On the Computational Complexity of Length- and Neighborhood-Constrained Path Problems

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Abstract

Finding paths in graphs is a fundamental graph-theoretic task. In this work, we study the task of finding a path with some constraints on its length and the number of vertices neighboring the path, that is, being outside of and incident with the path. Herein, we consider short and long path on the one side, and small and large neighborhoods on the other side—yielding four decision problems. We show that all four problems are NP-complete, even in planar graphs with small maximum degree. Moreover, we study all four variant when parameterized by the bound k on the length of the path, by the bound ℓ on the size of neighborhood, and by $k + \ell$.

Keywords: secludedness, fixed-parameter tractability, W-hardness, problem kernelization

1. Introduction

Finding paths (connecting two designated terminal vertices) is a fundamental problem in computer science. The task differs from finding short paths, which is tractable by folklore results, or long paths, being an NP-complete problem. In this work, we study short and long paths with small and large *open* neighborhoods. The open neighborhood of a path consists of all vertices that are not contained in the path but adjacent with at least one vertex in the path. Formally, we study the following 2×2 problems:

$\{S, L\} \times \{S, U\}$ Path
Input: An undirected graph G, two integers $k \geq 1$, $\ell \geq 0$.
Question: Is there a simple path P in G with open neighborhood $N := |N_G(V(P))|$ such that

Short Secluded Path (SSP):	$\|V(P)\| \leq k$ & $N \leq \ell$?
Long Secluded Path (LSP):	$\|V(P)\| \geq k$ & $N \leq \ell$?
Short Unsecluded Path (SUP):	$\|V(P)\| \leq k$ & $N \geq \ell$?
Long Unsecluded Path (LUP):	$\|V(P)\| \geq k$ & $N \geq \ell$?

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Table 1: Overview of our results: NP-c., W[1]/W[2]-h., p-NP-h., noPK abbreviate NP-complete, W[1]/W[2]-hard, para-NP-hard, no polynomial kernel, respectively. [a] (even on planar graphs, Thm. 1) [b] (even on planar graphs with maximum degree seven, Thm. 7)

Problem	Compl.	Parameterized Complexity		
		k	ℓ	$k+\ell$
(st-)SSP	NP-c.[a]	XP, W[1]-h. (Thm. 5)	p-NP-h.[a]	FPT (Thm. 3)/noPK[b]
(st-)LSP	NP-c.[a]	p-NP-h.[a]	p-NP-h.[a]	p-NP-h.[a]
(st-)SUP	NP-c.[a]	XP, W[2]-h. (Thm. 6)	*open*	FPT (Thm. 4)/noPK[b]
(st-)LUP	NP-c.[a]	p-NP-h.[a]	p-NP-h.[a]	*open*/noPK[b]

We also consider their so-called s-t variants: Herein, two distinct vertices are part of the input, and the question is whether there is an s-t path fulfilling the respective conditions. Note that herein $k \geq 2$, as at least s and t must be contained in the path. We indicate the s-t variants by using st as prefix.

Short paths with small neighborhoods are of considerable interest in network security [3]. Moreover, SUP and the k-Dominating Path problem [10] are related.[3]

Our Contributions. Our main results are summarized in Table 1. We prove $\{S, L\} \times \{S, U\}$ Path (and their s-t variants) to be NP-complete even on planar graphs with maximum degree five (seven). In ten-out-of-twelve cases, we settle the parameterized complexity of the four problems regarding their problem-own parameters number k of vertices in the path and size ℓ of the open neighborhood of the path. Regarding the parameter k, we have containment in XP for the "short" variants, and para-NP-hardness for the "long" variants. However, the "short" variants are—presumably—fixed-parameter intractable (W-hard) when parameterized by k. The only cases in which we identified fixed-parameter tractability are for the "short" variants when parameterized by the combined parameter $k + \ell$. Complementing this, we prove that a polynomial problem kernelization is—presumably—excluded, even in planar graphs with small maximum degree. Regarding the parameter ℓ, we identified in three of the four cases para-NP-completeness.

Related Work. Chechik et al. [3] introduced the Secluded Path problem, that, different to SSP, seeks to minimize the *closed* neighborhood of the path in question, where the closed neighborhood are all vertices that are either contained in the path or adjacent with a vertex in the path. They proved Secluded Path to be NP-hard on weighted or directed graphs of maximum degree four, and polynomial-time solvable in undirected unweighted graphs (note that we prove SSP to be NP-complete in this case). They study Secluded Path (and a "secluded" variant of the Steiner Tree problem) in the context of approximation algorithms.

[3]There is a straight forward reduction from k-Dominating Path to SUP: Add for each vertex a private neighbor, set $k' = k$ and $\ell = n$, where n denotes the number of vertices in the input graph.

Fomin et al. [8], building upon the work of Chechik et al. [3], studied the parameterized complexity of Secluded Path (in its weighted version). They prove Secluded Path to be W[1]-hard when parameterized by the length of the path (which refers to the value $k-1$ in SSP). Moreover, they prove Secluded Path to be in FPT when parameterized by the size of the closed neighborhood of the path (which refers to the value $k+\ell$ in SSP), but does not—presumably—admit a polynomial kernel when parameterized by the combined parameter size of the closed neighborhood of the path, treewidth and maximum degree of the underlying graph. We point out that in our proofs of our related results (see Theorem 5 and Theorem 7), we use ideas similar to those of Fomin et al. [8].

