The Complexity of Degree Anonymization by Graph Contractions*

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Abstract. We study the computational complexity of k-anonymizing a given graph by as few graph contractions as possible. A graph is said to be k-anonymous if for every vertex in it, there are at least $k-1$ other vertices with exactly the same degree. The general degree anonymization problem is motivated by applications in privacy-preserving data publishing, and was studied to some extent for various graph operations (most notable operations being edge addition, edge deletion, vertex addition, and vertex deletion). We complement this line of research by studying several variants of graph contractions, which are operations of interest, for example, in the contexts of social networks and clustering algorithms. We show that the problem of degree anonymization by graph contractions is NP-hard even for some very restricted inputs, and identify some fixed-parameter tractable cases.

1 Introduction

Motivated by concerns of data privacy in social networks, Clarkson et al. [10] introduced the general degree anonymization problem, defined as follows. Given an input graph G and an allowed operation O, the task is to transform G into a k-anonymous graph by performing as few O operations as possible; a graph is said to be k-anonymous if for every vertex in it, there are at least $k-1$ other vertices with exactly the same degree. This problem has been studied, both theoretically and practically, for several graph modification operations such as edge addition [10, 17, 20], edge deletion [8], vertex addition [9, 5], and vertex deletion [4]. This paper can be seen as complementing this line of research by considering graph contractions, as a natural graph modification operation, specifically, studying the (parameterized) complexity of this degree anonymization problem.

This paper also complements research done on the following problem: given an input graph $G=(V,E)$ and a family $\mathcal{F}$ of graphs, find a minimum-sized subset of edges $E' \subseteq E$, such that after contracting the edges in E', G would be in the family $\mathcal{F}$ (indeed, in our case, $\mathcal{F}$ is the family of all k-anonymous graphs). Asano and Hirata [1] defined a set of conditions on $\mathcal{F}$, which is sufficient for the NP-hardness of this problem. Others studied specific graph classes (as $\mathcal{F}$),

* A full version is available at `http://fpt.akt.tu-berlin.de/talmon/abcfv.pdf`.

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such as planar graphs [15], bipartite graphs [18], paths [18], trees [16], and d-regular graphs [3]. This last work is of particular interest, as the concept of k-anonymity is a generalization of the notion of regularity (in particular, a graph is n-anonymous if and only if it is regular).

Studying graph contractions in the context of degree anonymization is interesting for several reasons. First, some variants of contractions can preserve original properties of the input graph (for example, connectivity). Second, vertex contraction (where also non-adjacent vertices can be contracted), is the inverse operation of vertex cleaving (as defined by Oxley [22, Chapter 3]), which was studied in the context of degree anonymization by Bredereck et al. [5] (there, called *vertex cloning*). We mention also the relation of graph contractions to communities detection in social networks and to clustering (see, for example, Delling et al. [11]).

2 Preliminaries

We assume familiarity with standard notions regarding algorithms, computational complexity, and graph theory. For a non-negative integer z, we denote $\{1, \ldots, z\}$ by $[z]$.

2.1 Parameterized Complexity

An instance (I, k) of a parameterized problem consists of the "classical" problem instance I and an integer k being the *parameter* [13, 21]. A parameterized problem is called *fixed-parameter tractable* (FPT) if there is an algorithm solving it in $f(k) \cdot |I|^{O(1)}$ time, for an arbitrary computable function f only depending on the parameter k. In difference to that, algorithms running in $|I|^{f(k)}$ time prove membership in the class XP (clearly, $\mathsf{FPT} \subseteq \mathsf{XP}$). One can show that a parameterized problem L is (presumably) not fixed-parameter tractable by devising a *parameterized reduction* from a W[1]-hard or a W[2]-hard problem to L. A parameterized reduction from a parameterized problem L to another parameterized problem L' is a function that, given an instance (I, k), computes in $f(k) \cdot |I|^{O(1)}$ time an instance (I', k') such that $k' \leq g(k)$ and $(I, k) \in L \Leftrightarrow (I', k') \in L'$. A parameterized problem which is NP-hard even for instances for which the parameter is a constant is said to be Para-NP-hard.

