure 4.15 and constructed as follows. We assume that Player 1 reaches only the vertices in $[\alpha_1, \beta_1]$, colored in red, and the Player 2 reaches only the vertices in $[\alpha_2, \beta_2]$, colored in blue. We assume that Player 1 plays to the left of Player 2, so that the vertex set that both players reach is $[\alpha_2, \beta_1]$ (colored in both colors). For the sake of contradiction, we choose p_1 and p_2, such that not only p_1 but also p_2 is to the left of central vertex m of $[\alpha_2, \beta_1]$. By the assumption that p_1 and p_2 reach the vertices in $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$, certain edges exist in the temporal graph. We mark these edges in blue and red respectively. It can be observed, that by the blue edges, p_2 also reaches some of the blue colored vertices which contradicts that p_2 reaches only the red vertices. We illustrate this in Figure 4.15b.

In the following, we outline the idea of the proof. We already mentioned in Observation 4.31 that latest in step $\text{at}(p_1, p_2) - 1$, p_2 begins reaching one vertex in $[p_1, p_2]$ every step. By the condition that p_1 reaches at least one vertex which p_2 does not reach, we can show that also after p_2 has reached p_1, p_2 continues reaching one vertex to the left every step. We show this in Observation 4.32.



(a) We assume that p_1 reaches only the blue and that p_2 reaches only the red vertices. For the sake of contradiction, we chose p_1 and p_2 such that central vertex m of the vertices that are red and blue is to not in between the positions of the players. By the reachability assumption from p_1 and p_2, the blue and red edges must exist in the temporal graph.



(b) This figure illustrates the contradiction in the example from Figure 4.15a. By the blue edges, p_2 also reaches some of the blue vertices, i.e., all vertices until α_2'. This contradicts that p_2 reaches only the red vertices.

Figure 4.15: Contradictory example for the proof of Lemma 4.33. We illustrate the assumptions that we make in Figure 4.15a and the corresponding contradiction in Figure 4.15b.

Observation 4.32. *Let $\mathcal{F} = (V, E, \tau)$ be a monotonically growing temporal path forest and let $v_1, v_2 \in V$ with $v_1 < v_2$. Assume that v_1 reaches a vertex $\alpha_1 < v_1$ which v_2 does not reach. Then, latest in step $at(v_2, v_1)$, v_2 begins reaching one vertex to the left of v_1 every step.*

Proof. As v_2 does not reach α_1 but as v_1 reaches α_1, we conclude that v_1 reaches every vertex in $[\alpha_1, v_1]$ before v_2 reaches them. Since additionally, edges do not disappear over time, we conclude that the coloring of a vertex in $[\alpha_1, v_1 - 1]$ by v_2 can never be stopped by non-existing edges. Consequently, when v_2 reaches v_1, v_2 continues reaching one vertex to the left every step. □

By Observation 4.31 and Observation 4.32, we conclude that latest in step $\text{at}(p_1, p_2) - 1$, p_2 begins reaching one vertex to the left every step. Thereby, we can prove that p_2 reaches more vertices to the left of p_2 than p_1 reaches to the right of p_2. Consequently, there are more vertices between α_2 and p_2 than between p_2 and β_1, which implies that p_2 is to the right of the central vertex of $[\alpha_2, \beta_1]$. The same argumentation can be applied to position p_1, that is, p_1 must be to the left of the central vertex of $[\alpha_2, \beta_1]$. As a result, the central vertex of $[\alpha_2, \beta_1]$ must be in between positions p_1 and p_2. We prove this in the following lemma:

Lemma 4.33. *Let $\mathcal{F} = (V, E, \tau)$ be a monotonically growing temporal path forest. Let p_1, p_2 with $p_1 < p_2$ be two positions in $\mathcal{F}$, reaching the vertex sets Ω_1 and Ω_2. Assume that $\Omega_1 \cap \Omega_2 \neq \emptyset$, $\Omega_1 \not\subseteq \Omega_2$ and $\Omega_2 \not\subseteq \Omega_1$.*

