Technische Universität Berlin

Fakultät IV - Elektrotechnik und Informatik
Institut für Softwaretechnik und Theoretische Informatik
Fachgebiet Algorithmik und Komplexitätstheorie

Bachelorarbeit
im Studiengang Informatik

# On Nash Equilibria for a Competitive Diffusion Game on Hypercubes and Grids

Eingereicht von Manuela Hopp
Betreuer: Prof. Dr. Rolf Niedermeier
Vincent Froese

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## Eigenständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und eigenhändig, sowie ohne unerlaubte fremde Hilfe und ausschließlich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe.

Berlin, den 13.12.2015

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**Abstract**

This bachelor's thesis investigates the existence of Nash equilibria in competitive diffusion games on graphs. A diffusion game is a game where the players initially choose vertices of an undirected graph from which the information then spreads across the edges. The objective of every player is to maximize the number of vertices infected by her. Diffusion games have been studied on several graph classes. For diffusion games on hypercubes, the existence of a Nash equilibrium for four players is proven. Finally, we look at empirical studies that suggest conjectures for diffusion games for three players on hypercubes and for four players on grids.



**Zusammenfassung**

Diese Bachelorarbeit untersucht die Existenz von Nash-Gleichgewichten in Diffusionsspielen für mehrere Spieler auf Graphen. Ein Diffusionsspiel ist ein Spiel, in dem die Spieler zu Beginn Knoten eines ungerichteten Graphen auswählen, von denen sich die Information über die Kanten ausbreitet. Das Ziel jeder Spielerin ist es, die Anzahl der von ihr infizierten Knoten zu maximieren. Diffusionsspiele wurden bereits auf einigen Graphklassen untersucht. Für Diffusionsspiele auf Hyperwürfeln wird die Existenz eines Nash-Gleichgewichtes für vier Spieler bewiesen. Schließlich betrachten wir empirische Untersuchungen, die Vemutungen für Diffusionsspiele auf Hyperwürfeln für drei Spieler und auf Grids für vier Spieler nahelegen.

# Contents

# 1 Introduction

Social networks play an important role in human interactions and as a medium for spreading information. A social network is usually modelled as a graph where the nodes are individuals, groups or organizations. Two nodes are connected by an edge if there is some kind of social relationship between them. Information usually originates in one node or a small subset of nodes and then spreads through the network. This can also be exploited by viral marketing campaigns where a small subset of individuals is targeted in the hope that they spread the information about a product. The challenge is then to select nodes that set in motion a diffusion process that results in a large number of users affected. In such a setting there is typically more than one party competing for the same set of resources. Alon et al. [2] propose a game-theoretic model for the propagation of information in social networks. In this model, the network is represented by an undirected graph with the users as nodes. The players are the companies that wish to advertise a certain product. Initially, they choose a subset of nodes in the network which then propagate the information to their neighbors. This sets in motion a diffusion process where a user adopts the product of a company if some of its neighbors in the network have done so too. If the neighbors of a node have adopted different products, then that node is undecided and adopts no product at all and is hence removed from the game. If a user has adopted one product, then she will not adopt another later. The fact that nodes can be removed from the game means that paths between a player's initial position and a node may be "blocked", that is a node on a path becomes a standoff and therefore no longer participates in the diffusion process and therefore a node does not necessarily adopt the product of the player that has chosen the closest node. The question arises whether there are stable states, that is whether the players can choose nodes such that none of the players can gain more by changing her decision. Whether such states, called Nash-equilibria, exist obviously depends on the number of players and the structure of the network.

This bachelor's thesis looks at diffusion games on hypercubes and grids. We show that there is always a Nash equilibrium for four players on a hypercube and consider diffusion games for three players on hypercubes and for four players on grids.

## 1.1 Related work

The model of competitive diffusion games was introduced by Alon et al. [2] who also claimed that there is always a Nash equilibrium on graphs with diameter at most two. However, Takehara et al. [10] gave a counterexample of a graph with nine vertices and diameter two that does not admit a Nash equilibrium for two players. Roshanbin [8] showed that for two players, there is always a Nash equilibrium on paths, cycles and Cartesian grids. They also discussed Nash equilibria on unicyclic graphs. Bulteau et al. [3] proved that on sufficiently large grids, there is no Nash equilibrium for three players and that there is always a Nash equilibrium on cycles for any number of players and on paths except for three players on paths of length at least six. Small and Mason [9] showed that there is always a Nash equilibrium on trees for two players but not necessarily for more than two players. Etesami and Başar [6] proved that there are Nash equilibria for two players on hypercubes and lattices and that, in general, the decision whether there is a Nash equilibrium for two or more players is NP-hard. They also generalized the model by allowing players to choose more than one node initially. Ito et al. [7] studied diffusion games on weighed graphs. Related models are Voronoi games where a player gets all vertices that she is closest to and those vertices to which several players have the same distance are shared (see Ahn et al. [1] and Dürr and Thang [5]) and the wave propagation model proposed by Carnes et al. [4] where a node randomly chooses one of its neighbors and mimics her choice.

## 1.2 Organization

After an introductory example we start by giving the smallest graph on which there exists no Nash equilibrium for two players in Section 2. Section 3 deals with diffusion games on hypercube graphs. The main result is that there is a Nash equilibrium for four players on $d$-dimensional hypercubes (Theorem 1). We then consider diffusion games for three players on hypercubes and conjecture that there is a Nash equilibrium as well. Finally, we look at diffusion games for four players on grid graphs.

## 1.3 Preliminaries

An undirected graph is an ordered pair $G = (V, E)$ where $V$ is a set of vertices and $E \subseteq \{\{u, v\} \mid u \neq v \in V\}$ is a set of edges. We use the terms vertex and node interchangeably. The distance $\Delta(u, v)$ of two vertices $u, v \in V$ is the length of the shortest path between $u$ and $v$. A diffusion game $\Gamma = (G, k)$ is defined by an undirected graph $G = (V, E)$ and a number $k$ of players. Initially, each player $i$ chooses a vertex $v_i$ which is then colored in color $i$. If more than one player chooses the same vertex $v$, then $v$ is removed from the game. A *strategy profile* is a vector $(v_1, ..., v_k) \in V^k = V \times ... \times V$ where $v_i$ is the vertex that player $i$ has chosen at time $t = 0$.

