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Multistage Committee Elections: Beyond Plurality Voting

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Zusammenfassung

Wir untersuchen die parametrisierte Komplexität mehrstufiger Komiteewahlen nach den Regeln Veto, Bloc, Borda, Copeland, Chamberlin-Courant und Monroe, wobei wir eine Abfolge von Komitees mit sich ändernden Abstimmungsprofilen im Laufe der Zeit wählen wollen. In unserer konservativen Variante verlangen wir, dass sich mindestens eine bestimmte Anzahl von Kandidaten in den aufeinanderfolgenden Ausschüssen überschneidet, während die revolutionäre Variante sicherstellt, dass sich zumindest einige Kandidaten in den aufeinanderfolgenden Ausschüssen unterscheiden. Wir zeigen, dass fast alle Varianten bereits für eine konstante Anzahl von Agenten NP-schwer sind, während die auf Veto, Bloc, Borda und Copeland basierenden Varianten für eine konstante Anzahl von Zeitschritten polynomial lösbar werden. Wir beweisen, dass alle Varianten in XP und die konservative Varianten W-schwer in Bezug auf die maximale Größe der zu wählenden Ausschüsse sind. Die revolutionäre Variante ist in Fällen von Veto, Bloc, Borda und Copeland einfacher zu berechnen, da sie polynomial lösbar wird, wenn die Grenze für symmetrische Differenzen konstant ist und die konservative Variante dahingegen für eine konstante Grenze NP-schwer bleibt.

Abstract

We study the parameterized complexity of multistage committee elections under Veto, Bloc, Borda, Copeland, Chamberlin-Courant and Monroe rules, where we aim to elect a sequence of committees under changing voting profiles over time. In our conservative variant, we require that at least a certain number of candidates overlap in the consecutive committees, whereas the revolutionary variant ensures that at least some candidates differ in the consecutive committees. We show that almost all variants are NP-hard even for a constant number of agents while the variants based on Veto, Bloc, Borda and Copeland become polynomial-time solvable for a constant number of time steps. We prove that all variants are in XP and that the conservative variants are W-hard regarding the maximum size of the committees to elect. The revolutionary variant is easier to compute in cases of Veto, Bloc, Borda and Copeland, since it becomes polynomial-time solvable if the bound on symmetric differences is constant. The conservative variant, however, remains NP-hard even for a constant bound.

Contents

Chapter 1

Introduction

There are various situations where societies hold elections to make collective decisions and since these situations have different characteristics and expectations, various types of elections and voting rules are used in different settings. While choosing a leader or a common meeting slot requires a singlewinner election, where a society chooses a single alternative from a set of candidates, there are also many situations where societies have to choose, instead of a single alternative, a subset of alternatives, which we call a committee. These so-called multiwinner elections include for instance parliamentary elections, various business decisions (e.g., an Internet store has to decide which products to show on its homepage), or shortlisting tasks (prior to deciding who should receive an award, typically there is a procedure that finds the finalists) [Fal+17].

The classic multiwinner elections have been subject of some research in the past years [CHV18; Elk+17; Fal+17], that focus on selecting a single committee considering a single voting profile. This typical setting disregards the time dimension and fails to prevent concentration of power and lack of knowledge transfer in case of successive elections. An online setting can help to avoid such problems in some situations, nevertheless planning with all information at hand from the beginning is also necessary in various scenarios, therefore, we use an offline setting.

We focus on a multistage model as in the work by Bredereck, Fluschnik, and Kaczmarczyk [BFK20], where we seek successive committees with a desired quality for a sequence of profiles and allow agents to change their preferences over time. The conservative variant of our model allows us to lower-bound the number of coinciding committee members between two consecutive committees, whereas the revolutionary variant allows us to lower-bound the number of differing ones.

In contrast to Bredereck, Fluschnik, and Kaczmarczyk [BFK20] who consider only a single voting rule, i.e., Single Non-Transferable Vote, we aim at continuing their work line by expanding their model for various voting rules following different principles. We define our central problem as follows:

Π MULTISTAGE R VOTING (ΠMRV)

Input: A set $A = \{a_1, \ldots, a_n\}$ of agents, a set $C = \{c_1, \ldots, c_m\}$ of candidates, a sequence of τ voting profiles, and three integers $k \in \mathbb{N}$, $\ell \in \mathbb{N}_0$, and $x \in \mathbb{N}$.

Question: Is there a sequence $(C_1, \ldots, C_\tau)$ of committees $C_t \subseteq C$ such that for all $t \in \{1, \ldots, \tau\}$ it holds true that $|C_t| \leq k$ and $\text{score}_t^{\text{R}}(C_t) \geq x$, and for all $t \in \{1, \ldots, \tau - 1\}$ it holds true that

$$|C_t \Delta C_{t+1}| \leq \ell \qquad (\Pi \equiv \text{CONSERVATIVE});$$
$$|C_t \Delta C_{t+1}| \geq \ell \qquad (\Pi \equiv \text{REVOLUTIONARY}).$$

This problem definition actually defines a family of problems that differ in the type of voting profile used and the way the scores are computed. Depending on the voting rule, two types of voting profiles are mainly used, approval ballots and ranked ballots. Approval ballots let agents express their preferences by either approving or disapproving each candidate, and some approval-based rules like Veto and Bloc specify the number of candidates an agent can approve, whereas ranked ballots allow agents to rank candidates according to their preferences. Table 2.1 formally shows which type of voting profiles are used and the way to compute the scores for the rules used in this thesis.

We believe that we have included quite a variety of multiwinner rules in this work. In fact multiwinner variants of classic singlewinner rules Veto, Borda and Copeland are called k-Veto[1], k-Borda and k-Copeland, and choose an exact k-sized committee with the maximal score. Since k is an upper bound in our model and included in the problem input, we use Veto, Borda and Copeland to point out rules based on them for the sake of readability and simplicity. All rules we considered are explained below:

Veto: Every agent approves all but one candidate, and a committee's score is equal to the sum of the approvals of the candidates in the committee.

Bloc: Every agent approves the same number of candidates as the desired committee size, and a committee's score is equal to the sum of the approvals of the candidates in the committee.

Borda: Every agent ranks the candidates in order of preference. A candidate's score from every agent is the number of candidates ranked below it. A committee's Borda score is equal to the sum of scores of the candidates in the committee.

Copeland$^\alpha$: Every agent ranks the candidates in order of preference and all candidates are compared pairwise. If the strict majority of agents rank one of them above another, winner gets a score of 1 and if there is a tie, both of them gets a score of α. A committee's Copeland score is equal to the sum of the scores of the candidates in the committee from pairwise comparisons.

The Chamberlin-Courant and Monroe rules are used in this work as Elkind et al. [Elk+17] and Faliszewski et al. [Fal+17] defined them. All of the following rules have an utilitarian and an egalitarian variant, which are notated with γ_U and γ_E, respectively, and if it is not especially notated, both variants are meant. The score of a committee

[1] The name k-Veto is also used for the related singlewinner voting rule, where every agent disapproves exactly k candidates as in the work by Lin [Lin11].

corresponds to the sum of the individual scores from all agents in the utilitarian variant and the lowest score from an agent in the egalitarian variant.

α-Chamberlin-Courant (α-CC): Every agent approves any number of candidates and if any of the approved candidates are in the committee, the agent gives a score of 1 to that committee.

β-Chamberlin-Courant (β-CC): Every agent ranks the candidates in order of preference. The score a committee receives from an agent is the number of candidates ranked below the committee member that the agent ranks highest. This scoring is similar to Borda rule, but only depends on the highest ranked candidates.

An *assignment function* $\pi : A \to C$ maps every agent to a candidate; $\pi(a_i)$ is interpreted as the member of committee $C' = \pi(A) = \{\pi(a_i) \mid a_i \in A\}$ that represents a_i under the assignment π. The assignment π satisfies the *Monroe criterion* if $\left\lfloor \frac{|A|}{|\pi(A)|} \right\rfloor \leq \left|\pi^{-1}(c)\right| \leq \left\lceil \frac{|A|}{|\pi(A)|} \right\rceil$ for all $c \in \pi(A)$. The set of all assignment functions for a committee C' is denoted by $\Phi(C') = \{\pi : A \to C \mid \pi(A) = C'\}$ and $\Phi^M(C') \subseteq \Phi(C')$ denotes the ones that satisfy Monroe criterion.

α-Monroe (α-M): Every agent approves any number of candidates. An agent gives a score of 1 to a committee C' under the assignment π, if the committee member representing the agent under π is approved by the agent. The Monroe score of C' is the best possible score under an assignment $\pi \in \Phi^M(C')$.

β-Monroe (β-M): Every agent ranks the candidates in order of preference. The score a committee C' gets from an agent under the assignment π is equal to the number of candidates ranked below the committee member representing the agent under π. The Monroe score of C' is t best possible score under an assignment $\pi \in \Phi^M(C')$.

We now give an example to demonstrate a real-world situation in which our model can be used. This also gives us a small overview on the bounds of conservative and revolutionary variants.

Example 1. A cinema with two screening rooms shows two movies from different genres every week. The cinema manager wants to choose two film genres for the next three weeks considering the preferences of four of their best customers. The customers are more likely to watch movies from their two most preferred genres. Their preferences change from week to week and are given in Figure 1.1. We use our model under Bloc rule and Bloc scores of the genres for each week are given in Figure 1.2. The manager wants to choose two genres with a total score of at least 5. Possible pairs of genres are A-C and A-D for Week 1, A-D for Week 2, C-D and C-H for Week 3.

If the manager wants to show at least one different genre from the previous week in one of the screening rooms, we can use our revolutionary model with a symmetric difference $\ell = 2$. In that case, they have to show movies from genres Action and Comedy in Week 1. If they show movies from genres Action and Drama in Week 1 instead, they cannot choose two genres with a total score of at least 5 and a symmetric difference of 2 in Week 2. If we apply our revolutionary model with $\ell = 2$, a possible solution is to show movies from A-C in Week 1, A-D in Week 2, C-H in Week 3.