2.2 Graph Theory and Contractions

Given a graph $G = (V, E)$, which may have self-loops and parallel edges, we denote the degree of a vertex $v \in V$ by $\deg(v)$, and define $B_d = \{v \in V : \deg(v) = d\}$ as the set of vertices of degree d (called the *block* of degree d). As usual, we define the degree of a vertex v with x neighbors and y self-loops to be $x + 2y$ (in particular, we count a self-loop twice). We define a *path-star* of degree d and length l to be the graph consisting of one center vertex, connected to d disjoint paths of length l each (indeed, this is a spider graph with equal-length legs). A *caterpillar-tree* is a tree for which removing the leaves and their incident edges leaves a path graph (formally, a path graph is a tree with no vertices of degree larger than 2).

Given an undirected graph $G = (V, E)$ and two adjacent vertices, u and v, contracting the vertices u and v (usually referred to as contracting the edge $e = \{u, v\}$), means removing u and v from V, replacing them by one new vertex (denoted by $u \oplus v$), adjacent to exactly those vertices that were adjacent to u, v, or to both. The resulting graph is denoted by G/e. In general, given a set of edges $E_1 \subseteq E$, we denote by G/E_1 the graph obtained from G after contracting all the edges in E_1. A graph $G = (V, E)$ is said to be l-contractible to a graph $G' = (V', E')$ if there is a set of edges $E_1 \subseteq E$ of size at most l, such that $G/E_1 = G'$. It follows that $G = (V, E)$ is contractible to $G' = (V', E')$ if and only if there exists a *witness structure* $V = V_1 \cup \ldots \cup V_{|V'|}$, where each V_i is called a *witness set*, such that for each V_i (for $1 \leq i \leq |V'|$) the subgraph of G induced by V_i is connected and for each pair of witness sets, V_i and V_j ($1 \leq i \neq j \leq |V'|$) we have that $\{V_i, V_j\} \in E' \iff \exists v_i \in V_i, v_j \in V_j : \{v_i, v_j\} \in E$ (indeed, the vertices in each part V_i are contracted to form a single vertex). We denote by $\deg(V_i)$ the resulting degree of the vertex corresponding to the contraction of the witness set and we call graph G' the witness graph.

We also define the closely related operation of *vertex contraction*, which is defined similarly to edge contraction, with the only difference that it is allowed to contract non-adjacent vertices as well (indeed, the vertices consisting a witness set of a vertex contracted graph are not assumed to be connected). It is clear that a graph contraction operation can sometimes introduce self-loops and parallel edges. We define three variants of edge contractions and vertex contractions, differing by how self-loops and parallel edges are treated:

- *Simple Contraction:* Both self-loops and parallel edges are removed.
- *Hybrid Contraction:* Only self-loops are removed.
- *Non-Simple Contraction:* Nothing is removed.

For the Hybrid and Non-Simple variants, we allow the input graph to be non-simple. See Figure 1 for some examples.

2.3 Main Problem

Given an undirected input graph G, we are interested in k-anonymizing it by performing at most c edge contractions (where a graph is said to be k-anonymous if every vertex degree in it occurs at least k times; equivalently, if $\forall i \in [n] : |B_i| = 0 \vee |B_i| \geq k$).

> Degree Anonymization by Graph Contractions
> **Input:** An undirected graph $G = (V, E)$, a budget $c \in \mathbb{N}$, and an anonymization level $k \in \mathbb{N}$.
> **Question:** Can G be made k-anonymous by performing at most c contractions?

