Parameterized Computational Complexity of Dodgson and Young Elections

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joint work with

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Computational Social Choice

Election and Condorcet Winner

Election

Set of votes V, set of candidates C.
A vote is preference list (total order) over all candidates.

Condorcet winner

A candidate c that defeats every other candidate in pairwise comparison, that is, c is better in more than half of the votes.

Example:

vote 1:	a	$>$	c	$>$	b
vote 2:	c	$>$	a	$>$	b
vote 3:	b	$>$	c	$>$	a

$\Rightarrow$ c is the Condorcet winner!

Young and Dodgson

Problem

A Condorcet winner does not always exist.

Example:

vote 1:	a	>	b	>	c
vote 2:	c	>	a	>	b
vote 3:	b	>	c	>	a

then:

a is preferred to b

b is preferred to c

c is preferred to a

Young and Dodgson

Problem

A Condorcet winner does not always exist.

Way out: Take the candidate that is “closest” to a Condorcet winner. We consider two ways:

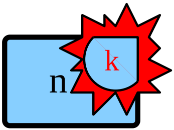
Dodgson and Young elections

In both we compute the score of every candidate and the candidate with highest/lowest score wins.

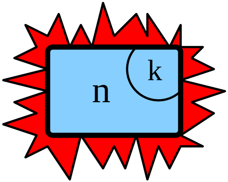
Both problems are NP-hard: Parameterized complexity?

Parameterized Complexity — Introduction I

Given an NP-hard problem with input size n and a parameter k.
Basic idea: Confine the combinatorial explosion to k.



instead of



Definition

A problem of size n is called *fixed-parameter tractable* with respect to a parameter k if it can be solved in $f(k) \cdot n^{O(1)}$ time.

Parameterized Complexity — Introduction II

Completeness program developed by Downey and Fellows [DOWNEY & FELLOWS 1999]

$$\text{FPT} \subseteq \overbrace{\text{W[1]} \subseteq \text{W[2]} \subseteq \ldots \subseteq \text{W[P]}}^{\text{Presumably fixed-parameter intractable}}$$

Assumption: FPT $\neq$ W[1]

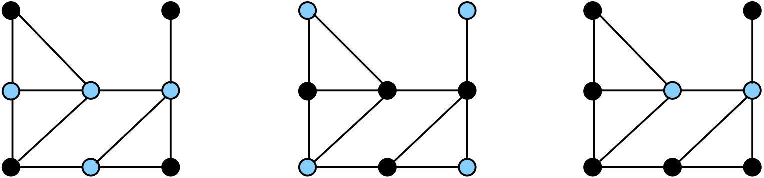
If W[1]=FPT then 3-SAT for a Boolean formula F with n variables can be solved in $2^{o(n)} \cdot |F|^{O(1)}$ time.

Parameterized Complexity — Introduction II

Presumably fixed-parameter intractable

$$\text{FPT} \subseteq \overbrace{\text{W[1]} \subseteq \text{W[2]} \subseteq \ldots \subseteq \text{W[P]}}$$

Prominent examples:



VERTEX COVER — FPT

INDEPENDENT SET — "W[1]−complete"

DOMINATING SET — "W[2]−complete"

Dodgson Score: Definition

Let a *switch* be the swapping of the two positions of two neighboring candidates in a vote.

Dodgson Score

Given: An election (V, C), a distinguished candidate $c \in C$, and an integer $k \geq 0$.
Question: Can c be made a Condorcet winner by at most k switches?

- NP-complete [Bartholdi III, Tovey & Trick, SCW 1989]
- Winner and ranking variant are Θ_2^p-complete [Hemaspaandra, Hemaspaandra, & Rothe, J. ACM 1997]
- Greedy heuristic with frequent success guarantee [Homan & L. Hemaspaandra, MFCS 2006]
- Approximability [McCabe-Dansted, Pritchard & Slinko, COMSOC 2006]

Parameterized Complexity of Dodgson Score

Number of votes n
Number of candidates m

Theorem

DODGSON SCORE can be solved in $O(2^k \cdot nk + nm)$ time.