The manager can also prefer to show at least one movie from one of the genres of previous week for consistency. We can then use our conservative variant with $\ell = 2$. In that case, they cannot show movies from genres Comedy and Horror in Week 3 after they show movies from genres Action and Drama in Week 2. If we apply our conservative

Customer	Week 1	Week 2	Week 3
Ann	C, D	A, D	A, C
Bob	A, C	A, H	D, H
Chuck	A, D	C, D	C, H
Don	A, H	A, D	C, D

Figure 1.1: Possible film genres are action A, comedy C, drama D, and horror H. Two most preferred (approved) genres of the customers is given for three weeks.

Genre	Week 1	Week 2	Week 3
Action	3	3	1
Comedy	2	1	3
Drama	2	3	2
Horror	1	1	2

Figure 1.2: The bloc scores of the film genres according to the preferences given in Figure 1.1.

model with $\ell = 2$, a possible solution is to show movies from A-D in Week 1, A-D in Week 2, C-D in Week 3.

1.1 Related work

Our work is closely related to the work by Bredereck, Kaczmarczyk, and Niedermeier [BKN20], who added time dimension to the classical multiwinner voting. However, they do not allow to set any limit on how many candidates can differ between successive committees and the agents cannot change the ballots they cast over the time. The work by Bredereck, Fluschnik, and Kaczmarczyk [BFK20] and ours use the same multistage model and only differ in the voting rules investigated. Hence, their work is the closest related to ours and we expand it by adapting their model to other multiwinner voting rules. Since we use a multistage model, other recently studied multistage problems [Bam+18; CTW20; Flu+20a; Flu+20b; GTW14] are also somewhat related.

The works by Freeman, Zahedi, and Conitzer [FZC17], and Parkes and Procaccia [PP13] are also based on a dynamic setting, that allows evolving voting profiles and successive elections. However, both works focus on singlewinner elections and use an online setting, despite the offline setting we use. Furthermore, agents report utilities instead of ballots in the former and are allowed to change their ballots to a limited extent in the latter.

We should also mention Aziz and Lee [AL18], who studied so-called subcommittee voting; they aimed to choose several subcommittees to form a single bigger committee. Nevertheless, their model does not have a time dimension and the subcommittees they choose are necessarily disjoint.

1.2 Our Contributions and Organization

We study the classical and parameterized complexity of multiwinner committee elections for the multiwinner rules mentioned above. We give an overview of our results in Tables 1.1 and 1.2 and the results for conservative and revolutionary variants are visualized in Figure 1.3. We start with introducing necessary definitions and notations in Chapter 2. In Chapter 3, we prove the NP-hardness and the para-NP-hardness regarding n of almost all our problem variants and the para-NP-hardness of the conservative variants regarding ℓ. In Chapter 4, we give an XP-algorithm with respect to parameter k and show that all variants are fixed-parameter tractable when parameterized by m. We present an XP-algorithm in Chapter 5 to show that the revolutionary variants are in XP with respect to ℓ. Although the Chamberlin-Courant and Monroe rules are para-NP-hard regarding τ, we discuss the parameterized complexity with respect to τ for other rules in Chapter 6 and give a dynamic programming algorithm to show that they are contained in XP. We conclude this thesis by summarizing our results and discussing the possible further research directions in Chapter 7.

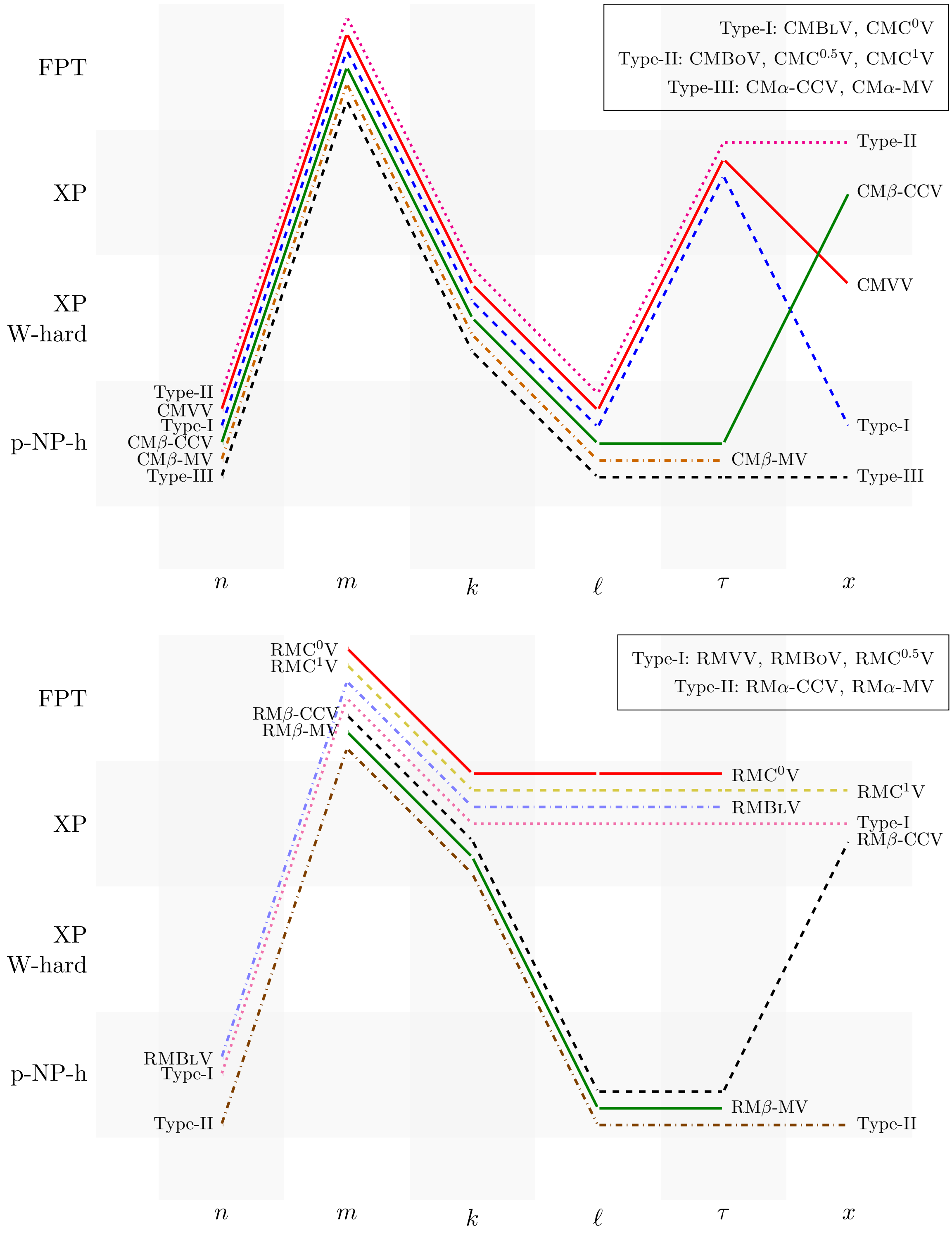
Table 1.1: The first part of overview of our results. First two columns describe the problems, parameterized complexity results of which are given in the following columns regarding the parameter indicated at the top. The voting rules are given in the first column. C stands for conservative variant and R stands for revolutionary variant in the second column. The following abbreviations are used for complexity classes: p-NP-h stands for para-NP-hard, W[1]-h stands for W[1]-hard and W[2]-h stands for W[2]-hard.

R	Π	n	m	k		ℓ	τ	x
Veto	C	p-NP-h Thm. 3.3	FPT Cor. 4.3	XP Thm. 4.1	W[1]-h Thm. 3.3	p-NP-h Thm. 3.3	XP Thm. 6.1	XP Cor. 4.7 W[1]-h Thm. 3.3
	R					XP Thm. 5.1		XP Cor. 4.7
Bloc	C	p-NP-h Thm. 3.9			W[2]-h Thm. 4.8	p-NP-h Thm. 3.9		p-NP-h Thm. 4.8
	R					XP Thm. 5.1		open
Borda	C	p-NP-h Thm. 3.13			W[2]-h Thm. 4.9	p-NP-h Thm. 3.13		XP Cor. 4.7
	R					XP Thm. 5.1		
Copeland^0	C	p-NP-h Thm. 3.13			W[2]-h Thm. 4.9	p-NP-h Thm. 3.13		p-NP-h Thm. 3.13
	R	open				XP Thm. 5.1		open
Copeland$^{0.5}$	C	p-NP-h Thm. 3.13			W[2]-h Thm. 4.9	p-NP-h Thm. 3.13		XP Cor. 4.7
	R					XP Thm. 5.1		
Copeland^1	C	p-NP-h Thm. 3.13			W[2]-h Thm. 4.9	p-NP-h Thm. 3.13		
	R	open				XP Thm. 5.1		

Table 1.2: The second part of overview of our results. First two columns describe the problems, parameterized complexity results of which are given in the following columns regarding the parameter indicated at the top. The voting rules are given in the first column. C stands for conservative variant and R stands for revolutionary variant in the second column. The following abbreviations are used for complexity classes: p-NP-h stands for para-NP-hard, W[1]-h stands for W[1]-hard and W[2]-h stands for W[2]-hard.

R	Π	n	m	k		ℓ	τ	x
α-CC	C	p-NP-h Thm. 3.19	FPT Cor. 4.3	XP Thm. 4.1	W[2]-h Thm. 3.19	p-NP-h Thm. 3.19	p-NP-h Cor. 3.18	p-NP-h Thm. 3.19
	R					p-NP-h Prop. 5.5		
β-CC	C	p-NP-h Thm. 3.25			W[2]-h Thm. 3.25	p-NP-h Thm. 3.25		XP Cor. 4.7
	R	open				p-NP-h Prop. 5.5		
α-M	C	p-NP-h Thm. 3.19			W[2]-h Thm. 3.19	p-NP-h Thm. 3.19		p-NP-h Thm. 3.19
	R					p-NP-h Prop. 5.5		
β-M	C	p-NP-h Thm. 3.25			W[2]-h Thm. 3.25	p-NP-h Thm. 3.25		open
	R	open				p-NP-h Prop. 5.5		

Figure 1.3: The first diagram depicts the results for the conservative variants and the second diagram depicts the results for the revolutionary variants.