1. *If the number of vertices in $\Omega_1 \cap \Omega_2$ is odd and if m is the central vertex of $\Omega_1 \cap \Omega_2$, then $p_1 < m < p_2$.*

2. *If the number of vertices in $\Omega_1 \cap \Omega_2$ is even and if m_l, m_r are the central vertices of $\Omega_1 \cap \Omega_2$ with $m_l < m_r$, then $p_1 \leq m_l < m_r \leq p_2$.*

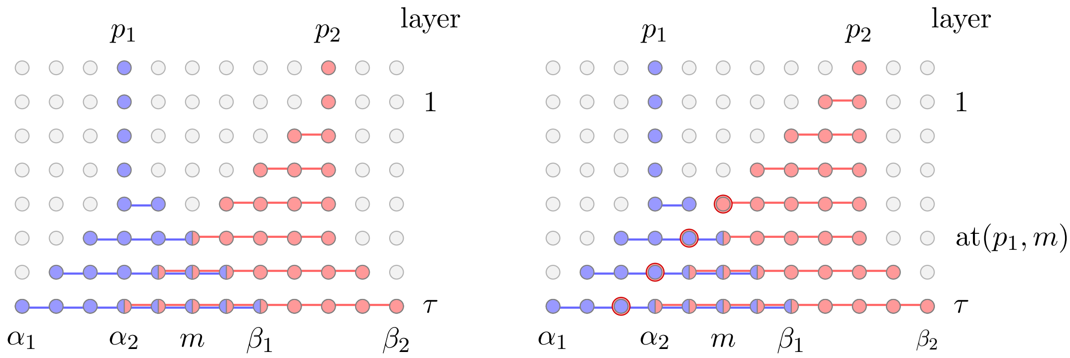
Proof. Assume that $\Omega_1 = [\alpha_1, \beta_1]$ and $\Omega_2 = [\alpha_2, \beta_2]$. Since $p_1 < p_2$, it follows that $\Omega_1 \cap \Omega_2 = [\alpha_2, \beta_1]$. Because of symmetry, it is enough to show that it holds $m < p_2$ if the number of vertices in $\Omega_1 \cap \Omega_2$ is odd and that it holds $m_r \leq p_2$ if the number of vertices in $\Omega_1 \cap \Omega_2$ is even. If p_1 does not reach p_2, that is $\beta_1 < p_2$, then it is clear that also $m < p_2$ and $m_r < p_2$ hold. Thus, we assume in the following that p_1 reaches p_2, that is $p_2 \leq \beta_1$.
By Observation 4.31 and Observation 4.32, latest in step $\text{at}(p_1, p_2) - 1$, p_2 begins reaching one vertex to the left every step. Since p_1 passes p_2 in order to reach β_1, at least $d(p_2, \beta_1)$ steps follow step $\text{at}(p_1, p_2)$. As a result, p_2 reaches at least $d(p_2, \beta_1) + 1$ vertices to the left. The leftmost reachable vertex from p_2 is α_2. We conclude that $d(\alpha_2, p_2) \geq d(p_2, \beta_1) + 1$, that is, $d(\alpha_2, p_2) > d(p_2, \beta_1)$. We consider two cases depending on the number of vertices in $[\alpha_2, \beta_1]$.

1. First, assume that the number of vertices in $[\alpha_2, \beta_1]$ is odd. For the sake of contradiction, assume that $p_2 \leq m$. Since m is the central vertex of $[\alpha_2, \beta_1]$ it follows that $d(\alpha_2, p_2) \leq d(p_2, \beta_1)$, which contradicts that $d(\alpha_2, p_2) > d(p_2, \beta_1)$. Consequently, it must hold that $p_2 > m$.

2. Otherwise, the number of vertices in $[\alpha_2, \beta_1]$ is even. For the sake of contradiction assume that $p_2 \leq m_l$. Since m_l is the central vertex of $[\alpha_2, \beta_1]$ that is closer to α_2, it follows that $d(\alpha_2, p_2) < d(p_2, \beta_1)$. This contradicts that $d(\alpha_2, p_2) > d(p_2, \beta_1)$. Consequently, it must hold that $p_2 > m_l$.