If after time step $t$, an uncolored vertex $v$ has at least one neighbor colored in color $i$ but no neighbors of any other color, then $v$ is colored in color $i$ in time step $t+1$. If $v$ has neighbors of different colors, then $v$ is called a *standoff* and removed from the game. The diffusion process ends when all remaining uncolored vertices are unreachable, that is there is no path from any colored vertex to the remaining uncolored vertices. The *payoff* $U_i(v_1, ..., v_k)$ of player $i$ is the number of vertices with color $i$ by the end of the diffusion process. A strategy profile $(v_1, ..., v_k)$ is a *Nash equilibrium* if no player $i$ can benefit from unilaterally changing her strategy, that is $U_i(v_1, ..., v_{i-1}, v', v_{i+1}, ..., v_k) \leq U_i(v_1, ...v_k)$ holds for all players $i \in \{1, ..., k\}$ and for all vertices $v' \in V$.

## 1.4 Introductory example

Consider a diffusion game for two players on the graph in Figure 1.

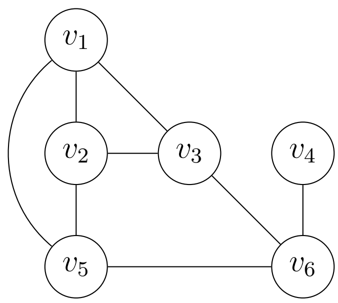
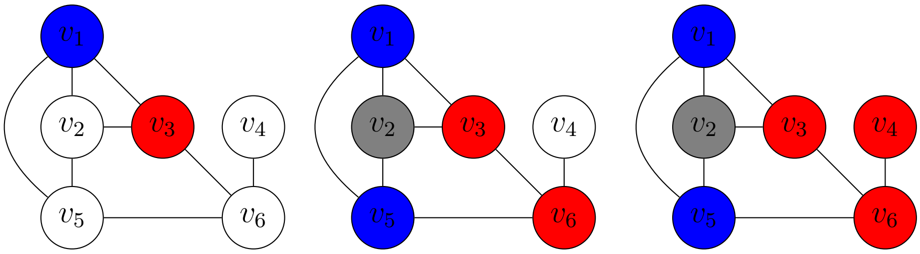



Figure 1: A graph on six vertices.

Figure 2 shows the diffusion process for the strategy profile $(v_1, v_3)$. At time step $t = 0$, the two players choose their initial vertices $v_1$ and $v_2$ which are then colored in their respective colors. At time $t = 1$, both players color the neighbors of their initially chosen vertices with $v_2$ becoming a standoff since it is a neighbor of $v_1$ and $v_3$ which have different colors. At time $t = 2$, the diffusion process ends because there are no more uncolored vertices left. Player 1's payoff is then $U_1(v_3, v_1) = 3$ and player 2's payoff is $U_2(v_3, v_1) = 2$. This strategy profile is not a Nash equilibrium since player 2 can improve by moving to $v_6$ where her payoff would be 3 instead of only 2 as before.
Figure 3 shows the payoff matrix for the two players. Each player's best responses are marked in their respective color. If for example the Player 1 (blue) has chosen vertex $v_1$, then the best response of Player 2 (red) is the maximum of the second value in the first row, in this case $v_3$. Nash equilibria are those strategy profiles where the players' positions are best responses to




(a) $t = 0$ (b) $t = 1$ (c) $t = 2$

Figure 2: The diffusion process on the graph from Figure 1 with the strategy profile $(v_3, v_1)$.

| Player 1 / Player 2 | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|---|---|---|---|---|---|---|
| $v_1$ | 0/0 | 1/1 | 2/3 | 4/2 | 2/3 | 2/2 |
| $v_2$ | 1/1 | 0/0 | 2/3 | 4/2 | 3/2 | 2/2 |
| $v_3$ | 3/2 | 3/2 | 0/0 | 4/1 | 1/1 | **3/3** |
| $v_4$ | 2/4 | 2/4 | 1/4 | 0/0 | 1/4 | 1/5 |
| $v_5$ | 3/2 | 2/3 | 1/1 | 4/1 | 0/0 | **3/3** |
| $v_6$ | 2/2 | 2/2 | **3/3** | 5/1 | **3/3** | 0/0 |

Figure 3: The two players' payoffs. The maximum values for each player are marked in their respective color. Nash equilibria are set bold.

each other, that is where none of the two players can improve by moving to some other vertex.

# 2 The smallest graph without a Nash equilibrium for two players

Using computer simulations, Bulteau et al. [3] found that there is always a Nash equilibrium for diffusion games for two players on graphs with at most seven vertices and that there is no Nash equilibrium in the game on the graph in Figure 4 with eight vertices. The following proof shows that there is indeed no Nash equilibrium in that game.

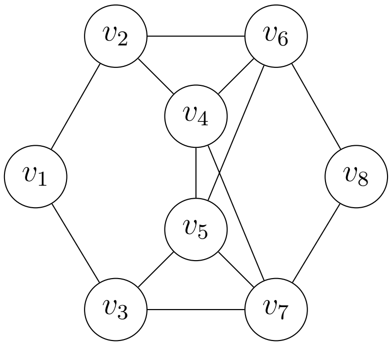



Figure 4: A graph on eight vertices for which there is no Nash-equilibrium in a diffusion game for two players.

**Proposition 1.** *There is no Nash equilibrium in a diffusion game for two players on the graph in Figure 4.*

*Proof.* Figure 5 shows the payoff matrix for the game for two players on the graph in Figure 4. The best options for player 1 are the maximum values of the first number in the column corresponding to player 2's choice (marked blue) while player 2's best choice is determined by maximizing the second number in the rows (marked red). As there is no combination of vertices that maximizes both values, there is no Nash equilibrium for two players. □

Proposition 1 shows that there is no Nash equilibrium in diffusion games on this graph, which means that for any strategy profile at least one of the players can improve by changing her decision. Now we look at how the players should decide in any situation. Figure 6 shows which vertices a player should choose depending on the other player's decision. If one player has chosen a vertex, then the other player's best answers are those vertices to which there is an arrow. If there was a Nash equilibrium, then there would be a cycle of length 2 because this would mean that there are two vertices in the graph that are the players' best answers to each other.