Hence: For a candidate that is close to be a Condorcet winner the Dodgson Score can be computed efficiently!

Proof: Dynamic programming algorithm

Dodgson: Idea of Dynamic Programming

Example:

vote 1 :	c_2	$>$	c_3	$>$	c	$>$	c_1
vote 2 :	c_1	$>$	c_2	$>$	c	$>$	c_3
vote 3 :	c_1	$>$	c_2	$>$	c	$>$	c_3
vote 4 :	c_2	$>$	c	$>$	c_3	$>$	c_1
vote 5 :	c_3	$>$	c_2	$>$	c_1	$>$	c

Deficit:

candidate c_1: $d_1 = 1$
candidate c_2: $d_2 = 3$
candidate c_3: $d_3 = 0$

Dodgson: Idea of Dynamic Programming

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Deficit:

candidate c_1: $d_1 = 1$
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3 Observations:

- Only switches that "improve" c need to be considered.
- One switch: Deficit of at most one candidate can be decreased.
  Number of "dirty" candidates is bounded by k.
- Decompose the solution by decomposing the deficits.

Dodgson: Idea of Dynamic Programming

Example:

vote 1 :	c_2	$>$	c_3	$>$	c	$>$	c_1
vote 2 :	c_1	$>$	c_2	$>$	c	$>$	c_3
vote 3 :	c_1	$>$	c_2	$>$	c	$>$	c_3
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vote 5 :	c_3	$>$	c_2	$>$	c_1	$>$	c

Deficit:

candidate c_1: $d_1 = 1$
candidate c_2: $d_2 = 3$
candidate c_3: $d_3 = 0$

Deficit list $(d_1, d_2, d_3) = (1, 3, 0)$.

Algorithm considers *partial deficit lists*.

Table entry: Minimum number of switches needed to achieve a partial deficit list in a set of votes.

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(0,2)$	$(1,2)$	$(0,3)$	$(1,3)$
$\{v_1\}$	∞	∞	∞	∞	∞	2	∞	0
$\{v_1, v_2\}$	∞	4	∞	3	2	1	2	0
$\{v_1, v_2, v_3\}$	...							

Dodgson: Idea of Dynamic Programming

Example:

vote 1 :	c_2	$>$	c_3	$>$	c	$>$	c_1
vote 2 :	c_1	$>$	c_2	$>$	c	$>$	c_3
vote 3 :	c_1	$>$	c_2	$>$	c	$>$	c_3
vote 4 :	c_2	$>$	c	$>$	c_3	$>$	c_1
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Deficit:

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We can show:

- Table size bounded by $2^k \cdot \#\text{votes}$.
- Computation of new entry in time linear in k.

Dodgson Score: Allowing ties

Two natural ways going from total to partial orders of the votes:

Transformation of

$$b = a > c \quad \text{into} \quad c > b = a$$

- requires one switch (Model 1).
- requires two switches (Model 2). Here one can choose upon which candidate to improve first.

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- requires two switches (Model 2). Here one can choose upon which candidate to improve first.

Theorem

DODGSON TIE SCORE 2 can be solved in $O(4^k \cdot nk + nm)$ time.

Theorem

DODGSON TIE SCORE 1 is $W[2]$-hard with respect to k.

Young Score: Definition

YOUNG SCORE

Given: An election (V, C), a candidate $c \in C$, an integer $s \geq 0$.
Question: A subset $V' \subseteq V$ of size at least s such that (V', C) has the Condorcet winner c?

DUAL YOUNG SCORE asks if c can become Condorcet winner by deleting at most k votes. [YOUNG, J. ECON. THEO. 1977]

Winner and ranking problems are Θ_2^p-complete.
[ROTHE, SPAKOWSKI, & VOGEL, TOCS 2003]

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Example:

vote 1:	b	>	a	>	c
vote 2:	a	>	c	>	b
vote 3:	a	>	c	>	b
vote 4:	c	>	a	>	b
vote 5:	b	>	c	>	a

"$a > c$" in 3 votes
"$c > a$" in 2 votes

$\Rightarrow$ delete two votes with "$a > c$"

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vote 3:	a	>	c	>	b
vote 4:	c	>	a	>	b
vote 5:	b	>	c	>	a

Delete vote 1 and 2 !