Chapter 2

Preliminaries

In this chapter, we first introduce the notations used in the thesis and provide a table containing an overview of voting rules studied in the thesis with the corresponding voting profiles and score computations. Then, we present the NP-hard problems and their so-called *half* variants.

We denote by $\mathbb{N}$ and $\mathbb{N}_0$ the natural numbers excluding and including zero, respectively. For a set S, $\mathcal{P}(S)$ denotes the set of all subsets of S, including the empty set and S itself and $\mathcal{P}_{=n}(S)$ denotes the set of subsets of S of cardinality exactly $n \in \mathbb{N}$.

Graph theory. An *undirected graph* is a pair $G = (V, E)$, with a set V of vertices and a set $E \subseteq \{\{v, w\} \mid v, w \in V, v \neq w\}$ of edges. A *directed graph* is a pair $D = (V, A)$, with a set V of vertices and a set $A \subseteq \{(v, w) \mid v, w \in V, v \neq w\}$ of arcs.

Let $D = (V, A)$ be a directed graph with $s, z \in V$. An *s-z path* in D is a sequence of vertices $(v_1, \ldots, v_n)$ with $v_1 = s$, $v_n = z$, $v_i \in V$ for all $i \in \{1, \ldots, n\}$, $(v_i, v_{i+1}) \in A$ for all $i \in \{1, \ldots, n-1\}$, and $v_i \neq v_j$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

Parameterized complexity theory. Let Σ be a finite alphabet. A *parameterized problem* is a language $L \subseteq \Sigma^* \times \mathbb{N}$. A parameterized problem L is *fixed-parameter tractable* if for any given instance (x, k) it can be decided whether $(x, k) \in L$ (`yes`-instance) or $(x, k) \notin L$ (`no`-instance) in $f(k) \cdot |x|^{\mathcal{O}(1)}$ time, where f is some computable function. The complexity class of fixed-parameter tractable problems is called FPT. The class XP contains the parameterized problems that can be solved in $|x|^{f(k)}$ time for any given instance (x, k), where f is some computable function. The problems in XP are also called *slicewise polynomial.*

W-hierarchy is a collection of classes of parameterized problems with

$$\text{FPT} = \text{W}[0] \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \ldots \subseteq \text{W}[P] \subseteq \text{XP}$$

and these inclusions are believed to be strict.

A parameterized problem is para-NP-hard with respect to parameter k if it is NP-hard even for some constant value of k. A parameterized problem in XP can only be para-NP-hard if P=NP.

Let P and Q be two parameterized problems. A *parameterized reduction* from P to Q is a function that computes from an instance (x, k) of P in $f(k) \cdot |x|^{\mathcal{O}(1)}$ time an instance (x', k') of Q, such that $(x, k) \in P$, if and only if $(x', k') \in Q$, and $k' \leq g(k)$, where f and g are some computable functions.

Social choice theory. We adapt basic notations from computational social choice [Bra+16] and multiwinner voting [Elk+17] to the multistage setting.

Let $A = \{a_1, \ldots, a_n\}$ denote a set of agents (also called voters) and $C = \{c_1, \ldots, c_m\}$ denote a set of candidates. An *approval ballot* cast by agent a_i at time step t is denoted by $u_t(a_i)$ and is a subset of candidates C, and contains the candidates that are approved by a_i at time step t. A *ranked ballot* cast by agent a_i at time step t is a strict linear order $\succ_{t,i}$ of C: $\succ_{t,i}$ is *transitive* (if $x \succ_{t,i} y$ and $y \succ_{t,i} z$ then $x \succ_{t,i} z$ for all $x, y, z \in C$), *semiconnex* ($x \succ_{t,i} y$ or $y \succ_{t,i} x$ or $x = y$ for all $x, y \in C$), *irreflexive* ($x \not\succ_{t,i} x$ for all $x \in C$) and *antisymmetric* (if $x \succ_{t,i} y$ and $x \neq y$ then $y \not\succ_{t,i} x$ for all $x, y \in C$).A *voting profile* $u_t : A \to \mathcal{P}(C)$ or $p_t = (\succ_{t,1}, \ldots, \succ_{t,n})$ specifies approval ballots or ranked ballots for each agent $a_i \in A$ at time step t. The *position* of a candidate $c \in C$ in the ballot $\succ_{t,i}$ is denoted by $\operatorname{pos}_{t,i}(c) = |\{c' \in C \mid c' \succ_{t,i} c\}| + 1$. A ballot $\succ_{t,i}$ is *single-peaked* with respect to a strict linear order $\sqsupset$ of C, if $d \succ_{t,i} e \succ_{t,i} f$ for $d = \arg\min_{c \in C} \operatorname{pos}_{t,i}(c)$ and for all $d, e \in C$ such that $c \sqsupset c' \sqsupset c''$ or $c'' \sqsupset c' \sqsupset c$. A profile is *single-peaked* if there exist a common linear order $\sqsupset$ on C such that every ballot of the profile is single-peaked with respect to $\sqsupset$. A profile is *single-crossing* with respect to a strict linear order of A with an ordered list $(a_1, \ldots, a_n)$, if for each couple $\{c, c' \in C \mid c \neq c'\}$ there is a $j \in \{1, \ldots, n\}$, such that $c \succ_{t,i} c'$ if $i \leq j$ and $c' \succ_{t,i} c$ if $i > j$ or $c' \succ_{t,i} c$ if $i \leq j$ and $c \succ_{t,i} c'$ if $i > j$. The *pairwise majority* relation $c >^{\mu}_{t} c'$ denotes that a strict majority of agents rank c over c' at time step t, i.e., $|\{a_i \in A \mid c \succ_{t,i} c'\}| > |\{a_i \in A \mid c' \succ_{t,i} c\}|$, and $c =^{\mu}_{t} c'$ denotes that the same number of agents rank c over c' at time step t as the number of agents that rank c' over c, i.e., $|\{a_i \in A \mid c \succ_{t,i} c'\}| = |\{a_i \in A \mid c' \succ_{t,i} c\}|$.

Further problems. We briefly discuss the following NP-hard problems and their special variants, from which we give polynomial-time many-to-one reductions throughout the thesis.

Vertex Cover

Input: An undirected graph G and an integer $k \in \mathbb{N}$.

Question: Is there a vertex set $X \subseteq V(G)$ such that $|X| \leq k$ and $e \cap X \neq \emptyset$ for all $e \in E(G)$?

Independent Set

Input: An undirected graph G and an integer $k \in \mathbb{N}$.

Question: Is there a vertex set $X \subseteq V(G)$ such that $|X| \geq k$ and $e \nsubseteq X$ for all $e \in E(G)$?

HITTING SET

Input: A set E of elements, a collection $\mathcal{C}$ of subsets of E and an integer $k \in \mathbb{N}$.

Question: Is there a set $X \subseteq E$ of elements such that $|X| \leq k$ and $S \cap X \neq \emptyset$ for all $S \in \mathcal{C}$?

We note that INDEPENDENT SET is known to be W[1]-complete [DF95] when parameterized by k and HITTING SET is W[2]-complete [DF12] when parameterized by k. HALF VERTEX COVER, HALF INDEPENDENT SET, and HALF HITTING SET are special variants of the problems mentioned above, where k is set to half the number of vertices or elements. As Bredereck, Fluschnik, and Kaczmarczyk [BFK20] also mentioned, it is not difficult to see that HALF VERTEX COVER is NP-complete: We can reduce any instance (G, k) of VERTEX COVER to HALF VERTEX COVER by adding a clique on $|V(G)| - 2k + 2$ vertices to G if $k < |V(G)|/2$, or adding enough isolated vertices to G until $k = |V(G)|/2$. Similarly, we can reduce any instance (G, k) of INDEPENDENT SET to HALF INDEPENDENT SET by adding a clique on $2k + 2 - |V(G)|$ vertices to G and increasing k by 1 if $k > |V(G)|/2$, or by adding enough isolated vertices to G and increasing k by the number of added vertices until $k = |V(G)|/2$. We can also reduce any instance $(E, \mathcal{C}, k)$ of HITTING SET to HALF HITTING SET. If $k < |E|/2$, we add $|E| - 2k$ elements to E, increase k by $|E| - 2k$, and add $\{e\}$ to $\mathcal{C}$ for every added element e. If $k > |E|/2$, we just add elements to E until $k = |V(G)|/2$.

Corollary 2.1. HALF VERTEX COVER, HALF INDEPENDENT SET, *and* HALF HITTING SET *are* NP-*complete.*

Table 2.1: Voting rules studied in this thesis with the corresponding voting profiles and formulas to compute the score of any committee C' at time step t.