□

In the following, we prove that if the number of vertices in $\Omega_1 \cap \Omega_2$ is even, then each player reaches the central vertex of $\Omega_1 \cap \Omega_2$ that is closer to her before the other player. For the odd case, we prove that the central vertex of $\Omega_1 \cap \Omega_2$ is reached from both players



(a) The players reach the central vertex of $\Omega_1 \cap \Omega_2$ at the same time.

(b) If p_2 would reach m before p_1, then p_2 would additionally reach $\alpha_2 - 1$.

Figure 4.16: Illustrations for the proof of Lemma 4.35 for the case that the number of vertices in $\Omega_1 \cap \Omega_2$ is odd.

at the same time. Both cases can be shown in a similar way. In the following, we give an overview of the proof for the case that the number of vertices in $\Omega_1 \cap \Omega_2$ is odd.

In Figure 4.16a, we give an example of a diffusion game with the considered strategy profiles, that is, the players play on positions such that $\Omega_1 \cap \Omega_2 \neq \emptyset$ and that $\Omega_1 \not\subseteq \Omega_2$ and $\Omega_2 \not\subseteq \Omega_1$. It can be observed that central vertex m of $\Omega_1 \cap \Omega_2$ is in between the positions of the players (as we have shown in Lemma 4.33) and that the players reach vertex m in the same step. We prove that the latter generally holds by contradiction. For illustration, we assume that p_2 reaches m before p_1. It can be observed in Figure 4.16b that thereby, at the latest in step $at(p_1, m) - 1$, p_2 begins reaching one vertex to the left of m every step. We show this in Lemma 4.34.

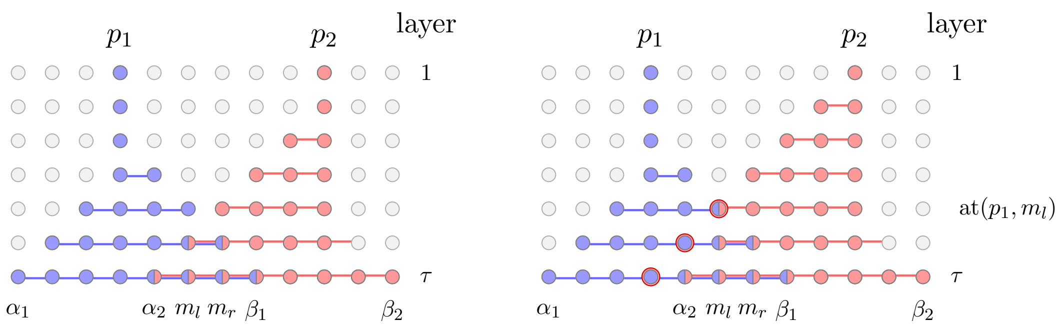
Lemma 4.34. *Let $\mathcal{F} = (V, E, \tau)$ be a monotonically growing temporal path forest. Let $v_1, m, v_2 \in V$ with $v_1 \leq m \leq v_2$. Assume that v_1 reaches a vertex $\alpha < v_1$ which v_2 does not reach.*

(i) Assume that v_2 reaches m before v_1. Then, latest in step $at(v_1, m) - 1$, v_2 begins reaching one vertex to the left of m every step.

(ii) Assume that v_2 reaches m in the same step as v_1. Then, latest in step $at(v_1, m)$, v_2 begins reaching one vertex to the left of m every step.