Parameterized Complexity of Dual Young Score

Theorem:

DUAL YOUNG SCORE is W[2]-hard with respect to the number of votes that have to be deleted.

Proof: Reduction from a Dominating Set variant.

Idea of Parameterized Reduction

Definition

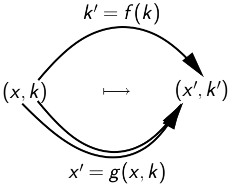
A *parameterized* problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet.

Let $L, L' \subseteq \Sigma^* \times \mathbb{N}$ be two parameterized problems.

$L \leq_{par} L'$ if

1.) $(x, k) \in L$ iff $(x', k') \in L'$.

2.)



We use the following observation:

Observation

By deleting one vote the distinguished candidate c either improves or worsens against every other candidate.

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Consider the following situation:

- A candidate a with $\#\text{votes}(\text{“}a > c\text{”}) = \#\text{votes}(\text{“}c > a\text{”})$.
- For the solution we have to delete k votes.

In the solution we must have “$a > c$” in $\lfloor k/2 \rfloor + 1$ votes.

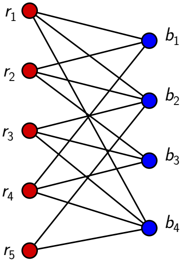
k/2-RedBlue-Dominating Set $\leq_{par}$ Young Score

k/2-RedBlue-Dominating Set

Input: Bipartite graph

Question: Is there a size-k subset of red vertices such that every blue vertex is dominated at least $\lfloor k/2 \rfloor + 1$ times?

Example:



$k = 3$, that means every blue vertex has to be dominated $\lfloor 3/2 \rfloor + 1 = 2$ times

W[2]-hardness can be shown by reduction from Dominating Set.

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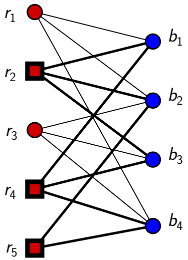

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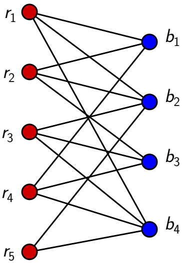
W[2]-hardness can be shown by reduction from Dominating Set.

k/2-RedBlue-Dominating Set $\leq_{par}$ Young Score

Construction:

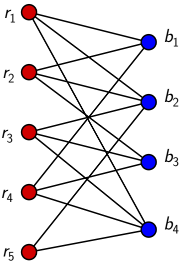
Set of votes: $V := V_1 \cup V_2$
Set of candidates: one candidate b_i for each blue vertex and the distinguished candidate c

V_1: one vote for each red vertex in which blue neighbors defeat c



V_1:
$v_{r1} : b_1 > b_2 > b_4 > c > b_3$
$v_{r2} : b_1 > b_2 > b_3 > c > b_4$
$v_{r3} : b_2 > b_3 > b_4 > c > b_1$
$v_{r4} : b_1 > b_3 > b_4 > c > b_2$
$v_{r5} : b_2 > b_4 > c > b_1 > b_3$

k/2-RedBlue-Dominating Set $\leq_{par}$ Young Score



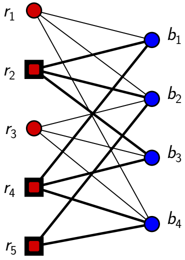
$$\left.\begin{array}{l} v_{r1} : b_1 > b_2 > b_4 > c > b_3 \\ v_{r2} : b_1 > b_2 > b_3 > c > b_4 \\ v_{r3} : b_2 > b_3 > b_4 > c > b_1 \\ v_{r4} : b_1 > b_3 > b_4 > c > b_2 \\ v_{r5} : b_2 > b_4 > c > b_1 > b_3 \end{array}\right\} V_1$$

$$\left.\begin{array}{c} \ldots\ldots \\ \ldots\ldots \end{array}\right\} V_2$$

V_2: set of votes such that in $V = V_1 \cup V_2$:

- For all blue candidates: $\#\text{votes}(b_i > c) = \#\text{votes}(c > b_i)$

k/2-RedBlue-Dominating Set $\leq_{par}$ Young Score



$$
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v_{r4} : b_1 > b_3 > b_4 > c > b_2 \\
v_{r5} : b_2 > b_4 > c > b_1 > b_3
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$$

$$
\left.\begin{array}{c}
\ldots\ldots \\
\ldots\ldots
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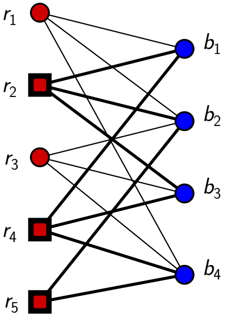
- For all blue candidates b_i: $\#\text{votes}(b_i > c) = \#\text{votes}(c > b_i)$

$\Rightarrow$ If we can delete only votes of V_1 to make c a Condorcet winner, then we have to remove a subset of votes such that each candidate is better than c in more than half of the votes!

k/2-RedBlue-Dominating Set $\leq_{par}$ Young Score



$$\left.\begin{array}{l} v_{r1} : d > e > b_1 > b_2 > b_4 > c > b_3 \\ v_{r2} : d > e > b_1 > b_2 > b_3 > c > b_4 \\ v_{r3} : d > e > b_2 > b_3 > b_4 > c > b_1 \\ v_{r4} : d > e > b_1 > b_3 > b_4 > c > b_2 \\ v_{r5} : d > e > b_2 > b_4 > c > b_1 > b_3 \end{array}\right\} V_1$$

$$\left.\begin{array}{c} \ldots\ldots \\ \ldots\ldots \end{array}\right\} V_2$$

V_2: set of votes such that in $V = V_1 \cup V_2$:

- For all blue candidates: $\#\text{votes}(b_i > c) = \#\text{votes}(c > b_i)$
- Add two candidates d and e such that
  - c has to improve on d and e in k votes.
  - only in votes of V_1 are d and e positioned better than c.

Young Score: Parameterized Complexity

Theorem

DUAL YOUNG SCORE is W[2]-complete with respect to the number of deleted votes.

- W[2]-hardness: Reduction from $k/2$-RedBlue-Dominating Set.
- W[2]-membership: Reduction to OPTIMAL LOBBYING. (W[2]-completeness proven by [CHRISTIAN ET AL., COMSOC 2006])

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Theorem

YOUNG SCORE is W[2]-complete with respect to the solution size.

- W[2]-hardness: Follows from the reduction of the Θ_2^p-completeness for the winner and ranking problems. [ROTHE, SPAKOWSKI, & VOGEL, TOCS 2003]
- W[2]-membership: Reduction to OPTIMAL LOBBYING.

Conclusion

Main observation

In contrast to classic complexity theory, the parameterized complexity of DODGSON SCORE and YOUNG SCORE differs.

Whereas DOGSON SCORE and both variants that allow ties are NP-complete, they have different parameterized complexity.

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Whereas DOGSON SCORE and both variants that allow ties are NP-complete, they have different parameterized complexity.

Note that the W[2]-completeness of Young Score has a positive aspect:

Remark

Controlling a Condorcet election by deleting a set of votes V' is W[2]-complete w.r.t. the size of V'.

Outlook

Open questions?

- Only W[2]-hardness of DODGSON TIE SCORE 1 is proven. Is it also W[2]-complete?
- Study other parameterizations.

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	DODGSON SCORE	YOUNG SCORE
# votes n	?	FPT (2^n)
# candidates m	FPT (ILP)	FPT (ILP)

ILP DODGSON SCORE: [BARTHOLDI III, TOVEY & TRICK, SCW 1989] and improved by [MACCABE-DANSTED 2006]
ILP YOUNG SCORE: [YOUNG, J. ECON. THEO. 1977]

Questions?