Bevern et al. [1] studied the problems of finding st-separators with small closed neighborhood ("secluded") and of finding small st-separators with small open neighborhood ("small secluded"). They motivated to distinguish between the size of the subgraph in question and the size of the open neighborhood. In addition, they studied several other classical optimization problems in their "secluded" and "small secluded" variant. Moreover, they also studied the Independent Set problem, being a maximization problem, in its "large secluded" variant.

2. Preliminaries

We use basic notation from graph theory [5] and parameterized complexity theory [4].

A path P is of length $\ell-1$ is a graph with vertex set $\{v_1,\dots,v_\ell\}$ and edge set $\{\{v_i,v_{i+1}\}\mid 1\le i<\ell\}$. We call v_1 and v_ℓ the endpoints of P, and hence also refer to P as a v_1-v_ℓ path. For a graph $G=(V,E)$, we denote by $N_G(W):=\{v\in V\setminus W\mid \exists w\in W:\{v,w\}\in E\}$ for any $W\subseteq V$ the *open* neighborhood of W in G. We say that path is k-short (k-long) if $|V(P)|\le k$ ($|V(P)|\ge k$). We say that path is ℓ-secluded (ℓ-unsecluded) if $|N_G(V(P))|\le\ell$ ($|N_G(V(P))|\ge\ell$). We denote by $\Delta(G)$ the maximum vertex-degree of G. We use $\mathcal{O}^*$-notation, which is the $\mathcal{O}$-notation hiding polynomial factors. For a problem Π and a parameter p, $\Pi(p)$ stands for "Π parameterized by p": E.g., SSP(k) stands for Short Secluded Path parameterized by the number k of vertices in the path.

3. All Variants are NP-complete

We show that even in planar graphs with small maximum degree, $\{S,L\}\times\{S,U\}$ Path is NP-complete, in some cases even if the requested size of the path and of the open neighborhood is constant.

Theorem 1. *The following problems are* NP*-complete, even on planar graphs:*
(a) SSP *even if* $\ell=0$ *and* $\Delta=3$*;*
(b) LSP *even if* $\ell=0$*,* $\Delta=3$*, and* $k=1$*;*
(c) SUP *even if* $\Delta=5$*;*
(d) LUP *even if* $\ell=0$ *and* $\Delta=3$*, or* $k=1$*;*

In the proof of Theorem 1, we give many-one reductions from the following NP-complete problem:

Planar Cubic Hamiltonian Path (PCHP)
Input: An undirected, planar, cubic, connected graph $G = (V, E)$.
Question: Is there a cycle in G that contains every vertex in V exactly once?

Proof. The containment in NP is immediate. Let (G) be an instance of PCHP. Let G' denote a copy of G. Denote by G'' the graph obtained from G' by adding for each vertex $v \in V$ two vertices to G' and making them adjacent only with v.

(a) & (d): Construct the instance $(G', k = n, \ell = 0)$. On the one hand, note that that G' admits a k-short ℓ-secluded path if and only if G admits a Hamiltonian path, as no neighboring vertices are allowed. On the other hand, note that that G' admits a k-long ℓ-unsecluded path if and only if G admits a Hamiltonian path, as all vertices are required to be contained in the path.

(b): Construct the instance $(G', k = 1, \ell = 0)$. Observe that G' admits a k-long ℓ-secluded path if and only if G admits a Hamiltonian path, as no neighboring vertices are allowed.

(c): Construct the instance $(G'', k = n, \ell = 2n)$. Observe that every path with at least $2n$ neighbors needs to contain all the vertices in G', as every path of length $1 \leq r \leq n$ has at most $2r + (n - r) = n + r \leq 2n$ neighbors. Hence, G'' admits a k-short ℓ-unsecluded path if and only if G admits a Hamiltonian path.

(d): Construct the instance $(G'', k = 1, \ell = 2n)$. Again, similar to (c), every path with at least $2n$ neighbors needs to contain all the vertices in G'. Analogously, G' admits a k-long ℓ-unsecluded path if and only if G admits a Hamiltonian path. □

We will show that the st-variants are NP-complete in the same restricted cases, that is, on planar graphs of small maximum degree.

Theorem 2. *Even on planar graphs with s-t being on the outerface, the followings hold: st-SUP is NP-complete. st-{SSP,LSP,LUP} are NP-complete for any constant $\ell \geq 0$ and $\Delta \geq 4$. For st-LSP additionally holds $k \geq 2$.*

Proof. We give a many-one reduction from Planar Cubic Hamiltonian Cycle (PCHC), which is PCHP where instead of asking for a Hamiltonian path, one asks for a Hamiltonian cycle.

Let $\mathcal{I} = (G)$ be an instance for PCHC, and let $c > 0$ be constant. We construct an instance $\mathcal{I}' = (G', s, t, k, \ell)$ as follows. Let G'' denote a copy of G, and let G' initially be G''. Consider a plane embedding of G such that x,y,z are incident to the outerface and y, z are neighbors of x. We add s and t to G, as well as the edges $\{s, x\}$ and $\{y, t\}, \{z, t\}$. Next, we add a set Z of c vertices to G' and make each vertex in Z adjacent only with s. Finally, we set $k = n+2$ and $\ell = c$. This finishes the construction of $\mathcal{I}'$. We exemplify the correctness via st-SSP.