Proof. By Observation 4.31, latest in step $\text{at}(v_1, m) - 1$, m begins reaching a vertex in $[v_1, m]$ every step. Assume that v_2 reaches m before v_1, i.e., latest in step $\text{at}(v_1, m) - 1$. Then, latest in step $\text{at}(v_1, m) - 1$, v_2 begins reaching a vertex in $[v_1, m]$ every step. By Observation 4.32, as soon as v_2 has reached v_1, v_2 continues reaching one vertex to the left every step. We conclude that latest in step $\text{at}(v_1, m) - 1$, v_2 begins reaching one vertex to the left of m every step. The proof for Case (ii) is analogously. □

By Lemma 4.34, we can show that if p_2 reaches m before p_1, then p_2 reaches more vertices to the left of m than p_1 reaches to the right of m. However, this contradicts



(a) Each player reaches the central vertex of $\Omega_1 \cap \Omega_2$ that is closer to her before the other player.

(b) If p_2 would reach m_l in the same step as p_1, then p_2 would additionally reach $\alpha_2 - 1$.

Figure 4.17: Illustrations for the proof of Lemma 4.35 for the case that the number of vertices in $\Omega_1 \cap \Omega_2$ is even.

that m is the only central vertex of the vertex set that both players reach, so that the players must instead reach m in the same step. We show this in Lemma 4.35. Note that the proof for the even case is very similar. We illustrate the proof for the even case in Figure 4.17.

Lemma 4.35. *Let $\mathcal{F} = (V, E, \tau)$ be a monotonically growing temporal path forest. Let p_1, p_2 with $p_1 < p_2$ be two positions in $\mathcal{F}$, reaching the vertex sets Ω_1 and Ω_2. Assume that $\Omega_1 \cap \Omega_2 \neq \emptyset$, $\Omega_1 \not\subseteq \Omega_2$ and $\Omega_2 \not\subseteq \Omega_1$.*

(i) If the number of vertices in $\Omega_1 \cap \Omega_2$ is odd, then central vertex m of $\Omega_1 \cap \Omega_2$ is reached from p_1 and p_2 in the same step.

(ii) If the number of vertices in $\Omega_1 \cap \Omega_2$ is even, then let m_l, m_r be the central vertices of $\Omega_1 \cap \Omega_2$ such that $m_l < m_r$. Then, m_l is reached from p_1 before it is reached from p_2, and m_r is reached from p_2 before it is reached from p_1.

Proof. Assume that $\Omega_1 = [\alpha_1, \beta_1]$ and $\Omega_2 = [\alpha_2, \beta_2]$. Since we assume that $p_1 < p_2$, it follows that $\Omega_1 \cap \Omega_2 = [\alpha_2, \beta_1]$. We consider two cases.

First, assume that the number of vertices in $\Omega_1 \cap \Omega_2$ is odd. By Lemma 4.33, it holds that $p_1 < m < p_2$. Without loss of generality and for the sake of contradiction, assume that p_2 reaches m before p_1. By Lemma 4.34, latest in step $\mathrm{at}(p_1, m) - 1$, p_2 begins reaching one vertex to the left of m every step. Since p_1 passes m in order to reach β_1, at least $d(m, \beta_1)$ steps follow step $\mathrm{at}(p_1, m)$. Consequently, p_2 reaches at least $d(m, \beta_1) + 1$ vertices to the left of m. The leftmost reachable vertex from p_2 is α_2. It follows that $d(\alpha_2, m) \geq d(m, \beta_2) + 1$. This contradicts that m is the only central vertex of $[\alpha_2, \beta_1]$. Thereby, p_2 cannot reach vertex m before p_1. We conclude that p_1 and p_2 reach vertex m in the same step.

Second, assume that the number of vertices in $\Omega_1 \cap \Omega_2$ is even. By Lemma 4.33, it holds that $p_1 \leq m_l < m_r \leq p_2$. Without loss of generality and for the sake of contradiction, assume that p_1 does not reach m_l before p_2, that is, p_2 reaches m_l in the same step as p_1 or before. By Lemma 4.34, latest in step at(p_1, m_l), p_2 begins reaching one vertex to the left of m_l every step. Since p_1 passes m_l in order to reach β_1, at least $d(m_l, \beta_1)$ steps follow the step in which p_1 reaches m_l. As a result, p_2 reaches at least $d(m_l, \beta_1)$ vertices to the left of m_l. The leftmost reachable vertex from p_2 is α_2. It follows that $d(\alpha_2, m_l) \geq d(m_l, \beta_2)$. This contradicts that m_l is the central vertex of $[\alpha_2, \beta_1]$ that is closer to α_2. Thereby, p_2 cannot reach vertex m_l before or in the same step as p_1. We conclude that p_1 reaches vertex m_l before p_2 and symmetrically that p_2 reaches vertex m_r before p_1. □