R	Voting profile	$\text{score}_t^{\text{R}}(C')$
VETO	$u_t : A \to \mathcal{P}_{=m-1}(C)$	$\sum_{a_i \in A} \lvert u_t(a_i) \cap C' \rvert$
BLOC	$u_t : A \to \mathcal{P}_{=k}(C)$	$\sum_{a_i \in A} \lvert u_t(a_i) \cap C' \rvert$
BORDA	$p_t = \{\succ_{t,1}, \ldots, \succ_{t,n}\}$	$\sum_{c \in C'} \sum_{a_i \in A} \big\lvert \{c' \in C \mid c \succ_{t,i} c'\} \big\rvert$
COPELAND$^\alpha$	$p_t = \{\succ_{t,1}, \ldots, \succ_{t,n}\}$	$\sum_{c \in C'} \Big(\big\lvert \{c' \in C \mid c >_t^\mu c'\} \big\rvert + \alpha \big\lvert \{c' \in C \setminus \{c\} \mid c =_t^\mu c'\} \big\rvert \Big)$
γ_E-α-CC	$u_t : A \to \mathcal{P}(C)$	$\begin{cases} 0, \text{ if } \lvert \{a_i \in A \mid u_t(a_i) \cap C' \neq \emptyset\} \rvert < \lvert A \rvert, \\ 1, \text{ if } \lvert \{a_i \in A \mid u_t(a_i) \cap C' \neq \emptyset\} \rvert = \lvert A \rvert \end{cases}$
γ_U-α-CC	$u_t : A \to \mathcal{P}(C)$	$\big\lvert \{a_i \in A \mid u_t(a_i) \cap C' \neq \emptyset\} \big\rvert$
γ_E-β-CC	$p_t = \{\succ_{t,1}, \ldots, \succ_{t,n}\}$	$\min_{a_i \in A} \max_{c \in C'} \big\lvert \{c' \in C \mid c \succ_{t,i} c'\} \big\rvert$
γ_U-β-CC	$p_t = \{\succ_{t,1}, \ldots, \succ_{t,n}\}$	$\sum_{a_i \in A} \max_{c \in C'} \big\lvert \{c' \in C \mid c \succ_{t,i} c'\} \big\rvert$
γ_E-α-M	$u_t : A \to \mathcal{P}(C)$	$\begin{cases} 0, \text{ if } \max_{\pi \in \Phi^M(C')} \lvert \{a_i \in A \mid \pi(a_i) \in u_t(a_i)\} \rvert < \lvert A \rvert, \\ 1, \text{ if } \max_{\pi \in \Phi^M(C')} \lvert \{a_i \in A \mid \pi(a_i) \in u_t(a_i)\} \rvert = \lvert A \rvert \end{cases}$
γ_U-α-M	$u_t : A \to \mathcal{P}(C)$	$\max_{\pi \in \Phi^M(C')} \lvert \{a_i \in A \mid \pi(a_i) \in u_t(a_i)\} \rvert$
γ_E-β-M	$p_t = \{\succ_{t,1}, \ldots, \succ_{t,n}\}$	$\max_{\pi \in \Phi^M(C')} \min_{a_i \in A} \lvert \{c \in C \mid \pi(a_i) \succ_{t,i} c\} \rvert$
γ_U-β-M	$p_t = \{\succ_{t,1}, \ldots, \succ_{t,n}\}$	$\max_{\pi \in \Phi^M(C')} \sum_{a_i \in A} \lvert \{c \in C \mid \pi(a_i) \succ_{t,i} c\} \rvert$

Chapter 3

Computational Hardness

In this chapter, we analyze the classical computational complexity of conservative and revolutionary variants of our problems and prove that most of them are NP-complete, even for small number of agents. We begin by showing that they are all in NP. The only non-trivial part of a certificate is calculating the scores.

Algorithms A.1 to A.5 are polynomial-time algorithms that compute scores in the cases of Veto, Bloc, Borda, Copeland, α-Chamberlin-Courant and β-Chamberlin-Courant rules (they can be found in Appendix A). It is not straightforward to compute the Monroe score of a committee, since we first seek for an assignment that satisfies the Monroe criterion and maximizes the score. Betzler, Slinko, and Uhlmann [BSU13] and Procaccia, Rosenschein, and Zohar [PRZ08] show that an assignment can be found and the score can be computed in polynomial-time in the number n of agents and the number m of candidates for α-Monroe and β-Monroe, respectively.

Corollary 3.1. *The score* $\text{score}_t^{\text{R}}(C)$ *of any committee* C *at any time step* t *can be computed*

(i) in $\mathcal{O}(nm^2)$ *time if* $\text{R} \in \{$Veto, Bloc, Borda, Copeland$^\alpha$, α-Chamberlin-Courant, β-Chamberlin-Courant$\}$*,*

(ii) in $\mathcal{O}(poly(n,m))$ *time if* $\text{R} \in \{\alpha$-Monroe, β-Monroe$\}$*.*

Now, we are set to prove that all problem variants studied in this thesis are in NP.

Lemma 3.2. Π Multistage R Voting *is in* NP *for all* $\text{R} \in \{$Veto, Bloc, Borda, Copeland, α-Chamberlin-Courant, β-Chamberlin-Courant, α-Monroe, β-Monroe$\}$*.*

Proof. A certificate for ΠMRV for all $\text{R} \in \{$Veto, Bloc, Borda, Copeland, α-Chamberlin-Courant, β-Chamberlin-Courant, α-Monroe, β-Monroe$\}$ is a sequence $(C_1, \ldots, C_\tau)$ of committees and verified if it holds true that

$$|C_t| \leq k \qquad \text{for all } t \in \{1, \ldots, \tau\}; \tag{3.1}$$

$$\text{score}_t^{\text{R}}(C_t) \geq x \qquad \text{for all } t \in \{1, \ldots, \tau\}; \tag{3.2}$$

$$|C_t \Delta C_{t+1}| \leq \ell \qquad \text{for all } t \in \{1, \ldots, \tau - 1\} \qquad (\Pi \equiv \text{Conservative}); \tag{3.3}$$

$$|C_t \Delta C_{t+1}| \geq \ell \qquad \text{for all } t \in \{1, \ldots, \tau - 1\} \qquad (\Pi \equiv \text{Revolutionary}). \tag{3.4}$$

Clearly, (3.1), (3.3) and (3.4) can be checked in polynomial time. Corollary 3.1 shows that (3.2) can also be checked in polynomial time. □

3.1 Veto

The first rule we studied is Veto and we start with exploring the classical computational complexity of Conservative Multistage Veto Voting (CMVV). Then we continue with an observation that sets bounds for the possible score of a committee. Lastly, by reducing the conservative variant to the revolutionary variant, we settle the classical computational complexity of Revolutionary Multistage Veto Voting (RMVV). We show that both variants are NP-hard even for two agents.

Theorem 3.3. Conservative Multistage Veto Voting *with* $\ell = 0$ *and* Revolutionary Multistage Veto Voting *with* $\ell = 2k$ *are* NP*-complete even for two agents, and* Conservative Multistage Veto Voting *is* W[1]*-hard when parameterized by* k *and* x *even for two agents and* $\ell = 0$.

First, we prove the NP-hardness and the W[1]-hardness with respect to k and x of CMVV giving a parameterized reduction from Independent Set. Then we prove the NP-hardness of RMVV reducing CMVV to RMVV.

Lemma 3.4. *From an instance* (G, k) *of* Independent Set*, one can compute in polynomial time an equivalent instance* (A, C, U, k, ℓ, x) *of* Conservative Multistage Veto Voting *with* $|A| = 2$, $\ell = 0$ *and* $x = 2k - 1$.

Proof. Let $I = (G = (V, E), k)$ be an instance of Independent Set, and let $E = \{e_1, \ldots, e_m\}$. We construct an instance $I' = (A, C, U, k, \ell, x)$ of CMVV in polynomial time as follows.

Construction: Let the set C of candidates be equal to V, and let the set A of agents be equal to $\{a_1, a_2\}$. We construct m voting profiles as follows. For all $t \in \{1, \ldots, m\}$, set $u_t(a_1) = C \setminus \{v\}$ and $u_t(a_2) = C \setminus \{w\}$, where $e_t = \{v, w\}$. Then set $\ell = 0$ and $x = 2k - 1$. This finishes the construction. An example of the construction is given in Figure 3.1.

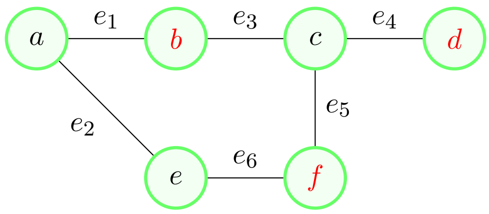
Correctness: We prove that I is a `yes`-instance if and only if I' is a `yes`-instance.

($\Rightarrow$) Let $X \subseteq V$ be an independent set of size k. We claim that the sequence $(C_1, \ldots, C_r)$ with $C_t = X$ for every $t \in \{1, \ldots, m\}$ is a solution to I'. Firstly, observe that $C_t \Delta C_{t+1} = \emptyset$ for every $t \in \{1, \ldots, m-1\}$, and that $|C_t| = k$ for every $t \in \{1, \ldots, m\}$. Since

$$\begin{aligned}
\mathrm{score}_t^{\textsc{Veto}}(C_t) &= \sum_{a_i \in A} |u_t(a_i) \cap C_t| \\
&= |u_t(a_1) \cap C_t| + |u_t(a_2) \cap C_t| \\
&= |(C \setminus \{v\}) \cap C_t| + |(C \setminus \{w\}) \cap C_t| \\
&= |C \cap C_t| - |\{v\} \cap C_t| + |C \cap C_t| - |\{w\} \cap C_t| \\
&= 2\,|C \cap C_t| - (|\{v\} \cap C_t| + |\{w\} \cap C_t|) \\
&= 2\,|C_t| - |\{v, w\} \cap C_t| \\
&= 2k - |e_t \cap C_t|
\end{aligned}$$

INDEPENDENT SET instance $I = (G, k)$ with $k = 3$ and

G :



CONSERVATIVE MULTISTAGE VETO VOTING instance $I' = (A, C, U, k, \ell, x)$ with $A = \{a_1, a_2\}$, $C = \{a, b, c, d, e, f\}$, $k = 3$, $\ell = 0$, $x = 5$, and

U :

t	a_i	$u_t(a_i)$	t	a_i	$u_t(a_i)$
1	a_1	$\{\underline{b}, c, d, e, f\}$	4	a_1	$\{a, b, \underline{d}, e, f\}$
	a_2	$\{\underline{a}, c, d, e, f\}$		a_2	$\{a, b, \underline{c}, e, f\}$
2	a_1	$\{b, c, d, \underline{e}, f\}$	5	a_1	$\{a, b, d, e, \underline{f}\}$
	a_2	$\{\underline{a}, b, c, d, f\}$		a_2	$\{a, b, \underline{c}, d, e\}$
3	a_1	$\{a, \underline{c}, d, e, f\}$	6	a_1	$\{a, b, c, d, \underline{f}\}$
	a_2	$\{a, \underline{b}, d, e, f\}$		a_2	$\{a, b, c, d, \underline{e}\}$

Figure 3.1: An instance of INDEPENDENT SET and the equivalent instance of CONSERVATIVE MULTISTAGE VETO VOTING, constructed as described in the proof of Lemma 3.4. Underlined candidates in every stage t refer to the endpoints of the corresponding edge e_t. Possible solutions are marked with red in both instances.

and $|e_t \cap X| \leq 1$ for every edge $e_t = \{v, w\}$, we have $\text{score}_t^{\text{VETO}}(C_t) \geq 2k - 1$. This proves the claim.