Let G admit a Hamiltonian cycle C. As G is cubic, vertex x has three neighbors including y and z, at least one of them is connected to x in the cycle C. Assume it is y (for z the arguments work analogously). Then, we construct an s-t path P as follows. We set $V(P) := \{s, t, V(C)\}$. Next, we set $E(P) := \{\{s, x\}, \{y, t\}\} \cup (E(C) \setminus \{x, y\})$. On a high level, P is starting at s, going to x and following cycle C, starting at the neighbor of x not being y, and ending at y, and finally taking the edge from y to t. Clearly, P is an s-t path and contains $n+2$ vertices. Moreover, P is also ℓ-secluded as $N_{G'}(V(P)) = Z$. It follows that $\mathcal{I}'$ is a YES-instance.

Conversely, let G' admit a k-short ℓ-secluded s-t path P. Note that $Z \subseteq N_{G'}(V(P))$ (and hence $V(P) \cap Z = \emptyset$). Moreover, since $|Z| = c$, $Z = N_{G'}(V(P))$, that is, P has no neighbors outside of Z. Since $x \in V(P)$, needs to contain x, P must contain all vertices in $V(G'')$. Moreover, P needs to contain either edge $\{y,t\}$ or edge $\{z,t\}$. Assume it is $\{y,t\}$ (for $\{z,t\}$ the arguments work analogously). Let $P' \subseteq P$ be the subpath of P with $V(P') = V(G'')$. Then $C := (V(P'), E(P') \cup \{y,x\})$ forms a cycle in G'' containing every cycle exactly once. It follows that $\mathcal{I}$ is a YES-instance.

Observe that for st-LSP, $k = n+2$ forces the path to visit all vertices in G, and hence the statement follows.

For st-SUP, modify $\mathcal{I}'$ as follows. For each vertex in $v \in V(G'')$, add a two vertices and make them adjacent only with v. Denote the obtained graph by G_+. Set $k' := k = n+2$, and $\ell' = 2n+c$. Consider the instance $\mathcal{I}_+ := (G_+, s, t, k', \ell')$. With the same arguments as in the proof of Theorem 1, the statement follows. Note here that $\Delta(G'') = \max\{c, 7\}$.

For st-LUP, we can use $\mathcal{I}_- = (G_+, s, t, k'', \ell')$, where we can set $k'' = 1$. □

4. Parameterized Complexity

Lemma 1. *There is a many-one reduction that maps any instance (G, k, ℓ) of $\{S,L\} \times \{S,U\}$ PATH in polynomial time to an instance (G', s, t, k', ℓ') of its s-t variant such that $k' = k+2$ and $\ell' = 2(\binom{|V(G)|}{2} - 1) + \ell$.*

Proof. Given a non-trivial instance $\mathcal{I} := (G, k, \ell)$, construct instance $\mathcal{I}' := (G', s, t, k', \ell')$ as follows. Let G' initially only consist of the (isolated) vertices s and t. Next, for each pair $\{v,w\} \in \binom{V(G)}{2}$, add a copy $G_{v,w}$ of G to G' and make s adjacent with v and t adjacent with w. Observe that every s-t path in G' must contain—except for s and t—only vertices in exactly one copy of G in G'. Hence, every s-t path has $2(\binom{|V(G)|}{2} - 1)$ unavoidable neighbors.

Let $\mathcal{I}$ be a YES-instance and let P be a path with endpoints v and w forming a solution to $\mathcal{I}$. Consider the copy $G_{v,w}$ in G', and let P^* denote the copy of P in $G_{v,w}$. Then the path $P' = (V(P^*) \cup \{s,t\}, E(P^*) \cup \{\{s,v\},\{w,t\}\})$ in G' forms a solution to $\mathcal{I}'$ as $||V(P)| - |V(P')|| = |k - k'| = 2$ and $||N_G(V(P))| - |N_{G'}(V(P'))|| = |\ell - \ell'| = 2(\binom{|V(G)|}{2} - 1)$.

Let $\mathcal{I}'$ be a YES-instance and let P' be a path forming a solution to $\mathcal{I}$. Let $N_{P'}(s) = \{x\}$ and $N_{P'}(t) = \{y\}$. and let $G_{v,w}$ be the copy of G with $V(G_{v,w}) \cap V(P') \neq \emptyset$. Let P^* denote P' restricted to $G_{v,w}$, that is, the path with vertex set $V(G_{v,w}) \cap V(P')$. Let P be the copy of P^* in G. We have $||V(P)| - |V(P')|| = |k - k'| = 2$ and $||N_G(V(P))| - |N_{G'}(V(P'))|| = |\ell - \ell'| = 2(\binom{|V(G)|}{2} - 1)$, and hence, P forms a solution to $\mathcal{I}$. □

As the many-one reduction given in Lemma 1 is also a parameterized reduction regarding the solution size k, we get the following.

Corollary 1. *The s-t variants of $\{S,L\} \times \{S,U\}$ PATH are W[i]-hard with respect to k whenever its general version is W[i]-hard with respect to k, for every $i \geq 1$.*

On the other hand, with a similar idea as in Lemma 1, one can see that positive results for the st-variants propagate to its counter part.