By Lemma 4.33 and Lemma 4.35, it is easy to conclude that the players color the same number of vertices in $\Omega_1 \cap \Omega_2$.

Lemma 4.7. *Let $(\mathcal{F}, 2)$ be a reduced diffusion game on a monotonically growing temporal path forest $\mathcal{F}$. Let (p_1, p_2) be a strategy profile and let Ω_1 and Ω_2 be the reachable vertex sets from p_1 and p_2. Assume that $\Omega_1 \cap \Omega_2 \neq \emptyset$ and that $\Omega_1 \not\subseteq \Omega_2$ and $\Omega_2 \not\subseteq \Omega_1$. Then, the players color the same number of vertices in $\Omega_1 \cap \Omega_2$.*

Proof. Without loss of generality, assume that $p_1 < p_2$. Let $\Omega_1 \cap \Omega_2 = [\alpha_2, \beta_1]$. We consider two cases.

First, assume that the number of vertices in $\Omega_1 \cap \Omega_2$ is odd. Let m be the central vertex of $\Omega_1 \cap \Omega_2$. By Lemma 4.35, the players reach vertex m at the same time. Since by Lemma 4.33, it holds that $p_1 < m < p_2$, it follows that Player 1 colors the vertices in $[\alpha_2, m-1]$ and Player 1 colors the vertices in $[m+1, \beta_1]$. Since m is the central vertex of $[\alpha_2, \beta_1]$, the players color the same number of vertices in $[\alpha_2, \beta_1]$.

Otherwise, the number of vertices in $[\alpha_2, \beta_1]$ is even. Let m_l and m_r be the central vertices of $\Omega_1 \cap \Omega_2$ such that $m_l < m_r$. By Lemma 4.35, Player 1 reaches m_l before Player 2 and Player 2 reaches m_r before Player 1. Since by Lemma 4.33, it holds that $p_1 \leq m_l < m_r \leq p_2$, it follows that Player 1 colors the vertices in $[\alpha_2, m_l]$ and that Player 2 colors the vertices in $[m_r, \beta_1]$. Consequently, the players color the same number of vertices in $[\alpha_2, \beta_1]$. □

Furthermore, we can conclude by Lemma 4.33 and Lemma 4.35 that a vertex between p_1 and p_2 is colored gray, if and only if the distance between α_2 and β_1 is even. We state this in Corollary 4.36.

Corollary 4.36. *Let $(\mathcal{F}, 2)$ be a reduced diffusion game on a monotonically growing temporal path forest $\mathcal{F}$. Let p_1, p_2 with $p_1 < p_2$ be two positions in $\mathcal{F}$, reaching the vertex sets $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$. Assume that $[\alpha_1, \beta_1] \cap [\alpha_2, \beta_2] \neq \emptyset$, $[\alpha_1, \beta_1] \not\subseteq [\alpha_2, \beta_2]$ and $[\alpha_2, \beta_2] \not\subseteq [\alpha_1, \beta_1]$. Then, a vertex in $[p_1, p_2]$ is colored gray if and only if $d(\alpha_2, \beta_1)$ is even.*

For the specific case that the players reach the same number of vertices, that they play on a vertex that is central in its reachable vertex set and that they reach an odd number of vertices, we can show an even more general statement. In this case, a vertex between the positions of the players is colored gray if and only if the distance of positions of the players is even. This is our last remaining open lemma of this section, i.e., Lemma 4.23 from Section 4.2.2.