($\Leftarrow$) Let $(C_1, \ldots, C_\tau)$ be a solution to I'. Note that since $\ell = 0$, $C_i = C_j$ for every $i, j \in \{1, \ldots, m\}$. We claim that $X := C_1$ is a independent set of G (note that $|X| \leq k$). Let $t \in \{1, \ldots, m\}$ be arbitrary but fixed. Since $\text{score}_t^{\text{VETO}}(C_t) = 2|C_t| - |e_t \cap C_t| = 2|X| - |e_t \cap X| \geq 2k - 1$, we know that $|X| \geq k$ and $|e_t \cap X| \leq 1$. It follows that $e_t \not\subseteq X$. Since t was chosen arbitrarily, the claim follows. □

W[1]-hardness follows directly from the above reduction and if we reduce the special variant HALF INDEPENDENT SET, which is also NP-hard, we get the following.

Corollary 3.5. CONSERVATIVE MULTISTAGE VETO VOTING *is* NP*-hard for two agents,* $\ell = 0$*,* $x = |C| - 1$*,* $k = |C|/2$*, and* W[1]*-hard when parameterized by* k *and* x*, even for two agents and* $\ell = 0$*.*

Before we continue with the revolutionary variant, we note an observation that will be helpful to bound x in the reduction from CMVV to RMVV.

Observation 3.6. *Let $I = (A, C, U, k, \ell, x)$ be a instance of* Π MULTISTAGE VETO VOTING. *For every committee $C' \subseteq C$ it holds that*

$$(|C'| - 1) \cdot |A| \leq \text{score}_t^{\text{VETO}}(C') \leq |C'| \cdot |A|.$$

Proof. The score of a committee $C' \subseteq C$ is at most $|C'| \cdot |A|$, even if all members of the committee are approved by all agents and at least $(|C'| - 1) \cdot |A|$, since every agent disapproves at most one candidate in the committee. □

We are ready to settle the classical computational complexity of RMVV and give the following reduction for this.

Lemma 3.7. *From an instance (A, C, U, k, ℓ, x) with $|A| \cdot k \geq x > |A| \cdot (k-1)$ (see Observation 3.6), $\ell = 0$, and $k = |C|/2$ of* CONSERVATIVE MULTISTAGE VETO VOTING, *one can compute in polynomial time an equivalent instance $(A, C', U', k', \ell', x')$ of* REVOLUTIONARY MULTISTAGE VETO VOTING *with $k' = |C'|/2$, $\ell' = 2k'$ and $|U'| = 2|U| + 1$.*

Proof. Let $I = (A, C, U, k, \ell, x)$ with $\ell = 0$, $k = |C|/2$ and τ profiles be an instance of CMVV. We construct instance $I' = (A, C', U', k', \ell', x')$ of RMVV with $\ell' = 2k'$ in polynomial time as follows.

Construction: Let the set of candidates C' be equal to $C \cup \{z, y\}$, where z, y are new candidates not in C. We construct $2 \cdot \tau + 1$ voting profiles as follows. For all $a \in A$ and all $t \in \{1, \dots, \tau\}$, set $u'_{2t-1}(a) = u_t(a) \cup \{z, y\}$ and $u'_{2t}(a) = C' \setminus \{z\}$. Moreover, set $u'_{2\tau+1}(a) = C' \setminus \{y\}$ for all $a \in A$. Finally, set $k' = k + 1$, $l' = 2k' = |C'|$ and $x' = x + |A|$. This finishes the construction.

Correctness: We prove that I is a `yes`-instance if and only if I' is a `yes`-instance. First, note that for every $t \in \{1, \dots, m\}$ and $C^* \subseteq C'$ it holds that

$$\begin{aligned}
\text{score}_{2t-1}^{\text{VETO}}(C^*) &= \sum_{a_i \in A} |u'_{2t-1}(a_i) \cap C^*| \\
&= \sum_{a_i \in A} \left|\big(u_t(a_i) \cup \{z, y\}\big) \cap C^*\right| \\
&= \sum_{a_i \in A} |u_t(a_i) \cap C^*| + \sum_{a_i \in A} |\{z, y\} \cap C^*| \\
&= \text{score}_t^{\text{VETO}}(C^* \cap C) + |A| \cdot |\{z, y\} \cap C^*|, \\
\text{score}_{2t}^{\text{VETO}}(C^*) &= \sum_{a_i \in A} \left|u'_{2t}(a_i) \cap C^*\right| \\
&= |A| \cdot \left|(C' \setminus \{z\}) \cap C^*\right| \\
&= |A| \cdot \big(\left|C' \cap C^*\right| - |\{z\} \cap C^*|\big) \\
&= |A| \cdot (|C^*| - |\{z\} \cap C^*|), \\
\text{and } \text{score}_{2\tau+1}^{\text{VETO}}(C^*) &= \sum_{a_i \in A} \left|u'_{2\tau+1}(a_i) \cap C^*\right| \\
&= |A| \cdot \left|(C' \setminus \{y\}) \cap C^*\right| \\
&= |A| \cdot \big(\left|C' \cap C^*\right| - |\{y\} \cap C^*|\big) \\
&= |A| \cdot (|C^*| - |\{y\} \cap C^*|).
\end{aligned}$$

(⇒) Let $(C_1, \ldots, C_\tau)$ be a solution to I such that $|C_1| = k$. Since $\ell = 0$, we have that $C_t = C_{t'}$ for every $t, t' \in \{1, \ldots, \tau\}$. We claim that $(C'_1, \ldots, C'_{2\tau+1})$ with $C'_{2t-1} = C_t \cup \{z\}$ for every $t \in \{1, \ldots, \tau+1\}$ and $C'_{2t} = (C \setminus C_t) \cup \{y\}$ for every $t \in \{1, \ldots, \tau\}$ is a solution to I'. Note that $|C'_t| = k + 1 = |C'|/2$ for every $t \in \{1, \ldots, 2\tau+1\}$. Since

$$\text{score}_{2t-1}^{\text{VETO}}(C'_{2t-1}) = \text{score}_t^{\text{VETO}}(C'_{2t-1} \cap C) + |A| \cdot \big|\{z, y\} \cap C'_{2t-1}\big| = \text{score}_t^{\text{VETO}}(C_t) + |A|$$

for all $t \in \{1, \ldots, \tau\}$ and $(C_1, \ldots, C_\tau)$ is a solution to I, we have $\text{score}_{2t-1}^{\text{VETO}}(C'_{2t-1}) \geq x + |A| = x'$ for all $t \in \{1, \ldots, \tau\}$. Moreover we have

$$\text{score}_{2t}^{\text{VETO}}(C'_{2t}) = |A| \cdot \big(\big|C'_{2t}\big| - \big|\{z\} \cap C'_{2t}\big|\big) = |A| \cdot (k+1) \geq x + |A| = x'$$

for all $t \in \{1, \ldots, \tau\}$ and

$$\text{score}_{2\tau+1}^{\text{VETO}}(C'_{2\tau+1}) = |A| \cdot \big(\big|C'_{2\tau+1}\big| - \big|\{y\} \cap C'_{2\tau+1}\big|\big) = |A| \cdot (k+1) \geq x + |A| = x'.$$

Lastly, as $C'_t \Delta C'_{t+1} = C'$ for every $t \in \{1, \ldots, 2\tau\}$, the claim follows.

(⇐) Let $(C_1, \ldots, C_{2\tau+1})$ be a solution to I'. First observe that, due to $\ell = |C'|$, we have that $C_t \Delta C_{t+1} = C'$ for every $t \in \{1, \ldots, 2\tau\}$. Since $k' = |C'|/2$, it follows that $C_{2t} = C_{2t'}$ for every $t, t' \in \{1, \ldots, \tau\}$ and $C_{2t-1} = C_{2t'-1}$ for every $t, t' \in \{1, \ldots, \tau+1\}$ and $|C_t| = k'$ for every $t \in \{1, \ldots, 2\tau+1\}$. Note that $x' = x + |A| > |A| \cdot (k-1) + |A| = |A| \cdot k$. Since

$$\text{score}_{2\tau+1}^{\text{VETO}}(C_{2\tau+1}) = |A| \cdot (|C_{2\tau+1}| - |\{y\} \cap C_{2\tau+1}|) = |A| \cdot (k + 1 - |\{y\} \cap C_{2\tau+1}|) \geq x'$$

and $x' > |A| \cdot k$, we have $y \notin C_{2\tau+1}$ and $y \notin C_{2t-1}$ for every $t \in \{1, \ldots, \tau\}$. Since

$$\text{score}_{2\tau}^{\text{VETO}}(C_{2\tau}) = |A| \cdot (|C_{2\tau}| - |\{z\} \cap C_{2\tau}|) = |A| \cdot (k + 1 - |\{z\} \cap C_{2\tau}|) \geq x'$$

and $x' > |A| \cdot k$, we have $z \notin C_{2\tau}$ and $z \in C_{2t-1}$ for every $t \in \{1, \ldots, \tau+1\}$. We claim that $(C'_1, \ldots, C'_\tau)$ with $C'_t = C_{2t-1} \setminus \{z\}$ for every $t \in \{1, \ldots, \tau\}$ is a solution to I. Note that $|C'_t| = k$ for all $t \in \{1, \ldots, \tau\}$ and $C'_t \Delta C'_{t+1} = \emptyset$ for all $t \in \{1, \ldots, \tau - 1\}$. Since $\text{score}_t^{\text{VETO}}(C'_t) = \text{score}_t^{\text{VETO}}(C_{2t-1} \cap C) = \text{score}_{2t-1}^{\text{VETO}}(C_{2t-1}) - |A| \cdot |\{z, y\} \cap C_{2t-1}| = \text{score}_{2t-1}^{\text{VETO}}(C_{2t-1}) - |A| \geq x' - |A| = x$, the claim follows. □

Combining Corollary 3.5 and Lemma 3.7, we get the following.

Corollary 3.8. Revolutionary Multistage Veto Voting *is* NP*-hard even for two agents, $\ell = 2k$, $x = |C| - 1$, and $k = |C|/2$.*

Theorem 3.3 now follows from Lemma 3.2 and Corollaries 3.5 and 3.8.