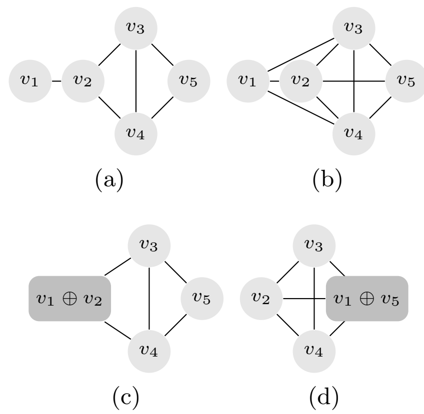

Figure 1: Example of 2-anonymizing an input graph. The input graph is depicted in (a), an optimal 2-anonymized graph with respect to edge addition is depicted in (b), an optimal 2-anonymized graph with respect to simple edge contraction or hybrid edge contraction is depicted in (c) (by contracting v_1 and v_2), and an optimal 2-anonymized graph with respect to non-simple vertex contraction is depicted in (d) (by contracting v_1 and v_5). Notice that there is no solution with respect to non-simple edge contraction, and the solution with respect to edge addition is less efficient than the solutions by edge contractions.

When the contraction operation is a simple (hybrid, non-simple) edge contraction operation, we denote the corresponding degree anonymization problem as SEC-A (respectively: HEC-A, NEC-A). Similarly, when the contraction operation is a simple (hybrid, non-simple) vertex contraction operation, we denote the corresponding degree anonymization problem as SVC-A (respectively: HVC-A, NVC-A).

Interestingly, it is not always possible to anonymize a graph by performing only graph contractions. As an example, consider n-anonymizing a complete graph which has one missing edge: as the input graph is not n-anonymized, at least one edge needs to be contracted, but then the number of remaining vertices will be strictly less than n, thus the graph cannot be further made n-anonymous. This phenomenon stands in contrast to anonymization by edge additions, as completing any graph, by adding all missing edges to it, makes it n-anonymous. However, some graphs can be anonymized more efficiently by using edge contractions rather than edge additions (see Figure 1 for an example).

2.4 Overview

We study the parameterized complexity of degree anonymization by graph contractions, considering the solution size c, the anonymity level k, and the maximum degree Δ, as the most natural parameters. From the variants defined in Section 2.3, we consider SEC-A and HEC-A as these are the most common (see, for example, Diestel [12, Chapter 1.7] and Wolle and Bodlaender [23]), and we consider NVC-A as it is equivalent to the underlying number problem (as defined in Section 3). We state some important points of our work:

- Contrary to degree anonymization by some other graph operations (for example, by edge addition), here even the underlying number problem (NVC-A) is NP-hard. Moreover, SEC-A, HEC-A, and NVC-A are NP-hard even on caterpillar trees.
- Parameterizing by either the solution size c, the maximum degree Δ, or the anonymity level k, does not help for tractability. However, combining Δ with c does help for tractability.
- Combining the maximum degree Δ with the anonymity level k helps for tractability for some variants of the problem, and we could show evidence suggesting otherwise for some other variants.

Table 1 gives an overview of our results. Due to space constraints, some of the proofs are omitted. Please refer to the full version (available at `http://fpt.akt.tu-berlin.de/talmon/abcfv.pdf`).

	solution size c	anonymization level k	maximum degree Δ
c	W-h[a] (Th. 3) XP (Obs. 1)	W-h[a] (Th. 3) XP (Obs. 1)	FPT (Th. 5)
k		Para-NP-h[a] (Th. 3)	FPT[b] (Cor. 1)
Δ			Para-NP-h[a] (Th. 4)

[a] Only for SEC-A and HEC-A.
[b] Only for NVC-A.

Table 1: Parameterized complexity landscape of Degree Anonymization by Graph Contractions. Rows and columns correspond to parameters, such that each cell corresponds to the combination of the corresponding parameters.

3 NP-hardness

We begin by considering NVC-A which, surprisingly, reduces to a number problem formed by the degrees in the input graph. This holds because (1) any two vertices can be contracted, and (2) the degree sequence of the resulting graph after performing a contraction only depends on the original degrees of the contracted vertices (this holds because self-loops and parallel edges are not removed). It follows that NVC-A is equivalent to the following number problem. Therein, a multiset of integers is k-anonymous if each integer in it occurs at least k times.