Lemma 4.23. *Let $(\mathcal{F}, 2)$ be a reduced diffusion game on a monotonically growing temporal path forest $\mathcal{F}$. Let p_1, p_2 with $p_1 < p_2$ be two optimal positions in $\mathcal{F}$, reaching the vertex sets Ω_1 and Ω_2. Assume that the number of vertices in Ω_1 and Ω_2 is odd. Assume that $\Omega_1 \cap \Omega_2 \neq \emptyset$. Then, a vertex in $[p_1, p_2]$ is colored gray if and only if $d(p_1, p_2)$ is even.*

Proof. Let L^* be the maximum number of vertices that are reachable from any vertex in $\mathcal{F}$. Since p_1 and p_2 are both optimal positions, it holds that $|\Omega_1| = |\Omega_2| = L^*$. Consequently, $\Omega_1 \not\subset \Omega_2$ and $\Omega_2 \not\subset \Omega_1$. Since L^* is odd and since there is only one central vertex of a vertex set of odd size, it follows that $\Omega_1 \neq \Omega_2$. We conclude that $\Omega_1 \cap \Omega_2 \neq \emptyset$, $\Omega_1 \not\subseteq \Omega_2$ and $\Omega_2 \not\subseteq \Omega_1$.

Assume that $\Omega_1 = [\alpha_1, \beta_1]$ and $\Omega_2 = [\alpha_2, \beta_2]$. Since $\Omega_1 \cap \Omega_2 \neq \emptyset$, we conclude that $d(p_1, p_2) = d(p_1, \beta_1) + d(\alpha_2, p_2) - d(\alpha_2, \beta_1)$. Since p_1 and p_2 are the only central vertices of Ω_1 and Ω_2 and since $|\Omega_1| = |\Omega_2|$, we conclude that $d(p_1, \beta_1) = d(\alpha_2, p_2)$. Consequently, $d(p_1, \beta_1) + d(\alpha_2, p_2)$ is even. Hence, $d(p_1, p_2)$ is even if and only if $d(\alpha_2, \beta_1)$ is even. By Corollary 4.36, a vertex in $[p_1, p_2]$ is colored gray if and only if $d(\alpha_2, \beta_1)$ is even. We conclude that a vertex in $[p_1, p_2]$ is colored gray if and only if $d(p_1, p_2)$ is even. □

Chapter 5

Conclusion

In this thesis, we introduced a game-theoretic model that applies competitive diffusion games, introduced by Alon et al. [Alo+10], to temporal graphs. We studied the existence of Nash equilibria in the game with two players on various *temporal* graph classes. We observed that even for simple temporal graph classes, i.e., temporal paths and temporal cycles, the existence of a Nash equilibrium cannot be guaranteed. This indicates that competitive diffusion games on temporal graphs are of a rather complex nature. Nevertheless, our study provides many interesting conclusions.

One reason for the non-existence of Nash equilibria in diffusion games on temporal graphs seems to be edges that exist in an early layer and then disappear. Players often prefer to play in the immediate surrounding of such a disappearing edge, in order to not loose the ability to color some part of the temporal graph. However, if the edge is not located somewhere around the center of the temporal graph, then the player playing close to it does not reach an equal number of vertices to one as to the other side, which makes the player vulnerable to loosing many vertices to the other player. This results in a situation where the players continue changing their positions such that no Nash equilibrium exists. We observed this behavior in diffusion games on two examples of temporal paths, presented in Section 3.1 and Section 3.3.

For temporal graphs where edges are not allowed to disappear over time, i.e., monotonically growing temporal graphs, we showed that a Nash equilibrium always exists if the underlying graph is a path or a cycle. Regarding future work, it would be interesting to see weather this also applies to other monotonically growing temporal graph classes, i.e., monotonically growing temporal Cartesian grids. For the analysis of further temporal graph classes, our results regarding an “optimal strategy” on a monotonically growing temporal cycle could be of interest. We observed that an “optimal” position in a monotonically growing temporal cycle includes reaching as many vertices as possible before the graph becomes a complete cycle. Additionally, in order to ensure the stability of the resulting strategy profile, the positions should reach a well-balanced number of vertices to one as to the other side. Furthermore, in Section 4.2.1, we observed a helpful connection between our game and a game where the utility of the players corresponds to the ratio of pay-offs, i.e., the difference in the number of vertices the players color.