3.2 Bloc

Under Bloc, every voter approves exactly the same number of candidates as the desired committee size. Faliszewski et al. [Fal+17] classify Bloc as an excellence-based multi-winner rule, which makes it a good choice when, for example, we are trying to select the applicants to be invited for an interview for an open position. In such so-called shortlisting processes, agents are typically experts who are trying to select the finalists with the highest quality.

In some approval-based multiwinner rules like SNTV and Veto, it is not determined which candidates to elect if all agents cast the same approval ballots. A clear advantage of Bloc compared to the rules mentioned is the easy determination of the winning committee in such a case. Thus, Bloc is considered as one of the most natural multiwinner rules. In this section, we aim to settle the classical computational complexity of Conservative Multistage Bloc Voting (CMBlV) and Revolutionary Multistage Bloc Voting (RMBlV).

Theorem 3.9. Conservative Multistage Bloc Voting *with* $\ell = 0$ *and* Revolutionary Multistage Bloc Voting *with* $\ell = 2k$ *are* NP*-complete even for two agents.*

We begin with CMBlV and give a reduction from Half Vertex Cover to show the NP-hardness of CMBlV, proving the first statement of Theorem 3.9.

Lemma 3.10. Conservative Multistage Bloc Voting *is* NP*-hard for two agents,* $\ell = 0$*,* $k = |C|/2$*, and* $x = k - 1$*.*

Proof. Let $I = (G = (V, E))$ be an instance of Half Vertex Cover, and let $E = \{e_1, \ldots, e_m\}$. We construct an instance $I' = (A, C, U, k, \ell, x)$ of CMBlV in polynomial time as follows.

Construction: Let the set C of candidates be equal to V, and let the set A of agents be equal to $\{a_1, a_2\}$. First, set $k = |C|/2$. We construct U with $r := m \cdot \binom{|C|-2}{2}$ voting profiles as follows. For each $i \in \{1, \ldots, m\}$ and $\{y, z\} \subseteq C \setminus e_i$, let $\{X_{i,\{y,z\}}, Y_{i,\{y,z\}}\}$ be an arbitrarily chosen partition of $C \setminus (e_i \cup \{y, z\})$ with $\left|X_{i,\{y,z\}}\right| = \left|Y_{i,\{y,z\}}\right| = k - 2$. Set a voting profile $u_{i,\{y,z\}}$ in U with $u_{i,\{y,z\}}(a_1) = e_i \cup X_{i,\{y,z\}}$ and $u_{i,\{y,z\}}(a_2) = e_i \cup Y_{i,\{y,z\}}$ for each $i \in \{1, \ldots, m\}$ and $\{y, z\} \subseteq C \setminus e_i$. Finally, set $\ell = 0$ and $x = k - 1$. An example of the construction is given in Figure 3.2.

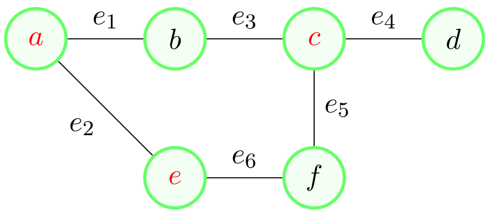
Correctness: We prove that I is a yes-instance if and only if I' is a yes-instance. ($\Rightarrow$) Let $X \subseteq V$ be a vertex cover of size $|V|/2 = k$. We claim that the sequence $(C_1, \ldots, C_r)$ with $C_t = X$ for every $t \in \{1, \ldots, r\}$ is a solution to I'. Firstly, observe that $C_t \Delta C_{t+1} = \emptyset$ for every $t \in \{1, \ldots, r-1\}$, and that $|C_t| \leq k$ for every $t \in \{1, \ldots, r\}$. Let $t \in \{1, \ldots, r\}$ be arbitrary but fixed and let $u_{j,\{y,z\}}$ be the voting profile in the t-th stage. Since

$$\begin{aligned}
\text{score}_t^{\text{Bloc}}(C_t) &= \sum_{a_i \in A} \left|u_{j,\{y,z\}}(a_i) \cap C_t\right| \\
&= \left|u_{j,\{y,z\}}(a_1) \cap C_t\right| + \left|u_{j,\{y,z\}}(a_2) \cap C_t\right| \\
&= \left|(e_j \cup X_{j,\{y,z\}}) \cap C_t\right| + \left|(e_j \cup Y_{j,\{y,z\}}) \cap C_t\right| \\
&= \left|(e_j \cup X_{j,\{y,z\}} \cup Y_{j,\{y,z\}}) \cap C_t\right| + |e_j \cap C_t| \\
&= |(C \setminus \{y, z\}) \cap C_t| + |e_j \cap C_t| \\
&= |C_t \setminus \{y, z\}| + |e_j \cap C_t|
\end{aligned}$$

and $e_j \cap X \neq \emptyset$, $\text{score}_t^{\text{Bloc}}(C_t\, |C_t \setminus \{y, z\}| + |e_j \cap C_t| \geq (k-2) + 1 = k - 1 = x$. Since t was chosen arbitrarily, the claim follows.

HALF VERTEX COVER instance $I = (G = (V, E))$ with

G :



CONSERVATIVE MULTISTAGE BLOC VOTING instance $I' = (A, C, U, k, \ell, x)$ with $A = \{a_1, a_2\}$, $C = \{a, b, c, d, e, f\}$, $k = 3$, $\ell = 0$, $x = 2$, and

U :

t		$u_{t,\{y,z\}}(a_i)$					
1	$\{y,z\}$	$\{c,d\}$	$\{c,e\}$	$\{c,f\}$	$\{d,e\}$	$\{d,f\}$	$\{e,f\}$
	a_1	$\{\underline{a},\underline{b},e\}$	$\{\underline{a},\underline{b},d\}$	$\{\underline{a},\underline{b},d\}$	$\{\underline{a},\underline{b},c\}$	$\{\underline{a},\underline{b},c\}$	$\{\underline{a},\underline{b},c\}$
	a_2	$\{\underline{a},\underline{b},f\}$	$\{\underline{a},\underline{b},f\}$	$\{\underline{a},\underline{b},e\}$	$\{\underline{a},\underline{b},f\}$	$\{\underline{a},\underline{b},e\}$	$\{\underline{a},\underline{b},d\}$
2	$\{y,z\}$	$\{b,c\}$	$\{b,d\}$	$\{b,f\}$	$\{c,d\}$	$\{c,f\}$	$\{d,f\}$
	a_1	$\{\underline{a},d,\underline{e}\}$	$\{\underline{a},c,\underline{e}\}$	$\{\underline{a},c,\underline{e}\}$	$\{\underline{a},b,\underline{e}\}$	$\{\underline{a},b,\underline{e}\}$	$\{\underline{a},b,\underline{e}\}$
	a_2	$\{\underline{a},\underline{e},f\}$	$\{\underline{a},\underline{e},f\}$	$\{\underline{a},d,\underline{e}\}$	$\{\underline{a},\underline{e},f\}$	$\{\underline{a},d,\underline{e}\}$	$\{\underline{a},c,\underline{e}\}$
3	$\{y,z\}$	$\{a,d\}$	$\{a,e\}$	$\{a,f\}$	$\{d,e\}$	$\{d,f\}$	$\{e,f\}$
	a_1	$\{\underline{b},\underline{c},e\}$	$\{\underline{b},\underline{c},d\}$	$\{\underline{b},\underline{c},d\}$	$\{a,\underline{b},\underline{c}\}$	$\{a,\underline{b},\underline{c}\}$	$\{a,\underline{b},\underline{c}\}$
	a_2	$\{\underline{b},\underline{c},f\}$	$\{\underline{b},\underline{c},f\}$	$\{\underline{b},\underline{c},e\}$	$\{\underline{b},\underline{c},f\}$	$\{\underline{b},\underline{c},e\}$	$\{\underline{b},\underline{c},d\}$
4	$\{y,z\}$	$\{a,b\}$	$\{a,e\}$	$\{a,f\}$	$\{b,e\}$	$\{b,f\}$	$\{e,f\}$
	a_1	$\{\underline{c},\underline{d},e\}$	$\{b,\underline{c},\underline{d}\}$	$\{b,\underline{c},\underline{d}\}$	$\{a,\underline{c},\underline{d}\}$	$\{a,\underline{c},\underline{d}\}$	$\{a,\underline{c},\underline{d}\}$
	a_2	$\{\underline{c},\underline{d},f\}$	$\{\underline{c},\underline{d},f\}$	$\{\underline{c},\underline{d},e\}$	$\{\underline{c},\underline{d},f\}$	$\{\underline{c},\underline{d},e\}$	$\{b,\underline{c},\underline{d}\}$
5	$\{y,z\}$	$\{a,b\}$	$\{a,d\}$	$\{a,e\}$	$\{b,d\}$	$\{b,e\}$	$\{d,e\}$
	a_1	$\{\underline{c},d,\underline{f}\}$	$\{b,\underline{c},\underline{f}\}$	$\{b,\underline{c},\underline{f}\}$	$\{a,\underline{c},\underline{f}\}$	$\{a,\underline{c},\underline{f}\}$	$\{a,\underline{c},\underline{f}\}$
	a_2	$\{\underline{c},e,\underline{f}\}$	$\{\underline{c},e,\underline{f}\}$	$\{\underline{c},d,\underline{f}\}$	$\{\underline{c},e,\underline{f}\}$	$\{\underline{c},d,\underline{f}\}$	$\{b,\underline{c},\underline{f}\}$
6	$\{y,z\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$	$\{b,c\}$	$\{b,d\}$	$\{c,d\}$
	a_1	$\{c,\underline{e},\underline{f}\}$	$\{b,\underline{e},\underline{f}\}$	$\{b,\underline{e},\underline{f}\}$	$\{a,\underline{e},\underline{f}\}$	$\{a,\underline{e},\underline{f}\}$	$\{a,\underline{e},\underline{f}\}$
	a_2	$\{d,\underline{e},\underline{f}\}$	$\{d,\underline{e},\underline{f}\}$	$\{c,\underline{e},\underline{f}\}$	$\{d,\underline{e},\underline{f}\}$	$\{c,\underline{e},\underline{f}\}$	$\{b,\underline{e},\underline{f}\}$

Figure 3.2: An instance of HALF VERTEX COVER and the equivalent instance of CONSERVATIVE MULTISTAGE BLOC VOTING, constructed as described in the proof of Lemma 3.10. Underlined candidates in every row t refer to the endpoints of the corresponding edge e_t. Possible solutions are marked with red in both instances.