| Player 1 / Player 2 | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ |
|---|---|---|---|---|---|---|---|---|
| $v_1$ | 0/0 | 2/4 | 2/4 | 2/5 | 2/5 | 2/4 | 2/4 | 3/3 |
| $v_2$ | 4/2 | 0/0 | 3/3 | 2/4 | 2/4 | 2/4 | 3/4 | 3/2 |
| $v_3$ | 4/2 | 3/3 | 0/0 | 2/4 | 2/4 | 3/4 | 2/4 | 3/2 |
| $v_4$ | 5/2 | 4/2 | 4/2 | 0/0 | 2/2 | 4/2 | 3/3 | 5/1 |
| $v_5$ | 5/2 | 4/2 | 4/2 | 2/2 | 0/0 | 3/3 | 4/2 | 5/1 |
| $v_6$ | 4/2 | 4/2 | 4/2 | 2/4 | 3/3 | 0/0 | 2/2 | 5/2 |
| $v_7$ | 4/2 | 4/3 | 4/3 | 3/3 | 2/4 | 2/2 | 0/0 | 5/2 |
| $v_8$ | 3/3 | 2/3 | 2/3 | 1/5 | 1/5 | 2/5 | 2/5 | 0/0 |

Figure 5: The payoffs of player 1 and player 2 depending on their positions. The maximum values for Player 1 are marked blue and the maximum values for player 2 are marked red. There is no Nash equilibrium in the game since the payoffs of the two players cannot be maximized at the same time.

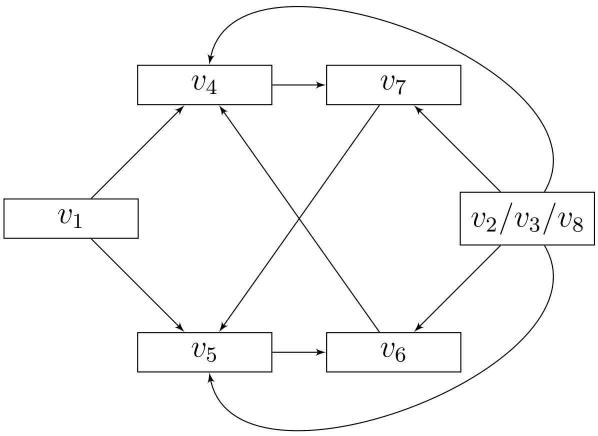



Figure 6: A representation of the players' best answers depending on the other player's choice. If one player has chosen a vertex, then the other player's best answers are those to which there is an arrow.

As Figure 6 shows, the vertices $v_4, v_7, v_5$, and $v_6$ form a cycle of length four to which all other vertices point. This means that if one player has chosen a vertex outside the cycle, then the other player should choose a vertex in the cycle in order to maximize her payoff. If, however, a player has chosen a vertex inside the cycle, then the other player can always improve by moving to the next vertex in the cycle. Hence, for all strategy profiles, there is always at least one player that can improve by changing her decision and therefore, there is no Nash equilibrium.

# 3 Hypercubes

As Etesami and Başar [6] showed, there is always a Nash equilibrium in diffusion games on hypercubes for two players. The following proof will show that there also exists a Nash equilibrium for four players.
Let $v \in \{0,1\}^d$. Then $v[i]$ denotes the $i$-th bit of $v$. Let the Hamming distance $d_H$ between $v$ and $w$ be the number of bits in which $v$ and $w$ differ, that is $d_H(v,w) = |\{j \in \{1,...,d\} \mid v[j] \neq w[j]\}|$. The Hamming distance satisfies the triangle inequality $\Delta(x,y) + \Delta(y,z) \leq \Delta(x,z)$ for all vertices $x, y$ and $z$.
Let $w_0(v)$ be the numbers of zeros in vertex $v$ and $w_1(v)$ be the number of ones, respectively.
A $d$-dimensional hypercube is an undirected graph $H_d = (V,E)$ with the vertex set $V = \{0,1\}^d$ such that two nodes are adjacent if and only if they differ in exactly one position, i.e. $E = \{\{u,v\} \mid d_H(u,v) = 1\}$.
If a player $i$ has distance $\Delta(p_i, v) =: \delta$ to a vertex $v$ and all other players $j$ have distance $\Delta(p_j, v) > \delta$ to $v$, then player $i$ has the unique shortest distance to $v$.

## 3.1 Nash equilibrium for four players

We will now show that the strategy profile $(0^d, 0^{d-1}1, 1^{d-1}0, 1^d)$, that is a profile where two players occupy two adjacent vertices and the other two are positioned on the opposite side of the hypercube, is a Nash equilibrium. Figure 7 shows this profile on a 3-dimensional hypercube where each player's payoff is one.

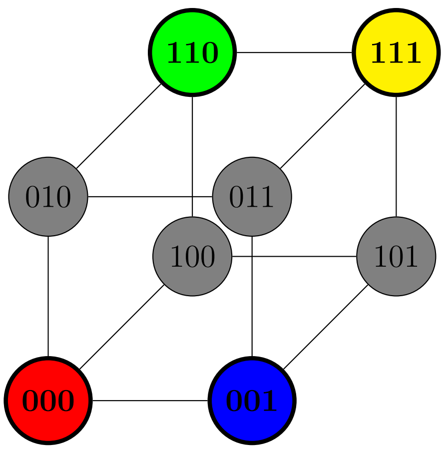



Figure 7: Nash equilibrium for four players on a 3-dimensional hypercube where each player's payoff is 1.

To prove that there is a Nash equilibrium on $H_d$ for four players, it will be shown that for the strategy profile $(p_1, 0^{d-1}1, 1^{d-1}0, 1^d)$ with $p_1 \in \{0,1\}^d$, a vertex will be colored by a player by the end of the diffusion process if and only if she has a unique shortest distance to that vertex. The first step to do that is to show that a vertex $v$ will be colored in color $i$ if Player $i$ has a unique shortest distance to it and that, with a strategy profile $(p_1, 0^{d-1}1, 1^{d-1}0, 1^d)$ with $p_1 \in \{0,1\}^d$, a vertex $v$ will not be colored by any player if multiple players have the same shortest distance to it.