An equivalent formulation of NVC-A
Input: A set $V = \{d_1, \ldots, d_n\}$ of n integers ($\forall i : 0 \leq d_i \leq \Delta$) and two integers $k, c \in \mathbb{N}$.
Question: Is there a partition $V = \bigcup_{j \in [z]} V_j$ (where $V_{j_1} \cap V_{j_2} = \emptyset$ for $1 \leq j_1 \neq j_2 \leq z$) such that the multiset $S = \{\sum_{d_i \in V_j} d_i : j \in [z]\}$ is k-anonymous and $\sum_{j \in [z]}(|V_j| - 1) \leq c$?

Informally, the above number problem is in the heart of the graph anonymization problem (for this reason we call it the underlying number problem). Interestingly, contrary to the situation for other operations (such as edge addition), here the underlying number problem is intractable (for formal correctness, we define the input of this number problem to be in unary; this does not cause problems, as we next prove a reduction from a *strongly* NP-hard problem).

Theorem 1. NVC-A *is* NP*-hard even on caterpillar trees.*

Proof. We provide a reduction from the following strongly NP-hard problem [14]:

Strictly Three Partition
Input: A set of numbers $S = \{a_1, \ldots, a_{3m}\}$ such that $\sum_{a_i \in S} a_i = mB$ and $\forall i \in [3m] : B/4 < a_i < B/2$.
Question: Are there m disjoint sets $S_1, \ldots, S_m$, each of size 3, such that $\forall j \in [m] : \sum_{a_i \in S_j} a_i = B$?

Given an instance for Strictly Three Partition, we create an instance for NVC-A. Intuitively, the idea is to create a set of $3m$ vertices, such that each number a_i would have a corresponding vertex whose degree is proportional to a_i. Then, we will add a distinguished vertex with degree proportional to B, making sure that the only way of anonymizing the block containing this distinguished vertex is by contracting m triplets of vertices corresponding to triplets of numbers, each of sum exactly m. Details follow.

We first scale the input numbers. Specifically, we define $a'_i = a_i \cdot mB$ and $B' = B \cdot mB$. We set $k := m+1$ and $c := 2m$. Then, for each number a'_i, we create a node $v_{a'_i}$ and connect it to a'_i paths of length $c+1$ (consisting of new vertices), such that $\deg(v_{a'_i}) = a'_i$ holds for each i. We add a path-star of degree B' and length $c+1$ (indeed, G is a forest; we can easily transform it into a caterpillar tree by placing all $v_{a'_i}$'s on a path together with the path-star, adjusting the number of additional new vertices connected to each $v_{a'_i}$ accordingly). The correctness proof is omitted, due to space constraints. Please refer to the full version. $\square$

We can show NP-hardness on caterpillar trees also for SEC-A and HEC-A.

Theorem 2. SEC-A *and* HEC-A *are both* NP*-hard even on caterpillar trees.*

Proof. We provide a reduction from the following strongly NP-hard problem [14]:

Numerical Matching with Target Sums
Input: Three sets of integers $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, and $C = \{c_1, \ldots, c_n\}$.
Question: Can the elements of A and B be paired such that, for each $i \in [n]$, c_i is the sum of the ith pair?

The variant where all $3n$ input integers are distinct is also known to be NP-hard in the strong-sense [19]. Without loss of generality, we assume that all input integers are greater than 3. Given an instance for Numerical Matching with Target Sums, we create an instance for SEC-A and HEC-A. Intuitively, the idea is to create a set of $k-1$ vertices for each c_i and a pair of vertices for each pair of a_i and b_j, such that the only possibility of anonymizing the vertices corresponding to the c_i's is to contract the correct pairs of a_i's and b_j's. Details follow.

We set $k := n - 1$ and $c := n$. We construct some *c-gadgets:* for each c_i, we create $k-1$ path-stars of degree $c_i - 2$ and length $c+1$. We construct some *ab-gadgets:* for each pair of integers, $i \in [n]$ and $j \in [n]$, we create two path-stars, one of degree a_i and another of degree b_j, both of length $c+1$, and connect them by an edge (indeed, the construction as such is a forest; we can transform it into a tree by arbitrarily connecting each pair of disconnected components by a path of length $c+1$). The correctness proof is omitted due to space constraints. Please refer to the full version. □

4 Non-structural parameters

Following the hardness results from the last section, we continue our quest for tractability by considering non-structural parameters. First, we observe that for constant solution size c we can simply enumerate all possible solutions, and conclude the following.