Lemma 2. *Any instance $\mathcal{I} = (G, k, \ell)$ of $\{S, L\} \times \{S, U\}$ PATH can be decided in $\mathcal{O}(|V(G)|^2 f(|\mathcal{I}|))$-time by an algorithm having access to an oracle deciding st-$\{S, L\} \times \{S, U\}$ PATH in $f(|\mathcal{I}|)$-time.*

Proof. Let $\mathcal{I} := (G, k, \ell)$ be a non-trivial instance. We can test for each candidate pair for s and t, that is, we call the st-variant on instance (G', s, t, k, ℓ) for every $\{s, t\} \in \binom{V(G)}{2}$, where G' denotes a copy of G. Observe that $\mathcal{I}$ is a YES-instance if and only if there is at least one $\{s, t\} \in \binom{V(G)}{2}$ such that (G', s, t, k, ℓ) is a YES-instance. □

Due to Lemma 2 we obtain the following.

Corollary 2. *$\{S, L\} \times \{S, U\}$ PATH is in FPT when parameterized by k and/or by ℓ whenever its s-t variant is in FPT when parameterized by k and/or by ℓ, respectively.*

Due to Corollaries 1 and 2, for positive result, we encounter the st-variants, and for negative results, the general versions.

4.1. Upper Bounds

Observation 1. *st-SSP and st-SUP are contained in XP when parameterized by k.*

Proof. We can test all $\sum_{i=2}^{k} \binom{n}{i} \leq k\binom{n}{k}$ subsets $S \subseteq V$ with $2 \leq |S| \leq k$, where n denotes the number of vertices in the input graph. For each such subset S, we can test in polynomial time whether they form an ℓ-secluded st-path. □

Proposition 1. *st-SSP and st-SUP admit a linear-time $\mathcal{O}(\Delta^{k+1})$-vertex problem kernelization and hence are in FPT when parameterized by $\Delta + k$.*

Proof. Let $\mathcal{I} = (G, s, t, k, \ell)$ be an instance of st-SSP or st-SSP, where $\Delta := \Delta(G)$. Start a breadth-first search rooted in s, and stop when depth $k + 1$ is explored, taking $O(n+m)$ time. Let $N_G^{k+1}(s) = N_G^{=0}(s) \uplus N_G^{=1}(s) \uplus \cdots \uplus N_G^{=k+1}(s)$ denote the vertex set explored through this step, where $N_G^{=i}(s)$ denote the vertices found at depth i, $0 \leq i \leq k$. Moreover, it holds true that $|N_G^{=i+1}(s)| \leq \Delta \cdot |N_G^{=i}(s)|$ for all $0 \leq i \leq k$. It follows that $|N_G^{k+1}(s)| = \sum_{i=0}^{k+1} |N_G^{=i}(s)| \leq \sum_{i=0}^{k+1} \Delta^i \leq (k + 1)\Delta^{k+1}$. Analogously, start a breadth-first search rooted in t, and stop when depth $k + 1$ is explored, taking $O(n + m)$ time. Finally, consider the intersection $N_{st} := N_G^{k+1}(s) \cap N_G^{k+1}(t)$ where $|N_{st}| \leq (k + 1)\Delta^{k+1}$, and let $G' = G[N_{st}]$. Our kernel then consists of $\mathcal{I}' = (G', s, t, k, \ell)$. Observe that every k-short st-path in G cannot contain vertices in $(V(G) \setminus N_{st}) \cup (N_G^{=k+1}(s)) \cup N_G^{=k+1}(t))$. Hence, for every k-short st-path P it holds true that $N_G(P) \subseteq N_{st}$. We conclude that $\mathcal{I}$ is equivalent to $\mathcal{I}'$. □

Theorem 3. *st-SSP admits an $\mathcal{O}^*((k + \ell)^k)$ time algorithm and hence is in FPT when parameterized by $k + \ell$.*

Proof. Let $(G = (V, E), s, t, k, \ell)$ be an instance of *st*-SSP. We partition $V = R \uplus B$ such that $R := \{v \in V \mid \deg(v) \geq k+\ell+1\}$. Clearly, no k-short ℓ-secluded s-t path can contain any vertex from v. We admit a BFS-like branching tree algorithm as follows. Starting at s, consider all neighbors of s and branch on vertices from B but not from R, that is, only on vertices of degree at most $k+\ell$, and proceed recursively. Stop branching at depth $k-1$ (s is by convention at depth zero). Clearly, every k-secluded s-t path of length k is found in the branching, and we can verify in polynomial time whether the found path is also ℓ-secluded (return YES in this case). As we only branch on vertices from B, we have at most $(k+\ell)^k$ nodes in our branching tree. If the whole branching tree is explored without returning YES, then return NO. Hence, we can decide $\mathcal{I}$ for *st*-SUP in $\mathcal{O}^*((k+\ell)^k)$ time. □

Observe that since every instance (G, k, ℓ) of SUP is trivial if $\Delta \geq \ell$, we get that SUP is in FPT when parameterized by $k + \ell$. For *st*-SUP, tractability also holds true.

Theorem 4. *st*-SUP *admits an* $\mathcal{O}^*((\ell+1)^k)$*-time algorithm and hence is in FPT when parameterized by* $k + \ell$*.*

Proof. Let $\mathcal{I} = (G = (V, E), s, t, k, \ell)$ be an arbitrary but fixed input instance to *st*-SUP. Our FPT-algorithm consists of two phases.