**Lemma 1.** *Let $(p_1, ..., p_n)$, $n \geq 2$, be a strategy profile. If $p_x \neq p_y$ for all players $x \neq y$, then a vertex $v$ will be colored in color $i$ by the end of the diffusion process if Player $i$ has the unique shortest distance to $v$, i.e. $\Delta(p_i, v) < \Delta(p_j, v)$ for all $j \neq i$.*

*Proof.* Let $v$ be a vertex and assume that Player $i$ has the unique shortest distance to $v$. If by the time Player $i$ reaches $v$, $v$ has already been colored by another player or is already a standoff, then there must be at least one player with a shorter path to $v$ which means that Player $i$ does not have the shortest distance to $v$. If $v$ becomes a standoff because there is some other Player $j$ competing for it, then $\Delta(p_j, v) \leq \Delta(p_i, v)$ which means that Player $i$ does not have the unique shortest distance to $v$. Thus, if Player $i$ has a unique shortest distance to $v$, then $v$ is colored in color $i$ by the end of the diffusion process. □

Lemma 1 states that a vertex is colored by a player if she has a unique shortest distance to it. However, the reverse statement is generally not true, that is even if multiple players have the same shortest distance to a vertex, that vertex does not necessarily become a standoff. Consider for example a 4-dimensional hypercube and the strategy profile $(0011, 0110, 1010, 1100)$ as shown in Figure 8. All players have distance 2 to the vertex 0000 but by the end of the diffusion process, that vertex is colored in blue because all neighbors of 0000 except for 0001 are standoffs. Yet, if exactly two players have the same shortest distance to a vertex $v$ and all other players have a longer shortest distance to $v$, then at least two players have a unique shortest distance to a neighbor of $v$ and therefore $v$ becomes a standoff:

**Lemma 2.** *Let $(p_1, ..., p_n)$, $n \geq 2$, be a strategy profile. If $p_x \neq p_y$ for all players $x \neq y$, then a vertex $v$ will be a standoff if two players $i$ and $j$ have the same shortest distance $\Delta(p_i, v) = \Delta(p_j, v) =: \delta$ to $v$ and $\Delta(p_l, v) > \delta$ for all other players $l$.*

*Proof.* Let $\delta := \Delta(p_i, v) = \Delta(p_j, v)$, $i \neq j$, and $\Delta(p_k, v) > \delta$ for all $k \notin \{i, j\}$. As $p_i$ and $p_j$ both have distance $\delta$ to $v$ and $p_i \neq p_j$, the distance between $p_i$

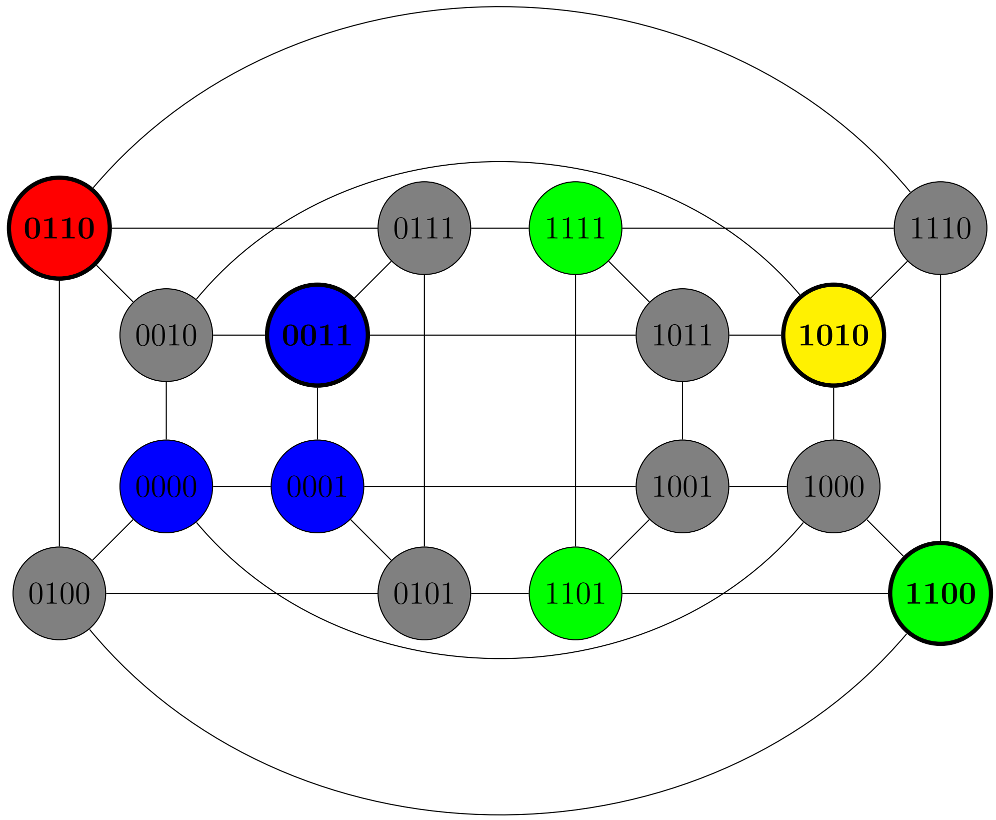



Figure 8: Diffusion game on a 4-dimensional hypercube with the strategy profile $(0011, 0110, 1010, 1100)$. Even though all players have distance 2 to the vertex 0000, it is colored blue by the end of the diffusion process because all neighbors of 0000 are standoffs except for 0001.

and $p_j$ is greater than 0 and even, so they differ in at least two bits. Let $s$ be the smallest number such that $p_i[s] \neq p_j[s] = v[s]$ and $t$ be the smallest number where $p_j[t] \neq p_i[t] = v[t]$. Let

$$x := v[1]...v[s-1]p_i[s]v[s+1]...v[d]$$

and

$$y := v[1]...v[t-1]p_j[t]v[t+1]...v[d].$$

It holds that $\Delta(p_i, x) = \delta - 1$ and $\Delta(p_j, x) = \delta + 1$. If any other Player $l$ had distance $\Delta(p_l, x) \leq \delta - 1$, then $\Delta(p_l, v) \leq \delta$. Hence, Player $i$ has a unique shortest distance to $x$. Analogously, Player $j$ has a unique shortest distance to $y$ as $\Delta(p_j, y) = \delta - 1$ and $\Delta(p_i) = \delta + 1$. Thus, after $\delta - 1$ time steps, two neighbors of $v$ are colored in two different colors. Therefore, $v$ will be a standoff. $\square$

Lemma 2 shows that a vertex $v$ becomes a standoff if exactly two players have the same shortest distance to $v$ and all other players have a longer shortest distance to $v$. The same is not generally true for more than two players for all strategy profiles, as for example Figure 8 shows.
Yet, with the strategy profile $(p_1, p_2, p_3, p_4) := (p_1, 0^{d-1}1, 1^{d-1}0, 1^d)$, $p_1 \in \{0,1\}^d$, a vertex to which exactly three players have the same shortest distance becomes a standoff.