In Section 3.2, we showed that for temporal paths, the existence of a Nash equilibrium can also be guaranteed if every edge existing in some layer also exists in the last layer.

In particular, we showed that on superset temporal paths, every Nash equilibrium is a strategy profile where both players play in the middle of the temporal path. We further outlined that, after weakening conditions of "superset" property, the existence a Nash equilibrium can no longer be guaranteed.

The proof for the existence of Nash equilibria in superset temporal paths is based on the two fundamental arguments that in a superset temporal path every vertex is reachable from every position and that in a temporal path a player cannot "pass" a vertex colored by the other player. We conjecture that also in other superset temporal graph classes where players cannot "pass" a vertex colored by the other player a Nash equilibrium always exists, as, for instance, in superset temporal trees. It would be interesting to analyze this in future work.

We analyzed competitive diffusion games on temporal graphs with only two players although we defined the game for any number of players. It would be interesting to investigate the existence of Nash equilibria in the game with more than two players. However, our preliminary studies indicate that for any number of players larger than two, a Nash equilibrium cannot generally be guaranteed if the underlying graph is a path or a cycle, even if edges are not allowed to disappear over time.

Apart from further analyzing our model, variations of it could be considered. In Section 4.2.1, we considered a diffusion game where the utility of a player is not defined by the number of vertices the player colors but by the ratio of pay-offs, that is, the difference in the number of vertices the players color. Diffusion games with this adapted utility function could generally be of interest in, for instance, presidential election campaigns. In such voting mechanisms, non-voters are normally disregarded. Consequently, a candidates strategy is not to gain many votes in general, but rather to gain a large number of votes compared to all other candidates. Notably this modified diffusion game is a zero-sum game, which makes it intuitively simpler to analyze (Maschler, Solan, and Zamir [MSZ13]). Based on a first consideration, it seems that for all temporal graph classes we analyzed, the found Nash equilibria are also Nash equilibria in the corresponding zero-sum game. However, for diffusion games where the number of gray vertices is typically large, completely different results could be obtained.

In the original version of competitive diffusion games, Alon et al. [Alo+10] assumed that the propagation process continues until the coloring of the vertices does not change between consecutive steps. Applying this model to temporal graphs, we equivalently assumed that the propagation process continues (with the last layer of the temporal graph) until no more vertices are colored. However, another natural model would assume that the propagation process stops as soon as the last layer has been colored for the first time. Thereby, the number of steps the diffusion game propagates is already defined at the beginning of the game by the given temporal graph. This could be interesting for applications where it is desired to have maximal influence at a certain time in the future. Regarding the existence of Nash equilibria in this modified diffusion game, we observe that the positive impact of the "superset" property, that is, that every vertex is reachable from every position, no longer holds. Thus, it is clear that a Nash equilibrium no longer exists on every superset temporal path. Furthermore, we observe that for temporal graphs with a large number of vertices compared to the number of layers, a player does not have a chance to color all vertices of the temporal graph anyway. This results in less competition between the players, so that a Nash equilibrium might be found very easily. Moreover, for temporal graphs that have a low number of vertices compared to the number of layers, this modified diffusion game is likely to give similar results as our model.

Another promising variation of our model could allow the players to choose multiple vertices at the beginning of the game, or even to let the players choose one additional vertex in every time step. These are reasonable assumptions for modeling the influence of a company in a social network, since companies rather continuously influence various members of a social network in different time periods, instead of only influencing once or only one member.

Finally, all mentioned variations of our model (a different propagation process, a different utility or a different number of vertices a player can color) yield also relevant variations of competitive diffusion games on non-temporal graphs.

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