In the first phase, we identify the set of vertices with high degree, formally, we consider the set $R := \{v \in V \mid \deg(v) \geq \ell + 2\}$. For each $v \in R$, we do the following. Check whether there is an k-secluded s-t path containing v. If yes, then we can return YES as $\mathcal{I}$ is a YES-instance: There is an k-secluded s-t path (of minimal length) containing v having at least ℓ neighbors. We can check whether there is a k-secluded s-t path containing v in polynomial time, by solving the following minimum-cost flow problem. We construct the following directed graph D as follows. Let D be initially empty. First, add a source vertex σ and a sink vertex τ. Next, for each vertex $w \in V$, add two vertices w_+ and w_-, as well as the arc (w_+, w_-) and set the cost and capacity to one. For each $\{u, w\} \in E$, add the two arcs (u_-, w_+) and (w_-, u_+), and set for each the cost to zero and the capacity to one. Next, add the arcs (s_-, τ) and (t_-, σ) with cost zero and capacity one. Finally, add the arc (σ, v_-) with cost zero and capacity two. We denote the set of vertices and the set of arc of D by W and A, respectively. We claim that D admits a flow of value two with cost at most $k-1$ if and only if there is a k-secluded s-t path containing v in G. Note that minimum-cost flow can be solved through e.g. linear programming.

($\Rightarrow$) Let D admits a flow f of value two with cost at most $k-1$. As all capacities are integral, we can assume that f is integral. Let $F = \{(w_+, w_-) \in A \mid f((w_+, w_-)) = 1\}$. Observe that $|F| \leq k-1$. We claim that $U = \{u \in V \mid (u_+, u_-) \in F\} \cup \{v\}$ forms a k-short s-t path P in G containing v. Note that since $f(a) \in \{0, 1\}$ for all $a \in A \setminus \{(\sigma, v_-)\}$, we can derive an v-s path P_s on the one hand, and an v-t path P_t on the other hand from f. Observe that by construction of D, P_s and P_t are vertex-disjoint. It follows that P is an s-t path P in G containing v. Finally, as $|U| = |F| + 1 \leq k$, we have that P is also k-short.

($\Leftarrow$) Let P be a k-secluded s-t path containing v in G. We denote $V(P) = \{u^1, \ldots, u_{k'}\}$ and $E(P) = \{\{u^i, u^{i+1}\} \mid 1 \leq i < k'\}$, where $u_1 = s$, $u_{k'} = t$

and $k' \leq k$. Note that there is some index $x \in [k']$ with $u^x = v$. We construct a function $f : A \to \{0,1,2\}$ as follows. Set $f((\sigma, v_-)) := 2$, $f((s_-, \tau)) := 1$ and $f((t_-, \sigma)) := 1$. Finally, set

$$f((u,u')) := \begin{cases} 1, & \text{if } \exists j \in [k'] \setminus \{x\} : (u,u') = (u^j_+, u^j_-) \text{ or} \\ & \exists j \in \{x, \dots, k'-1\} : (u,u') = (u^j_-, u^{j+1}_+) \text{ or} \\ & \exists j \in \{2, \dots, x\} : (u,u') = (u^j_-, u^{j-1}_+) \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, f is an σ-τ flow of value two. As $|V(P)| \leq k' \leq k$ and f assigns one to exactly $k'-1$ arcs of cost one each, f has cost at most $k-1$.

In the second phase, we admit a BFS-like branching-tree algorithm as follows. Note that the set $B := V \setminus R$ only consists of vertices of degree at most $\ell + 1$. Starting at s, consider all neighbors of s and branch on vertices from B but not from R, that is, only on vertices of degree at most $\ell + 1$, and proceed recursively. Stop branching at depth $k-1$ (s is by convention at depth zero). Clearly, every k-secluded s-t path of length k is found in the branching, and we can verify in polynomial time whether the found path is also ℓ-unsecluded (return YES in this case). As we only branch on vertices from B, we have at most $(\ell+1)^k$ nodes in our branching tree. If the whole branching-tree is explored without returning YES, then return NO. Hence, we can decide $\mathcal{I}$ for st-SUP in $\mathcal{O}^*((\ell+1)^k)$ time. □

4.2. Lower Bounds

In the previous section, we proved SSP to be solvable in $\mathcal{O}^*(2^{k\log(k+\ell)})$-time (Theorem 3) and SUP to be solvable in $\mathcal{O}^*(2^{k\log(\ell+1)})$-time (Theorem 4). Due to the reductions given in Theorem 1, assuming the *Exponential Time Hypothesis (ETH)* [9] holds true, we cannot essentially improve the running times for SHORT SECLUDED PATH and SHORT UNSECLUDED PATH regarding the parameter $k + \ell$.

Corollary 3. *Unless the ETH breaks, (st-)*SSP$(k+\ell)$ *and (st-)*SUP$(k+\ell)$ *are not solvable in* $\mathcal{O}^*(2^{o(k+\ell)})$*-time.*

Proof. In the many-reductions given in Theorem 1, we have that $k + \ell \in \mathcal{O}(n)$, where n denotes the number of vertices in the input graph. The statement then follows by the fact that HAMILTONIAN PATH is not solvable in $\mathcal{O}^*(2^{o(n)})$-time unless the ETH breaks [4]. □

Due to Proposition 1, we know that both SSP(k) and SUP(k) are contained in XP. Our two following results show that containment in FPT when parameterized by k only is excluded for SSP (unless FPT = W[1]) and for SUP (unless FPT = W[2]).