($\Leftarrow$) Let $(C_1, \dots, C_r)$ be a solution to I'. Note that since $\ell = 0$, $C_i = C_j$ for all $i, j \in \{1, \dots, r\}$. We claim that $X := C_1$ is a vertex cover of G (note that $|X| \le k$). Suppose towards a contradiction that this is not the case. Hence, there is a $j \in \{1, \dots, m\}$ such that $e_j \cap X = \emptyset$. Let $u_{j,\{y,z\}}$ be a voting profile with $\{y, z\} \subseteq X$ and $t \in \{1, \dots, r\}$ be the corresponding stage. Thus, it holds that

$$\text{score}_t^{\text{Bloc}}(C_t) = |C_t \setminus \{y,z\}| + |e_j \cap C_t| = |C_t \setminus \{y,z\}| = k - 2 < k - 1 = x,$$

contradicting the fact that $(C_1, \dots, C_r)$ is a solution. Hence, the claim follows. □

Next, we give a reduction from a special case of CMBlV, which is proved to be NP-hard in Lemma 3.10, to RMBlV to prove the NP-hardness of RMBlV.

Lemma 3.11. *From an instance (A, C, U, k, ℓ, x) with $\ell = 0$ and $k = |C|/2$ of* Conservative Multistage Bloc Voting, *one can compute in polynomial time an equivalent instance (A, C, U', k, ℓ', x) of* Revolutionary Multistage Bloc Voting *with $\ell' = 2k$ and $|U'| = 2|U|$.*

Proof. Let $I = (A, C, U, k, \ell, x)$ be an instance of CMBlV with $\ell = 0$, $k = |C|/2$, and τ profiles. We construct instance $I' = (A, C, U', k, \ell', x)$ of RMBlV with $\ell' = 2k$ and $U' = \{u'_1, \dots, u'_{2\tau}\}$ in polynomial time as follows.

Construction: First, set $u'_{2t-1}(a_i) = u_t(a_i)$ and $u'_{2t}(a_i) = C \setminus u_t(a_i)$ for all $t \in \{1, \dots, \tau\}$ and $a_i \in A$. We note that $|u'_t(a_i)| = k$ for all $t \in \{1, \dots, 2\tau\}$ and $a_i \in A$. Finally, set $\ell' = 2k = |C|$. This finishes the construction.

Correctness: We prove that I is a `yes`-instance if and only if I' is a `yes`-instance.

($\Rightarrow$) Let $(C_1, \dots, C_\tau)$ be a solution to I such that $|C_1| = k$. Since $\ell = 0$, we have that $C_t = C_{t'}$ for every $t, t' \in \{1, \dots, \tau\}$. We claim that $(C'_1, \dots, C'_{2\tau})$ with $C'_{2t-1} = C_t$ and $C'_{2t} = C \setminus C_t$ for every $t \in \{1, \dots, \tau\}$ is a solution to I'. Note that $|C'_t| = k$ for every $t \in \{1, \dots, 2\tau\}$. Since

$$\begin{aligned}
\text{score}_{2t-1}^{\text{Bloc}}(C'_{2t-1}) &= \sum_{a_i \in A} \left| u'_{2t-1}(a_i) \cap C'_{2t-1} \right| \\
&= \sum_{a_i \in A} |u_t(a_i) \cap C_t| \\
&= \text{score}_t^{\text{Bloc}}(C_t) \\
\text{and } \text{score}_{2t}^{\text{Bloc}}(C'_{2t}) &= \sum_{a_i \in A} \left| u'_{2t}(a_i) \cap C'_{2t} \right| \\
&= \sum_{a_i \in A} |(C \setminus u_t(a_i)) \cap (C \setminus C_t)| \\
&= \sum_{a_i \in A} (|C| - |u_t(a_i)| - |C_t| + |u_t(a_i) \cap C_t|) \\
&= \sum_{a_i \in A} |u_t(a_i) \cap C_t| \\
&= \text{score}_t^{\text{Bloc}}(C_t)
\end{aligned}$$

for all $t \in \{1, \ldots, \tau\}$ and $(C_1, \ldots, C_\tau)$ is a solution to I, we have $\text{score}_t^{\text{BLOC}}(C'_t) \geq x$ for all $t \in \{1, \ldots, 2\tau\}$. Lastly, as $C'_t \Delta C'_{t+1} = C'$ for every $t \in \{1, \ldots, 2\tau - 1\}$, the claim follows.

($\Leftarrow$) Let $(C_1, \ldots, C_{2\tau})$ be a solution to I'. First observe that, due to $\ell = |C|$, we have that $C_t \Delta C_{t+1} = C$ for every $t \in \{1, \ldots, 2\tau - 1\}$. Since $k = |C|/2$, it follows that $C_{2t} = C_{2t'}$ and $C_{2t-1} = C_{2t'-1}$ for every $t, t' \in \{1, \ldots, \tau\}$ and $|C_t| = k$ for every $t \in \{1, \ldots, 2\tau\}$. We claim that $(C'_1, \ldots, C'_\tau)$ with $C'_t = C_1$ for every $t \in \{1, \ldots, \tau\}$ is a solution to I. Note that $|C'_t| = k$ for all $t \in \{1, \ldots, \tau\}$. By construction, $\text{score}_t^{\text{BLOC}}(C'_t) = \text{score}_{2t-1}^{\text{BLOC}}(C_1) \geq x$ for all $t \in \{1, \ldots, \tau\}$, and as $C'_t \Delta C'_{t+1} = \emptyset$ for all $t \in \{1, \ldots, \tau - 1\}$, the claim follows. ☐

Combining Lemmas 3.10 and 3.11, we get the following.

Corollary 3.12. REVOLUTIONARY MULTISTAGE BLOC VOTING *is* NP*-hard even for two agents, $\ell = 2k$, $k = |C|/2$, and $x = k - 1$.*

Theorem 3.9 now follows from Lemmas 3.2 and 3.10 and Corollary 3.12.

3.3 Borda and Copeland

In this section, we discuss the classical computational complexity of the problem variants based on two well-known voting rules, Borda and Copeland. Their multiwinner versions, namely k-Borda and k-Copeland are also excellence-based rules like Bloc and focus on candidate excellence rather than the diversity of the selected committee [Fal+17].

Borda belongs to a class of voting rules called scoring rules. Each scoring rule can be described by a score vector $(w_1, \ldots, w_m)$. This means that each agent awards w_1 points to its top-ranked candidate, w_2 to its second-ranked, and so on. We note that employing the same construction used in the proofs in this section we can also prove the NP-hardness of the problems based on scoring rules with any score vector $(w_1, \ldots, w_m)$ such that $w_1, w_2 > w_i$ and $w_i + w_{m+3-i} = d$ for all $i \in \{3, \ldots, m\}$ and a constant $d \in \mathbb{R}$.

Theorem 3.13. *Following problems are* NP*-complete even for two agents:*

(i) CONSERVATIVE MULTISTAGE BORDA VOTING *with $\ell = 0$,*

(ii) CONSERVATIVE MULTISTAGE COPELAND^0 VOTING *with $\ell = 0$ and $x = 1$,*

(iii) CONSERVATIVE MULTISTAGE COPELAND$^\alpha$ VOTING *with $\ell = 0$, if $\alpha > 0$,*

(iv) REVOLUTIONARY MULTISTAGE BORDA VOTING *with $\ell = 2k$,*

(v) REVOLUTIONARY MULTISTAGE COPELAND$^{0.5}$ VOTING *with $\ell = 2k$.*

We start with proving *(i)*, *(ii)* and *(iii)* of Theorem 3.13 and reduce HALF VERTEX COVER to CONSERVATIVE MULTISTAGE BORDA VOTING (CMBoV) and CONSERVATIVE MULTISTAGE COPELAND$^\alpha$ VOTING (CMC$^\alpha$V) to prove their NP-hardness.

Lemma 3.14. CONSERVATIVE MULTISTAGE BORDA VOTING, CONSERVATIVE MULTISTAGE COPELAND$^{0.5}$ VOTING *and* CONSERVATIVE MULTISTAGE COPELAND^1 VOTING *are* NP*-hard even for two agents, $\ell = 0$ and $k = |C|/2$.* CONSERVATIVE MULTISTAGE COPELAND^0 VOTING *is* NP*-hard even for two agents, $\ell = 0$, $k = |C|/2$ and $x = 1$.*

Proof. Let $I = (G = (V, E))$ be an instance of HALF VERTEX COVER, and let E be equal to $\{e_1, \dots, e_m\}$. We construct an instance $I' = (A, C, P, k, \ell, x)$ of CMRV with R $\in$ {BORDA, COPELAND$^\alpha$} in polynomial time as follows.

Construction: Let the set C of candidates be equal to V, and the set A of agents be equal to $\{a_1, a_2\}$. We construct m voting profiles as follows. For all $e_t = \{u, w\} \in E$, set $u \succ_{t,1} w$, $w \succ_{t,2} u$, $y \succ_{t,i} z$ for every $y \in \{u, w\}$, $z \in C \setminus \{u, w\}$, $i \in \{1, 2\}$ and rank candidates $c, c' \in C \setminus \{u, w\}$ in any order among themselves such that $c \succ_{t,1} c'$ if and only if $c' \succ_{t,2} c$. Finally, set $k = |C|/2$, $\ell = 0$ and

$$x = \begin{cases} \frac{|C|}{2} \cdot (|C| - 3) + 1, & \text{if R} = \text{BORDA}, \\ \frac{|C|}{2} \cdot \alpha\,(|C| - 3) + 1, & \text{if R} = \text{COPELAND}^\alpha. \end{cases}$$

An example of the construction is given in Figure 3.3.