**Lemma 3.** *For a strategy profile* $(p_1, p_2, p_3, p_4) := (p_1, 0^{d-1}1, 1^{d-1}0, 1^d)$ *with* $p_1 \in \{0,1\}^d$, *a vertex* $v$ *will not be colored by any player if* $p_1 \notin \{p_2, p_3, p_4\}$ *and three players have the same shortest distance to* $v$.

*Proof.* As a permutation of the 1s and 0s of the vertices of a hypercube does not change the graph, there is no loss of generality in assuming that $p_1 := abc$ with $a := 1^k$, $b := 0^{d-k-1}$ and $c \in \{0,1\}$ with $0 \le k \le d-1$. Since $\Delta(p_3, p_4) = 1$, Players 3 and 4 can never have the same distance to any vertex. Let $v := efg$ be a vertex with $e \in \{0,1\}^k$, $f \in \{0,1\}^{d-k-1}$, $g \in \{0,1\}$, and $v \notin \{p_1, p_2, p_3, p_4\}$. The distances are the following:

$$\begin{aligned}
\Delta(p_1, v) &= w_0(e) + w_1(f) + w_{|c-1|}(g),\\
\Delta(p_2, v) &= w_1(e) + w_1(f) + w_0(g),\\
\Delta(p_3, v) &= w_0(e) + w_0(f) + w_1(g),\\
\Delta(p_4, v) &= w_0(e) + w_0(f) + w_0(g).
\end{aligned}$$

The two cases that have to be considered are $p_1 = ab0$ and $p_1 = ab1$.

**Case 1.** $p_1 = ab0$.
Let $v$ be a vertex such that $\Delta(p_1, v) = \Delta(p_2, v) = \Delta(p_3, v) =: \delta$ and $\Delta(p_4, v) > \delta$. Then $g = 0$ because otherwise Player 4 would be closer to $v$ than Player 3, and $w_0(e) = w_1(e) + 1$ because

$$\Delta(p_1, v) = w_0(e) + w_1(f) + w_1(0) = w_1(e) + w_1(f) + w_0(0) = \Delta(p_2, v).$$

Also, $w_0(f) = w_1(f)$ must hold since

$$\Delta(p_1, v) = w_0(e) + w_1(f) + w_1(0) = w_0(e) + w_0(e) + w_1(0) = \Delta(p_3, v).$$

Let $l$ be the smallest number such that $v[l] = p_1[l] \neq p_3[l]$. Then Player 2 has a unique shortest distance to

$$v' := v[1]...v[d-1]1,$$

and Player 3 has a unique shortest distance to

$$v'' := v[1]...v[l-1]p_3[l]v[l+1]...v[d].$$

So at time $\delta - 1$, at least two neighbors of $v$ have different colors. Therefore, $v$ will be a standoff.
Now, let $v$ be a vertex such that $\Delta(p_1, v) = \Delta(p_2, v) = \Delta(p_4, v) =: \delta$ and $\Delta(p_3, v) > \delta$. Then $g = 1$ as otherwise Player 3 would be closer to $v$ than Player 4. Also, $w_0(e) = w_1(e) + 1$ because

$$\Delta(p_1, v) = w_0(e) + w_1(f) + w_1(1) = w_1(w) + w_1(f) + w_0(1) = \Delta(p_2, v),$$

and $w_0(f) = w_1(f)$ since

$$\Delta(p_1, v) = w_0(e) + w_1(f) + w_1(1) = w_0(e) + w_0(f) + w_0(1) = \Delta(p_4, v).$$

Let $l$ be the smallest number such that $v[l] = p_2[l] \neq p_1[l]$ and $m$ be the smallest number such that $p_4[m] \neq p_1[m] = v[m]$. Then Player 1 has a unique shortest distance to

$$v' := v[1]...v[d-1]0,$$

and Player 2 has a unique shortest distance to

$$v'' := v[1]...v[l-1]p_2[l]v[l+1]...v[d].$$

So after time step $\delta - 1$, at least two neighbors of $v$ have different colors. Therefore, $v$ will be a standoff.

**Case 2.** $p_1 = ab1$.
Let $v$ be a vertex such that $\Delta(p_1, v) = \Delta(p_2, v) = \Delta(p_3, v) := \delta$ and $\Delta(p_4, v) > \delta$. Then $g = 0$ because otherwise Player 4 would be closer to $v$ than Player 3, and $w_0(e) = w_1(e)$ since

$$\Delta(p_1, v) = w_0(e) + w_1(f) + w_0(0) = w_1(e) + w_1(f) + w_0(g) = \Delta(p_2, v).$$

Also, $w_1(f) + 1 = w_0(f)$ since

$$\Delta(p_1, v) = w_0(e) + w_1(f) + w_0(0) = w_0(e) + w_0(f) + w_1(0) = \Delta(p_4, v).$$

Let $l$ be the smallest number such that $v[l] = p_1[l] \neq p_3[l]$ and $m$ be the smallest number such that $v[m] = p_1[m] \neq p_2[m]$. Then Player 3 has a unique shortest distance to

$$v' := v[1]...v[l-1]p_3[l]v[l+1]...v[d]$$

and Player 2 has a unique shortest distance to

$$v'' := v[1]...v[m-1]p_2[m]v[m+1]...v[d].$$

So after time step $\delta - 1$, at least two neighbors of $v$ have different colors. Therefore, $v$ will be a standoff.

Now let $v$ be a vertex such that $\Delta(p_1, v) = \Delta(p_2, v) = \Delta(p_4, v) =: \delta$ and $\Delta(p_3, v) > \delta$. Then $g = 1$ as otherwise Player 3 would be closer to $v$ than Player 4. Also, $w_0(e) = w_1(e)$ because

$$\Delta(p_1, v) = w_0(e) + w_1(f) + w_0(1) = w_1(e) + w_1(f) + w_0(1) = \Delta(p_2, v)$$

and $w_0(f) = w_1(f)$ since

$$\Delta(p_1, v) = w_0(e) + w_1(f) + w_0(1) = w_0(e) + w_0(f) + w_0(1) = \Delta(p_4, v).$$

Let $l$ be the smallest number such that $v[l] = p_1[l] \neq p_2[l]$ and define $m$ as the smallest number such that $v[m] = p_1[m] \neq p_4[m]$. Then Player 2 has a unique shortest distance to

$$v' := v[1]...v[l-1]p_2[l]v[l+1]...v[d]$$

and Player 4 has a unique shortest distance to

$$v'' := v[1]...v[m-1]p_4[m]v[m+1]...v[d].$$

So after time step $\delta - 1$, at least two neighbors of $v$ have different colors. Therefore, $v$ will be a standoff.