Theorem 5. SSP *is* W[1]*-hard with respect to* k*.*

In the following proof, we consider the CLIQUE problem: Given an undirected graph G and an integer $k \in \mathbb{N}$, decide whether G contains a k-clique, where a k-clique is a graph on at least k vertices such that each pair of vertices is adjacent. CLIQUE parameterized by the solution size k is a classical W[1]-complete problem [6, 7].

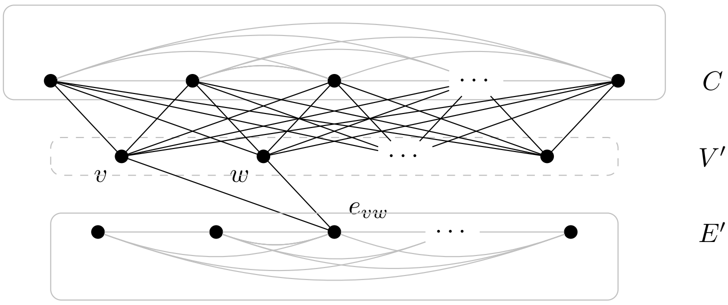

Figure 1: G'

Proof. We give an FPT-reduction from Clique parameterized by the solution size. Let $(G = (V, E), k)$ be an instance of Clique. We construct the instance (G', k', ℓ) of SSP as follows (refer to Figure 1 for an illustration).

Construction: Let G' be initially empty. We add a copy V' of V to G (if $v \in V$, we denote its copy in V' by v'). Moreover, for each $e \in E$, we add the vertex v_e to G' (denote the vertex set by E'). If $e = \{v, w\} \in E$, then we add the edges $\{v_e, v'\}$ and $\{v_e, w'\}$ to G'. Next, add the vertex set C consisting of $|E| + k + 1$ vertices to G'. Finally, make C and E' a clique. Set $k' = \binom{k}{2}$ and $\ell = |E| - k' + k$. This finishes the construction.

Correctness: We prove that G contains a k-clique if and only if G' admits a k'-short ℓ-secluded path.

($\Rightarrow$) Let G contain a k-clique $G[K]$ with $|K| = k$ and edge set $F \subseteq E$. Denote by K' and F' the vertices in V' and E' corresponding to K and F, respectively. Then construct the k'-short ℓ-secluded path P. Let P be an arbitrary ordering of the vertices in F except one. Recall that E' forms a clique, and hence P can be constructed this way. Note that P contains k' vertices. The neighborhood $N_{G'}(P)$ of P contains $|E| - k'$ vertices in E', and k vertices in V' (recall that K forms a clique in G). Hence, P is a k'-short ℓ-secluded path.

($\Leftarrow$) Let G' admit a k'-short ℓ-secluded path P. First, observe that P contains no vertex in $V' \cup C$, as otherwise $|N_{G'}(P)| \geq |E| + k + 1 - k' \geq \ell$, yielding a contradiction. Hence, P only contains vertices in E'. As P contains at most k' vertices and E' forms a clique in G', $|N_{G'}(P) \cap E'| \geq |E| - k'$. It follows that $N_{G'}(P)$ contains at most k vertices $K' \subseteq V'$. If $\binom{k}{2}$ edges are incident with k vertices, it follows that the vertex set K corresponding to K' forms a k-clique in G. □

Theorem 6. SUP *is* W[2]*-hard with respect to* k*.*

In the following proof, we consider the Red-Blue Dominating Set (RBDS) problem: Given an undirected graph $G = (V = R \uplus B, E)$ and an integer $k \in \mathbb{N}$, decide whether G contains a *red k-dominating set*, where a red k-dominating set is a subset $V' \subseteq R$ with $|V'| \leq k$ such that each vertex in B is adjacent to at least one vertex in V'. RBDS parameterized by the solution size k is a W[2]-complete problem [6, 7].

Proof. We give a many-one FPT-reduction from Red-Blue Dominating Set (RBDS) when parameterized by the solution size. Let $(G = (V = R \uplus B, E), k)$

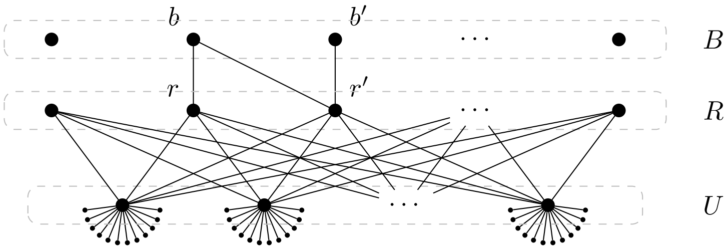

Figure 2: G'

be an instance of RBDS(k). We construct the instance (G', k', ℓ) of SUP as follows (refer to Figure 2 for an illustration).

Construction: Let G' be initially empty. Add a copy of G to G'. Next add the vertex set $U = \{u_1, \ldots, u_{k+1}\}$ to G'. Connect every vertex in R' with every vertex in U via an edge (i.e. $R \cup U$ forms a biclique). Finally, for each vertex $u \in U$, add n^2 vertices making each adjacent only to u. Denote by H all the vertices introduced in the previous step. Set $k' = 2k+1$ and $\ell = k \cdot n^2 + 2n - k$. This finishes the construction.