Observation 1 Degree Anonymization by Graph Contractions *is* XP *with respect to c.*

However, there is no hope for fixed-parameter tractability with respect to this parameter, as even combining it with the anonymity level k does not help for tractability.

Theorem 3. *Both* SEC-A *and* HEC-A *are* NP*-hard and* W*-hard with respect to c, even if $k = 2$.*

Proof. For SEC-A, we provide a reduction from the following W[2]-hard problem, parameterized by the solution size [13]:

Set Cover
Input: Sets $S_1, \ldots, S_m$ containing elements from $x_1, \ldots, x_n$ and $h \in \mathbb{N}$.
Question: Is there a set of at most h sets covering all elements?

Given an instance for Set Cover, we create an instance for SEC-A. We set $k := 2$ and $c := h$. For each x_i we create a new vertex x'_i. For each S_j we create two new vertices, S'_j and S''_j, and connect them by an edge. Each S'_j and S''_j (corresponding to a set S_j) are connected to all x'_i's which correspond to elements $x_i \in S_j$. We add several paths of length $c+1$ to each x'_i such that the degree of each x'_i will be $f(i) = i(c+1)+2$. Similarly, we add several paths of length $c+1$ to each S'_j and S''_j, such that the degree of each S'_j and S''_j will be $f(n+1)$. For every $i \in [n]$ and $z \in [h]$, we add a path-star of degree $f(i) - z$ and length $c+1$. We add k path-stars of degree $f(n+1)$ and length $c+1$ in order to anonymize the vertices corresponding to the sets.

Given a set cover, it is possible to anonymize the input graph, by contracting together each pair of S'_j and S''_j which correspond to a set S_j in the cover: the degrees of each x'_i will decrease by the number of sets covering it, therefore the graph would be anonymized. For the other direction, notice that each x_i needs to be anonymized, Therefore, by a simple exchange argument, a solution must correspond to a set cover.

For HEC-A, we provide a reduction from the following W[1]-hard problem, parameterized by the solution size h (an h-coloring is a function $color : V \to [h]$, assigning to each vertex v a color $\mathrm{color}(v) \in [h]$) [13]:

Multi-Colored Clique
Input: An undirected graph $G = (V, E)$ and an h-coloring of its vertices.
Question: Is there a clique of size h including vertices of all h colors?

Cai [6] showed that Multi-Colored Clique remains hard even on regular graphs. We assume, without loss of generality, that there are no monochromatic edges. Given an instance for Multi-Colored Clique, we create an instance for HEC-A. We define the following function, $f(i) = 2^i \cdot 2\binom{h}{2}$, whose domain is the set of colors (that is, $i \in [h]$).

We set $k := 2$ and $c := h-1$. For every vertex v, we add $(f(\mathrm{color}(v)) - \deg(v))$ paths of length $c+1$ such that the degree of each vertex colored in color $i \in [h]$ is $f(i)$. We construct $k+1$ copies of this modified graph. We add $k-1$ path-stars of degree $((\sum_{i\in[h]} f(i)) - 2\binom{h}{2})$ and length $c+1$.

Given a multi-colored clique of size h, it is possible to contract the vertices of the clique into one vertex: the degree of the new vertex will be equal to the degree of the $k-1$ path-stars, resulting in an anonymized graph, due to the $k+1$ copies.