$\square$

Lemma 3 shows that a vertex $v$ becomes a standoff if the strategy profile is $(p_1, 0^{d-1}1, 1^{d-1}0, 1^d)$ with $p_1 \in \{0,1\}^d$ such that exactly three players have the same shortest distance to $v$ while the forth player has a longer distance. With the strategy profile $(p_1, 0^{d-1}1, 1^{d-1}0, 1^d)$, no more than three players can have the same shortest distance to a vertex since $\Delta(p_3, p_4) = 1$ and therefore $\Delta(p_3, v) = \Delta(p_4, v)+1$ or $\Delta(p_4, v) = \Delta(p_3, v)+1$ for all vertices $v \in V$. Hence, by Lemmas 1, 2, and 3, in any profile $(p_1, 0^{d-1}1, 1^{d-1}0, 1^d)$ with $p_1 \in \{0,1\}^d$, every player gets exactly the vertices to which she has a unique shortest distance. Now we show that the strategy profile $(0^d, 0^{d-1}1, 1^{d-1}0, 1^d)$ is a Nash equilibrium. Figure 9 shows this Nash equilibrium on a 4-dimensional hypercube.

**Theorem 1.** *In a diffusion game for four players on a d-dimensional hypercube, the strategy profile* $(p_1, p_2, p_3, p_4) := (0^d, 0^{d-1}1, 1^{d-1}0, 1^d)$ *is a Nash equilibrium and the payoff for any player j is*

$$U_j(p_1, p_2, p_3, p_4) = \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{d-1}{k}.$$

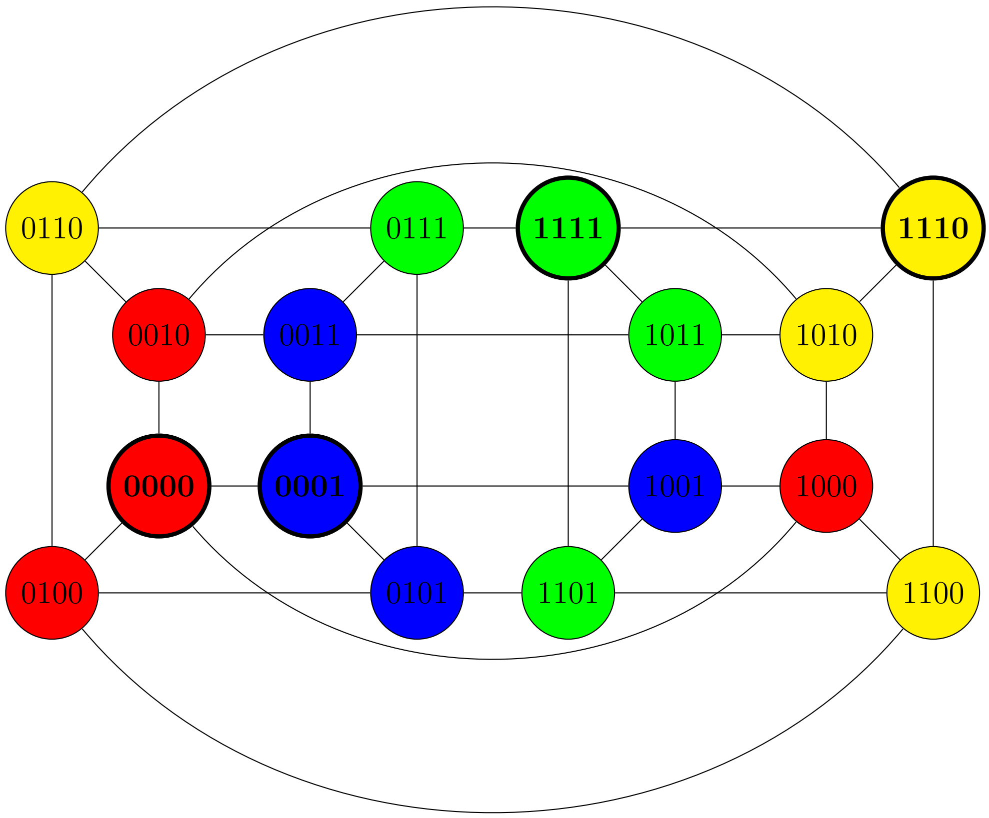



Figure 9: Nash equilibrium for four players on a 4-dimensional hypercube.

*Proof.* By Lemmas 1, 2, and 3, all players get exactly the vertices to which they have a unique shortest distance among all players.
Let $(0^d, 0^{d-1}1, 1^{d-1}0, 1^d)$ be a strategy profile. Since $\Delta(p_2, v) < \Delta(p_1, v)$ for all $v \in \{w \in \{0,1\}^d \mid w[d] = 1\}$, only vertices where the last bit is 0 are colored by Player 1 by the end of the diffusion process.

Let $v \in \{w0 \mid w \in \{0,1\}^{d-1}\}$. Since $\Delta(p_3, v) < \Delta(p_4, v)$, only Players 1 and 3 compete for these vertices. Player 1's distance to $v$ is then $\Delta(p_1, v) = w_1(v)$ and Player 3's distance is $\Delta(p_3, v) = w_0(w)$. Hence, the vertices colored in color 1 by the end of the diffusion process are exactly those where the last bit is zero and $w_0(w) < w_1(w)$, that is, $w_1(w) < \frac{d-1}{2}$. Player 1's playoff is therefore

$$U_1(0^d, 0^{d-1}1, 1^{d-1}0, 1^d) = \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{d-1}{k}.$$

The two cases are that the dimension of the hypercube is even or odd.