Correctness: We prove that G admits a red k-dominating set if and only if G' admits a k'-short ℓ-unsecluded path.

($\Rightarrow$) Let $W \subseteq V$ be a red k-dominating set in G with $|W| = k$. Let $W' = \{w'_1, w'_2, \ldots, w'_k\} \subseteq V'$ denote the vertices in V' corresponding to the vertices in W. We claim that the path $P := (u_1, w'_1, u_2, w'_2, \ldots, w'_k, u_{k+1})$ is a k'-short ℓ-unsecluded path in G'. First observe that the number of vertices in P is $k' = 2k + 1$. As W is a dominating set in G, $N_{G'}(W') = V'' \cup U$. Moreover, $N_{G'}(U) = V' \cup H$. As P consists exactly of the vertices in $W' \cup U$, we have $|N_{G'}(W' \cup U)| = |N_{G'}(W')| - |U| + |N_{G'}(U)| - |W'| = n + k \cdot n^2 + (n - k) = \ell$.

($\Leftarrow$) Let P be a k'-short ℓ-unsecluded path in G'. The first observation is that P contains all vertices from U as P has more than $k \cdot n^2$ neighbors. The second observation is that P needs to alternate between the vertices in V' and U as P only contains $k' = 2k + 1$ vertices and all of U. It follows that P contains exactly k vertices W' in V'. As $V(P) \cup N'_G(V(P)) = V(G')$, the vertex set W' dominate all the vertices in V''. It follows that the set $W \subseteq V$ corresponding to W' forms a k-dominating set in G'. ☐

We proved SSP($k + \ell$) (Theorem 3) and SUP($k + \ell$) (Theorem 4) to be contained in FPT. We next prove that, presumably, we do not expect any of the two problems to admit a problem kernel of polynomial size, even on planar graphs with small maximum degree.

Theorem 7. *Unless* coNP $\subseteq$ NP / poly, *(st-)*$\{S, L\} \times \{S, U\}$ Path *does not admit a polynomial problem kernel with respect to $k + \ell$ even on planar graphs with maximum degree seven.*

Proof. We employ the OR-composition framework [2]. An easy application (taking the disjoint union of the graphs) proves the statement for $\{S, L\} \times \{S, U\}$ Path. Hence, we next consider the *st*-variants. Let $\{\mathcal{I}_i = (G_i, s_i, t_i, k, \ell) \mid 1 \leq i \leq p\}$ be a set of p input instances, where p is a power of two, and G_i is planar, is of maximum degree five, and allows for an embedding with s, t being on the outer face.

(st-SSP) We construct the instance $\mathcal{I}' = (G', s, t, k', \ell')$ as follows. Let G' initially empty. We add two binary trees T_s and T_t with root s and t, respectively, where each tree has p leaves all being at the same depth. Let $\sigma_1, \ldots, \sigma_p$ denote the leaves of T_s enumerated through an post-order depth-first search. Similarly, let $\tau_1, \ldots, \tau_p$ denote the leaves of T_t enumerated through an post-order depth-first search. Next, for each $i \in [p]$, add copy G_i' of G_i to G', and add the edges $\{\sigma_i, s_i\}$ and $\{t_i \tau_i\}$. Finally, for each $i \in [p]$, subdivide the edges $\{\sigma_i, s_i\}$ and $\{t_i \tau_i\}$ each k times, and denote the vertices by $\sigma_i^1, \ldots, \sigma_i^k$ resulted from the subdivision from $\{\sigma_i, s_i\}$, enumerated by the distance from σ_i, and by $\tau_i^1, \ldots, \tau_i^k$ resulted from the subdivision from $\{\tau_i, t_i\}$, enumerated by the distance from t_i. For simplicity, we also denote σ_i and s_i by σ_i^0 and σ_i^{k+1}, respectively, and t_i and τ_i by τ_i^0 and τ_i^{k+1}, respectively, This finishes the construction of G'. Observe that one can embed both T_s and T_t such that when adding the edge set $\{\{\sigma_i, \tau_i\} \mid 1 \leq i \leq p\}$, the resulting graph is crossing free and s and t are on the outer face. As each G_i is planar and allows for an embedding with s, t being on the outer face, it follows that G' is planar with s and t being on the outer face. Moreover, note that $\Delta(G') \leq 1 + \max_{1 \leq i \leq p} \Delta(G_i)$. Finally, set $k' := 3k + 2(\log(p) + 1)$ and $\ell' := \ell + 2\log(p)$. We next prove that $\mathcal{I}'$ is a YES-instance if and only if there is at least one $i \in [p]$ such that $\mathcal{I}_i$ is a YES-instance.