For the other direction, notice that contracting edges of a path-star does not change its degree. Moreover, as there are no monochromatic edges, we can only contract edges of different colors. Due to the way we defined $f(i)$, the only possible way of reaching the degree of the path-star (that is, $\sum_{i\in[h]} f(i) - 2\binom{h}{2}$) is by contracting a multi-colored clique, because all colors are needed for the first part (that is, $\sum_{i\in[h]} f(i)$) and all edges between the colors are needed for the second part (that is, $2\binom{h}{2}$). $\square$

5 Structural parameters

We go on to consider the maximum degree $\varDelta$ of the input graph, as a natural structural parameter. For example, the edge addition variant admits an FPT-algorithm and even a polynomial kernel with respect to $\varDelta$ [17]. In contrast to this, we next show that in case of edge contractions parameter $\varDelta$ alone does not help for tractability. The reductions, from Vertex Cover on Cubic Graphs (for SEC-A) and Partition into Triangles (for HEC-A) are omitted due to space constraints.

Theorem 4. *Both* SEC-A *and* HEC-A *are para-*NP*-hard with respect to $\varDelta$.*

Contrary to the above hardness results, combining the maximum degree with the solution size does help for tractability, for all variants of Degree Anonymization by Graph Contractions.

Theorem 5. Degree Anonymization by Graph Contractions *is* FPT *with respect to $(\varDelta, c)$.*

Proof. Consider a yes-instance for Degree Anonymization by Graph Contractions. There exists a set E' of at most c edges such that contracting them would result in a k-anonymous graph. Consider the set V' of vertices, containing all the endpoints of the edges in E', including also all of their neighbors (formally, $V' := N[\{u, v | \{u, v\} \in E'\}]$, where $N[V_1]$ denotes the closed neighborhood of $V_1 \subseteq V$). As each edge has two endpoints and each vertex has at most $\varDelta$ neighbors, it follows that $|V'| \leq 2c(\varDelta + 1)$. Consider the set V'' containing all vertices whose degree will be changed as a result of contracting the edges in E'. Roughly speaking, as it holds that $V'' \subseteq V'$, it is enough to find the subgraph induced by V'.

To this end, we consider all possible graphs H containing at most $2c(\varDelta + 1)$ vertices. For each such graph H, we consider all possible sets C of at most c edges to be contracted. For each such pair of a graph H and a set C, we compute the degree changes in H incurred by contracting the edges in C. If these degree changes make the graph k-anonymous, then we try to find this graph H as a subgraph in G. This step can be performed by using, for example, the result by Cai et al. [7]. ◻

We consider now the combined parameter $\varDelta$ and k. The situation here is more involved. First, for NVC-A, we can bound c in these parameters by a function dependent only on $\varDelta$ and k.

Lemma 1. *For any yes-instance (V, k, c) of* NVC-A *it holds that (V, k, c'), with $c' = k \cdot (\varDelta \cdot \varDelta!)^{\varDelta}$, is also a yes-instance.*

Proof. Let (V, k, c) be a yes-instance of NVC-A and denote by $c_{\text{opt}} \leq c$ the smallest number such that (V, k, c_{opt}) is still a yes-instance. Moreover, let the partition $P = \{V_1, \ldots, V_i\}$ of V be a solution which corresponds to c_{opt} (that is, P is the witness structure corresponding to a solution of (V, k, c_{opt})). In the following we define two operations on P with the property that applying each of them, when it is applicable, results in another solution with less than c_{opt} contractions. Since we show that at least one of them is applicable in case $c_{\mathrm{opt}} > k \cdot (\Delta \cdot \Delta!)^{\Delta}$, this proves Lemma 1.

To formally describe our operations, we associate with each witness set V_i a witness vector $\vec{v_i} \in \mathbb{N}^{\Delta}$ with $\vec{v_i}[j]$ being equal to the number of vertices of degree j in the witness set V_i. The *degree* of a witness set is defined to be the sum of the degrees of the vertices in the witness set (that is, the degree of the vertex corresponding to contracting all of the vertices in the witness set).

Operation 1: This operation is applicable to P if there are at least k witness sets in P, all of equal degree, such that in each of them, say V_i, there is at least one j with $\vec{v_i}[j] \geq \Delta!$. If there exists such a collection of witness sets, then consider such a collection P which is maximal with respect to containment, and do the following. For each witness set V_i in this collection, let j be an integer with $\vec{v_i}[j] \geq \Delta!$. remove $(\Delta!/j)$-many vertices of degree j from V_i (notice that $\Delta!/j$ is always an integer), and form a new witness set containing these vertices.