**Case 1.** $d$ is even. Then, for the strategy profile $(0^d, 0^{d-1}1, 1^{d-1}0, 1^d)$ Player 1's playoff is

$$\begin{aligned} U_1(0^d, 0^{d-1}1, 1^{d-1}0, 1^d) &= \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{d-1}{k} = \sum_{k=0}^{\frac{d-2}{2}} \binom{d-1}{k} \\ &= \frac{1}{2} \sum_{k=0}^{d-1} \binom{d-1}{k} = 2^{d-2}. \end{aligned}$$

Now we show that Player 1 can never get more than $2^{d-2}$ vertices. Consider a strategy profile $(x, y, \overline{x}, \overline{y})$ with $x \neq y \in \{0,1\}^d$ and $\overline{x}, \overline{y}$ the bitwise complements of $x$ and $y$. Since this profile is symmetric, $x$ has a unique shortest distance to exactly as many vertices as $\overline{x}$, and $y$ has a unique shortest distance to exactly as many vertices as $\overline{y}$. Assume that $y$ and $\overline{y}$ together are closest to more than $2 \cdot 2^{d-2} = 2^{d-1}$ vertices. Then $x$ and $\overline{x}$ are also closest to more than $2^{d-1}$ vertices. This is a contradiction, since a $d$-dimensional hypercube only has $2^d$ vertices. Since a vertex $v \in V$ cannot have the same distance to $x$ and $\overline{x}$, $x$ is closest to at most $2^{d-2}$ vertices.
Now consider the strategy profile $(p_1, p_2, p_3, p_4) = (p_1, 0^{d-1}1, 1^{d-1}0, 1^d)$ with $p_1 \in \{0,1\}^d$ and $p_1 \notin \{p_2, p_3, p_4\}$. Clearly, $p_3 = \overline{p_2}$. Since Player 1 does not have a unique shortest distance to any of the vertices closest to $\overline{p_1}$, Player 1's payoff is never greater than $2^{d-2}$.
Thus, $(0^d, 0^{d-1}1, 1^{d-1}0, 1^d)$ is a Nash equilibrium with

$$U_1(0^d, 0^{d-1}1, 1^{d-1}0, 1^d) = \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{d-1}{k} = 2^{d-2}.$$

**Case 2.** $d$ is odd. Then, for the strategy profile $(0^d, 0^{d-1}1, 1^{d-1}0, 1^d)$ Player 1's playoff is

$$U_1(0^d, 0^{d-1}1, 1^{d-1}0, 1^d) = \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{d-1}{k} = \sum_{k=0}^{\frac{d-1}{2}} \binom{d-1}{k}$$
$$= 2^{d-2} - \frac{1}{2}\binom{d-1}{\frac{d-1}{2}}.$$

Again, we show that this is the maximum payoff possible for Player 1. Consider a strategy profile $(x, y, \overline{x}, \overline{y})$ with $x \neq y \in \{0,1\}^d$ and $\overline{x}, \overline{y}$ the bitwise complements of $x$ and $y$ where $x \neq y \in \{0,1\}^d$ and $\overline{x}, \overline{y}$ are the bitwise complements of $x$ and $y$. Since $d$ is odd, one of $\Delta(x, y)$ and $\Delta(x, \overline{y})$ is odd and one is even. Without loss of generality assume that $\Delta(x, y)$ is even and $\Delta(x, \overline{y})$ is odd. Without loss of generality assume that $x = 0^d$ and $y = ab$ with $a = 1^\alpha$ and $b = 0^{d-\alpha}$ for some even $\alpha \leq d-1$. Then there exist vertices $v = ef$ with $e \in \{0,1\}^\alpha$ and $f \in \{0,1\}^{d-\alpha}$ such that $\Delta(x, v) = \Delta(y, v) =: \delta$, $\Delta(\overline{x}, v) > \delta$ and $\Delta(\overline{y}, v) > \delta$, that is $\delta \leq \lfloor \frac{d}{2} \rfloor$. Then it must hold that $w_1(e) = \frac{\alpha}{2}$ and $w_0(e) = \frac{\alpha}{2}$, and $w_1(v) \leq \lfloor \frac{d}{2} \rfloor$ because otherwise, $v$ is closer to $\overline{x}$. Hence, $w_1(f) \leq \lfloor \frac{d}{2} \rfloor - \frac{\alpha}{2}$. Thus, there are

$$\binom{\alpha}{\frac{\alpha}{2}} \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor - \frac{\alpha}{2}} \binom{d-1}{k} = \binom{\alpha}{\frac{\alpha}{2}} 2^{d-\alpha-1}$$

vertices where $\delta = \Delta(x, v) = \Delta(y, v)$, $\Delta(\overline{x}, v) > \delta$ and $\Delta(\overline{y}, v) > \delta$. We use the following identity:

$$\binom{2n}{n} = \frac{(2n)!\, 2^{2n}}{n!\, n!\, 2^{2n}} = 2^{2n} \frac{(2n)!}{(n!\, 2^n)^2} = 2^{2n} \frac{(2n-1)!\,!}{n!\, 2^n} = 2^{2n} \frac{(2n-1)!\,!}{(2n)!\,!}$$
$$= 2^{2n} \frac{1 \cdot 3 \ldots \cdot (2n-1)}{2 \cdot 4 \cdot \ldots \cdot 2n}.$$

Then,

$$\binom{\alpha}{\frac{\alpha}{2}} 2^{d-\alpha-1} = 2^\alpha \cdot 2^{d-\alpha-1} \cdot \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (\alpha - 1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot \alpha} = 2^{d-1} \cdot \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (\alpha - 1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot \alpha}$$
$$\geq 2^{d-1} \cdot \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (d-2)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (d-1)} = \binom{d-1}{\frac{d-1}{2}},$$

and therefore

$$\binom{\alpha}{\frac{\alpha}{2}} \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor - \frac{\alpha}{2}} \binom{d-1}{k} \geq \binom{d-1}{\frac{d-1}{2}}.$$

Now assume that $x$ is closest to more than $2^{d-2} - \frac{1}{2}\binom{d-1}{\frac{d-1}{2}}$ vertices. Then $x$ and $\overline{x}$ together are closest to more than $2^{d-1} - \binom{d-1}{\frac{d-1}{2}}$ vertices. Since this profile is symmetric, $y$ and $\overline{y}$ are also closest to more than $2^{d-1} - \binom{d-1}{\frac{d-1}{2}}$ vertices. Thus, there are more than $2^d - 2\binom{d-1}{\frac{d-1}{2}}$ vertices to which $x$, $\overline{x}$, $y$ or $\overline{y}$ has a unique shortest distance. This is a contradiction, since there are at least $\binom{d-1}{\frac{d-1}{2}}$ vertices $v$ such that $\delta = \Delta(x,v) = \Delta(y,v)$, $\Delta(\overline{x},v) > \delta$ and $\Delta(\overline{y},v) > \delta$ and because of symmetry at least $\binom{d-1}{\frac{d-1}{2}}$ vertices $v$ such that $\delta_2 = \Delta(\overline{x},v) = \Delta(\overline{y},v)$, $\Delta(x,v) > \delta_2$ and $\Delta(y,v) > \delta_2$. Now consider the strategy profile $(p_1,p_2,p_3,p_4) = (p_1, 0^{d-1}1, 1^{d-1}0, 1^d)$ with $p_1 \in \{0,1\}^d$ and $p_1 \notin \{p_2,p_3,p_4\}$. Clearly, $p_3 = \overline{p_2}$. Since Player 1 does not have a unique shortest distance to any of the vertices closest to $\overline{p_1}$, Player 1's payoff is never greater than $2^{d-2} - \frac{1}{2}\binom{d-1}{\frac{d-1}{2}}$.
Thus, $(0^d, 0^{d-1}1, 1^{d-1}0, 1^d)$ is a Nash equilibrium with

$$U_1(0^d, 0^{d-1}1, 1^{d-1}0, 1^d) = \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{d-1}{k} = 2^{d-2} - \frac{1}{2}\binom{d-1}{\frac{d-1}{2}}.$$