($\Leftarrow$) Let $i \in [p]$ such that $\mathcal{I}_i$ is a YES-instance, and let P be an k-short ℓ-secluded s_i-t_i path in G. Let P' denote its copy in G_i'. Let $P_{s,i}$ denote the unique path with endpoints s and σ_i in T_s. Note that $|V(P_{s,i})| = \log(p) + 1$. Similarly, Let $P_{t,i}$ denote the unique path with endpoints t and τ_i in T_t. Note that $|N_{T_s}(P_{s,i})| = \log(p)$, as each vertex in $P_{s,i}$ except s and σ_i are of degree three in T_s, and s has one unique neighbor not in $P_{s,i}$. With the same argument, we have $|N_{T_t}(P_{i,t})| = \log(p)$. Let $V_P := V(P) \cup V(P_{s,i}) \cup V(P_{i,t}$ and $E_P := E(P) \cup E(P_{s,i}) \cup E(P_{i,t}) \cup \bigcup_{j=0}^{k} \{\{\sigma_i^j, \sigma_i^{j+1}\}\} \cup \bigcup_{j=0}^{k} \{\{\tau_i^j, \tau_i^{j+1}\}\}$. We claim that the path $Q = (V_P, E_P)$ is a k'-short ℓ'-secluded st-path in G'. By construction, Q is a k'-short st-path in G'. Moreover, we have $|N_{G'}(Q)| = |N_{T_s}(P_{s,i})| + |N_{T_t}(P_{i,t})| + |N_{G_i'}(P')| \leq 2\log(p) + \ell = \ell'$.

($\Rightarrow$) Let $\mathcal{I}'$ be a YES-instance, and let P be a k'-short ℓ'-secluded s-t path in G'. We claim that there is a subpath $P' \subseteq P$ such that P' is a k-short ℓ-secluded s_i-t_i path in G_i, for some $i \in [p]$. Observe that P must contain at least one leaf in T_s and one leaf in T_t. Hence, $|V(P) \cap V(T_s)| \geq \log(p) + 1$ and $|V(P) \cap V(T_s)| \geq \log(p) + 1$. Moreover, $s_i \in V(P)$ if and only if $t_i \in V(P)$, as P has only endpoints s and t, and $\{s_i, t_i\}$ separates $V(G_i) \setminus \{s_i, t_i\}$ from $V(G') \setminus V(G_i)$. Hence, let $i \in [p]$ such that $\sigma_i \in V(P)$ (and hence $\tau_i \in V(P)$). Let P' be the subpath of P with endpoints s_i and t_i. Clearly, $V(P') \subseteq V(G_i')$. We claim that P' is a k-short ℓ-secluded s_i-t_i path in G_i' (and hence, also in G_i). First, suppose $|V(P')| > k$. Then we have $|V(P)| \geq |V(P) \cap V(T_s)| + |V(P) \cap V(T_t)| + |V(P')| + 2k > 3k + 2(\log(p) + 1) = k'$, contradicting the fact that P is a k'-short s-t path in G'. Next, we claim that there is no $j \in [p] \setminus \{i\}$ such that $s_j \in V(P)$ (and hence, $t_j \in V(P)$). Suppose not. Then $|V(P)| \geq |V(P) \cap V(T_s)| + |V(P) \cap V(T_t)| + 4k > 3k + 2(\log(p) + 1) = k'$, again contradicting the fact that P is a k'-short s-t path in G'. It follows that $T_s[V(P) \cap V(T_s)]$ is the unique path in T_s with endpoints s and σ_i, and $T_t[V(P) \cap V(T_t)]$ is the unique path in T_t with endpoints t and τ_i. Moreover, $|N_{T_s}(V(P))| = |N_{T_t}(V(P))| = \log(p)$. Finally, suppose that $|N_{G'_i}(V(P'))| > \ell$. Then we have $|N_{G'}(V(P))| = |N_{T_s}(V(P))| + |N_{T_t}(V(P))| + |N_{G'_i}(V(P'))| > \ell + 2\log(p) = \ell'$, contradicting the fact that P is a ℓ'-secluded s-t path in G'. We conclude that P' is a k-short ℓ-secluded s_i-t_i path in G_i, and hence, $\mathcal{I}_i$ is a YES-instance.

(st-SUP) The construction is exactly the same as for st-SSP. The crucial observation is, again, that every k'-short ℓ'-unsecluded s-t path P in G' only contains s_i (and t_i) for exactly one $i \in [p]$.

(st-LSP) Let $\mathcal{I}'$ as in the construction for st-SSP. Make each vertex of the binary trees a star with $2\log(p) + \ell + 1$ leaves, and denote by G'' the graph obtained from G' in this step. Set $\ell'' := 2(\log(p)+1)\cdot(2\log(p)+\ell+1)+\ell+2\log(p)$. This forces every k'-long ℓ''-secluded s-t path P in G' to only contain $\log(p)$ vertices in each of the binary trees, as otherwise such a path P would contain at least $2(\log(p) + 1) \cdot (2\log(p) + \ell + 1) + (2\log(p) + \ell + 1) > \ell'$ neighbors.

(st-LUP) There is an straight-forward polynomial parameter transformation from LONGEST st-PATH on planar graphs with maximum degree three [2]. Note there herein, we set $\ell = 0$. □

5. Conclusion and Outlook

We conclude that in all four variants remain NP-complete in planar graphs with small vertex degree. The "short" and "long" variants are distinguishable through their parameterized complexity regarding k. We conjecture that all four variants are pairwise distinguishable through the parameterized complexity regarding the parameters k, ℓ, and $k + \ell$. To resolve this conjecture, the parameterized complexity of SUP(ℓ) and LUP($k + \ell$), that we left open, has to be settled.

As a further research direction, we find it interesting to investigate the problem of finding small/large secluded/unsecluded (sub-)graphs different to paths. For instance, the class of trees could be an interesting next candidate in this context. Note that herein, the large secluded variant is polynomial-time solvable.

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