We introduced at least k new witness sets, all being of degree exactly $\Delta!$, and we decreased the degree of each of the initial witness sets by the same number $\Delta!$. Since there are at least k of such witness sets, it follows that performing this operation results in a partition of V that is still a solution for (V, k, c_{opt}), while requiring less edge contractions than P requires.

Operation 2: This operation is applicable to P if there is a collection of at least k witness sets in P, such that the witness sets in the collection all have the same witness vector, and this witness vector is of hamming weight of at least 2 (that is, these are not singletons). If such a collection exists, then choose an arbitrary integer j occurring in this witness vector. Then, for each witness set V_i in this collection, remove one vertex of degree j from V_i and form a new witness set containing only this vertex of degree j (that is, form a new singleton witness set).

Since there are at least k witness sets where a vertex of the same degree j is cut out from them, the resulting partition is a solution for V which requires less edge contractions than P requires.

Applicability: It remains to argue that in case of $c_{\mathrm{opt}} > k \cdot (\Delta \cdot \Delta!)^{\Delta}$, at least one of the two operations described above is applicable. First, assume that P contains a witness set V_i of degree at least $(\Delta \cdot \Delta!)$. Then, since P is k-anonymous, it holds that there are at least k witness sets of the same degree, which is at least $(\Delta \cdot \Delta!)$. It follows that each of these witness sets must contain at least one integer j which occurs at least $\Delta!$ times in it. Thus, Operation 1 is applicable.

So, let us assume now that the degree of each witness set in P is at most $(\Delta \cdot \Delta!)$. Then, we have that there are at most $(\Delta \cdot \Delta!)^{\Delta}$ different witness vectors, none of them with degree greater or equal to $(\Delta \cdot \Delta!)$. Hence, if P contains at least $k \cdot (\Delta \cdot \Delta!)^{\Delta}$ witness sets of size at least two, then Operation 2 is applicable.

Finally, a solution for which $c_{\mathrm{opt}} > k \cdot (\Delta \cdot \Delta!)^{\Delta}$ edge contractions have been performed either contains a set of size at least $(\Delta \cdot \Delta!)$ or it contains at least

$$\frac{k \cdot (\Delta \cdot \Delta!)^{\Delta}}{(\Delta \cdot \Delta!)} = k \cdot (\Delta \cdot \Delta!)^{\Delta}$$

witness sets of size at least two. □

Using the above Lemma, we can show the following.

Corollary 1. NVC-A *is* FPT *with respect to* (Δ, k).

Proof. For a given instance (V, k, c) of NVC-A we decide the instance $(V, k, \min\{c, k\cdot(\Delta\cdot\Delta!)^{\Delta}\})$ using the FPT-algorithm with respect to (Δ, c) (Theorem 5). By Lemma 1, these two instances are equivalent and the corresponding running time proves fixed-parameter tractability with respect to (Δ, k). □

6 Conclusion

We investigated the (parameterized) complexity of degree anonymization by several variants of graph contractions. We showed that most of the variants are intractable even on very restricted graph classes (indeed, even the underlying number problem) and we could identify some fixed-parameter tractable cases.

For further research, one could consider related graph operations, such as *structure contraction* (contracting a whole subgraph at unit cost), *edge twisting* (see [22, Chapter 3]), and *vertex dissolution* (see [22, Chapter 3]).

Bazgan and Nichterlein [2] studied graph anonymization with edge/vertex deletions from the viewpoint of approximation algorithms, while mainly obtaining inapproximability result for the variant of minimizing the number of edit operations. One way of extending this line of research would be to study whether their results transfer to edge contractions and to look at different notions of approximations. For example, partially anonymizing an input graph (only *some* of the vertices are anonymized) or almost anonymizing an input graph (for each vertex, there are at least $k-1$ other vertices of *roughly* the same degree).

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