□

## 3.2 Nash equilibrium for three players

Computer simulations show that if one of the players from the Nash equilibrium for four players which was shown in the previous section is removed, the resulting strategy profile is still a Nash equilibrium. This leads to the following conjecture:

**Conjecture 1.** *In a diffusion game for three players on a d-dimensional hypercube, the strategy profile* $(p_1,p_2,p_3) := (0^d, 0^{d-1}1, 1^d)$ *is a Nash equilibrium.*

The difference to the four player case is that this strategy profile is not symmetric and therefore we have to show for all three players that they cannot improve by changing their decision. Since $\Delta(0^d, 0^{d-1}1) = 1$, there is no vertex $x \in V$ such that $\Delta(0^d, v) = (0^{d-1}1, v)$. Thus, it follows from Lemma 1 and Lemma 2 that with a strategy profile $(0^d, 0^{d-1}1, p_3)$ where $p_3 \in \{0,1\}^d$, a vertex is colored by Player 3 if and only if she has a unique shortest distance to it. For strategy profiles $(p_1, 0^{d-1}1, 1^d)$ with $p_1 \in \{0,1\}^d$ and $(0^d, p_2, 1^d)$ with $p_2 \in \{0,1\}^d$, an argument similar to that of Lemma 3 can be made, that is, that for these strategy profiles, a vertex becomes a standoff if three players have the same shortest distance to it. With Lemmas 1 and 2, this means that a vertex is colored by Player $i$ if and only if Player $i$ has a unique shortest distance to that vertex.

# 4 Grids

Roshanbin [8] showed that for $n, m \in \mathbb{N}$, there is always a Nash equilibrium on a $n \times m$ grid for two players. Bulteau et al. [3] showed that there is no Nash equilibrium for three players if $n \geq 5$ and $m \geq 5$.
A grid graph $G_{n \times m}$, $n, m \in \mathbb{N}$, is an undirected graph with the vertex set

$$V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \ 0 \leq x < n, 0 \leq y < m \right\}$$

and edge set

$$E = \left\{ \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\} \middle| \ x_1 = y_1 \wedge x_2 = y_2 + 1 \vee x_1 = y_1 + 1 \wedge x_2 = y_2 \right\}.$$

**Proposition 2.** *There is a Nash equilibrium for 4 players on $G_{5\times 5}$.*

*Proof.* We show that the strategy profile

$$p = \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix} \right),$$

as shown in Figure 10, is a Nash equilibrium. Since this strategy profile is symmetric, it suffices to show that players 1 and 2 cannot improve. Figure 11 shows the payoff matrices for players 1 and 2. Since $U_i(p) = 6$ for all players $i$ and six is the maximum value in both payoff matrices, $p$ is a Nash equilibrium. $\square$

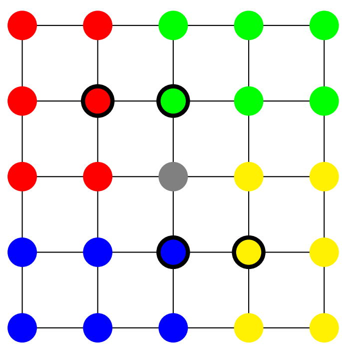


Figure 10: Nash equilibrium on $G_{5\times 5}$.

|  | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 4 | 3 | 6 | 3 | 4 |
| 1 | 3 | **6** | 4 | 6 | 5 |
| 2 | 5 | 0 | 3 | 0 | 3 |
| 3 | 3 | 4 | 3 | 0 | 2 |
| 4 | 3 | 3 | 3 | 3 | 1 |

(a) Player 1's payoff matrix

|  | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 3 | 3 | 3 | 3 |
| 1 | 5 | 0 | 3 | 4 | 3 |
| 2 | 3 | **6** | 1 | 0 | 3 |
| 3 | 5 | 4 | 6 | 0 | 2 |
| 4 | 3 | 5 | 3 | 5 | 5 |

(b) Player 2's payoff matrix

Figure 11: The payoff matrices of players 1 and 2.

Computer simulations suggest that the following holds:

**Conjecture 2.** *If $n \geq 6$ and $m \geq 6$, then there is no Nash equilibrium for four players on $G_{n\times m}$.*

In their proof for the three player case, Bulteau et al. [3] distinguish the cases where the players play far from each other, that is there are two players that differ by at least four in some coordinate, and where they play within a $3 \times 3$ subgrid. For the first case, they consider two subcases, namely whether one player strictly controls the others, that is

$$\begin{aligned} &\forall i \neq j: \; x_i < x_j \wedge y_i < y_j \\ \text{or } &\forall i \neq j: \; x_i < x_j \wedge y_i > y_j \\ \text{or } &\forall i \neq j: \; x_i > x_j \wedge y_i < y_j \\ \text{or } &\forall i \neq j: \; x_i > x_j \wedge y_i > y_j, \end{aligned}$$

or not. For the second case, they distinguish all possible positions the players can take within a $3 \times 3$ subgrid. For the four player case, a similar proof seems possible. However, the number of subcases increases.

## 5 Conclusion

We studied competitive diffusion games for three and four players on hypercubes, showing that there is always a Nash equilibrium for four players and conjecturing that there is a Nash equilibrium for three players as well. We then looked at diffusion games for four players on grids where we showed that there is a Nash equilibrium on a $5 \times 5$ grid for four players and conjectured that there is no Nash equilibrium if $m \geq 6$ and $n \geq 6$.
There are still several open questions left, some of which shall be mentioned here. It is still open whether there are Nash equilibria in diffusion games for more than four players on hypercubes. Another open question is whether there are Nash equilibria on grids for a higher number of players as well as what the minimum number of players is such that there is a Nash equilibrium on a given grid.

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