Correctness: We prove that I is a yes-instance if and only if I' is a yes-instance. First, note that for every $t \in \{1, \dots, m\}$ and $C' \subseteq C$ it holds that

$$\begin{aligned} \text{score}_t^{\text{BORDA}}(C') &= \sum_{c \in C'} \sum_{a_i \in A} \left|\{c' \in C \mid c \succ_{t,i} c'\}\right| \\ &= \sum_{c \in C'} \left(\left|\{c' \in C \mid c \succ_{t,1} c'\}\right| + \left|\{c' \in C \mid c \succ_{t,2} c'\}\right|\right) \\ &= \sum_{c \in C' \cap e_t} (|e_t \setminus \{c\}| + 2\,|C \setminus e_t|) + \sum_{c \in C' \setminus e_t} |C \setminus (e_t \cup \{c\})| \\ &= \left|C' \cap e_t\right| (2\,|C| - 3) + \left|C' \setminus e_t\right| (|C| - 3) \end{aligned}$$

$$\begin{aligned} \text{and } \text{score}_t^{\text{COPELAND}^\alpha}(C') &= \sum_{c \in C'} \left(\left|\{c' \in C \mid c >_t^\mu c'\}\right| + \alpha \left|\{c' \in C \setminus \{c\} \mid c =_t^\mu c'\}\right|\right) \\ &= \sum_{c \in C' \cap e_t} \left(\left|\{c' \in C \mid c >_t^\mu c'\}\right| + \alpha \left|\{c' \in C \setminus \{c\} \mid c =_t^\mu c'\}\right|\right) \\ &\quad + \sum_{c \in C' \setminus e_t} \left(\left|\{c' \in C \mid c >_t^\mu c'\}\right| + \alpha \left|\{c' \in C \setminus \{c\} \mid c =_t^\mu c'\}\right|\right) \\ &= \sum_{c \in C' \cap e_t} (|C \setminus e_t| + \alpha\,|e_t \setminus \{c\}|) + \sum_{c \in C' \setminus e_t} (|\emptyset| + \alpha\,|C \setminus (e_t \cup \{c\})|) \\ &= \left|C' \cap e_t\right| \cdot (|C| - 2 + \alpha) + \left|C' \setminus e_t\right| \cdot \alpha(|C| - 3). \end{aligned}$$

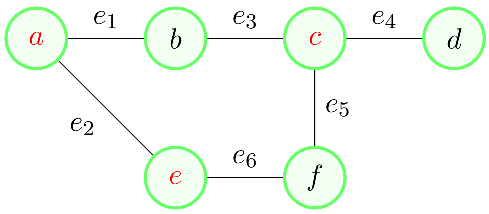
($\Rightarrow$) Let $X \subseteq V$ be a vertex cover of size $|V|/2$. We claim that the sequence $(C_1, \dots, C_m)$ with $C_t = X$ for every $t \in \{1, \dots, m\}$ is a solution to I'. Firstly, observe that $C_t \Delta C_{t+1} = \emptyset$ for every $t \in \{1, \dots, m-1\}$, and that $|C_t| \le k$ for every $t \in \{1, \dots, m\}$. Since $X \cap e_t \neq \emptyset$ for every $t \in \{1, \dots, m\}$, we have

$$\text{score}_t^{\text{BORDA}}(C_t) = \begin{cases} 2\,|C| - 3 + \left(\frac{|C|}{2} - 1\right)(|C| - 3), & \text{if } |X \cap e_t| = 1, \\ 4\,|C| - 6 + \left(\frac{|C|}{2} - 2\right)(|C| - 3), & \text{if } |X \cap e_t| = 2, \end{cases} \ge x \text{ and}$$

$$\text{score}_t^{\text{COPELAND}^\alpha}(C_t) = \begin{cases} (|C| - 2 + \alpha) + \left(\frac{|C|}{2} - 1\right) \cdot \alpha(|C| - 3), & \text{if } |X \cap e_t| = 1, \\ 2\,(|C| - 2 + \alpha) + \left(\frac{|C|}{2} - 2\right) \cdot \alpha(|C| - 3), & \text{if } |X \cap e_t| = 2, \end{cases} \ge x.$$

HALF VERTEX COVER instance $I = (G = (V, E))$ with

G :



CONSERVATIVE MULTISTAGE R VOTING instance $I' = (A, C, P, k, \ell, x)$ with

$A = \{a_1, a_2\}$, $C = \{a, b, c, d, e, f\}$, $k = 3$, $\ell = 0$, $x = \begin{cases} 10, & \text{if R = BORDA}, \\ 9\alpha + 1, & \text{if R = COPELAND}^\alpha, \end{cases}$

P :

t	a_i	e_t	$C \setminus e_t$	t	a_i	e_t	$C \setminus e_t$
1	a_1	$a \succ b \succ$	$c \succ d \succ e \succ f$	4	a_1	$c \succ d \succ$	$a \succ b \succ e \succ f$
	a_2	$b \succ a \succ$	$f \succ e \succ d \succ c$		a_2	$d \succ c \succ$	$f \succ e \succ b \succ a$
2	a_1	$a \succ e \succ$	$b \succ c \succ d \succ f$	5	a_1	$c \succ f \succ$	$a \succ b \succ d \succ e$
	a_2	$e \succ a \succ$	$f \succ d \succ c \succ b$		a_2	$f \succ c \succ$	$e \succ d \succ b \succ a$
3	a_1	$b \succ c \succ$	$a \succ d \succ e \succ f$	6	a_1	$e \succ f \succ$	$a \succ b \succ c \succ d$
	a_2	$c \succ b \succ$	$f \succ e \succ d \succ a$		a_2	$f \succ e \succ$	$d \succ c \succ b \succ a$

Figure 3.3: An instance of HALF VERTEX COVER and the equivalent instances of CONSERVATIVE MULTISTAGE BORDA VOTING and CONSERVATIVE MULTISTAGE COPELAND$^\alpha$ VOTING, constructed as described in the proof of Lemma 3.14. Possible solutions are marked with red in both instances.

This proves the claim.

($\Leftarrow$) Let $(C_1, \ldots, C_m)$ be a solution to I'. Note that since $\ell = 0$, $C_i = C_j$ for every $i, j \in \{1, \ldots, m\}$. We claim that $X := C_1$ is a vertex cover of G (note that $|X| \leq k$). Suppose towards a contradiction that this is not the case. Hence, there is a $t \in \{1, \ldots, m\}$ such that $X \cap e_t = \emptyset$. This means that

$$\text{score}_t^{\text{BORDA}}(C_t) = |C_t \setminus e_t|\,(|C| - 3) \leq k \cdot (|C| - 3) = \frac{|C|}{2} \cdot (|C| - 3) < x \text{ and}$$

$$\text{score}_t^{\text{COPELAND}^\alpha}(C_t) = |C_t \setminus e_t| \cdot \alpha(|C| - 3) \leq k \cdot \alpha\,(|C| - 3) = \frac{|C|}{2} \cdot \alpha\,(|C| - 3) < x,$$

contradicting the fact that $(C_1, \ldots, C_m)$ is a solution. Hence, the claim follows. □

Next, to prove the NP-hardness of RMBoV and CMC$^{0.5}$V, we give a reduction from their conservative variants. We first note an observation which will be used in the subsequent lemma.

Observation 3.15. *Let* $I = (A, C, P, k, \ell, x)$ *be an instance of* Π MULTISTAGE R VOTING *with* R ∈ {BORDA, COPELAND$^{0.5}$}*. For any time step* t *and* $C' \subseteq C$ *it holds that*

$$\mathrm{score}_t^{\mathrm{R}}(C') + \mathrm{score}_t^{\mathrm{R}}(C \setminus C') = \mathrm{score}_t^{\mathrm{R}}(C) = \begin{cases} \frac{|C|}{2} |A| (|C| - 1), & \textit{if } \mathrm{R} = \textsc{Borda}, \\ \frac{|C|}{2} (|C| - 1), & \textit{if } \mathrm{R} = \textsc{Copeland}^{\alpha}. \end{cases}$$

Proof. Since the score of a committee C' is equal to the sum of the scores of candidates it contains, the sum of the scores of C' and $C \setminus C'$ is equal to the sum of the scores of all candidates and this sum is constant for a given instance. In case of a ΠMBoV instance, each agent gives a total score of $\frac{|C|}{2}(|C|-1)$, so that the sum of the scores of all candidates is equal to $\frac{|C|}{2} |A| (|C| - 1)$. In case of a ΠMC$^{\alpha}$V instance, the sum of the scores of all candidates is equal to the number of pairwise comparisons, i.e., $\binom{C}{2} = \frac{|C|}{2}(|C| - 1)$. □

We are ready to settle the classical computational complexity of RMBoV and CMC$^{0.5}$V. We give the following reduction.

Lemma 3.16. *From an instance* (A, C, P, k, ℓ, x) *with* $\ell = 0$ *and* $k = |C|/2$ *of* CONSERVATIVE MULTISTAGE R VOTING*, one can compute in polynomial time an equivalent instance* (A, C, P', k, ℓ', x) *of* REVOLUTIONARY MULTISTAGE R VOTING *with* $\ell' = 2k$ *and* $|P'| = 2|P|$ *for all* R ∈ {BORDA, COPELAND$^{0.5}$}.

Proof. Let $I = (A, C, P, k, \ell, x)$ with $\ell = 0$, $k = |C|/2$ and τ profiles be an instance of CMRV. We construct instance $I' = (A, C, P', k, \ell', x)$ of RMRV with $\ell' = 2k$ in polynomial time as follows.

Construction: For all $t \in \{1, \ldots, \tau\}$ set $p'_{2t-1} = p_t$. Next, for profile p'_{2t}, $t \in \{1, \ldots, \tau\}$, set $c \succ'_{2t,i} c'$ if and only if $c' \succ_{t,i} c$ for every $c, c' \in C$, $a_i \in A$. Finally set $\ell' = 2k = |C|$. This finishes the construction.

Correctness: We prove that I is a yes-instance if and only if I' is a yes-instance. First, note that for every $t \in \{1, \ldots, m\}$ and $C' \subseteq C$ with $|C'| = |C|/2$ it holds that

$$\begin{aligned} \mathrm{score}_{2t}^{\textsc{Borda}}(C') &= \sum_{c \in C'} \sum_{a_i \in A} \left|\{c' \in C \mid c \succ'_{2t,i} c'\}\right| \\ &= \sum_{c \in C'} \sum_{a_i \in A} \left|\{c' \in C \mid c' \succ_{t,i} c\}\right| \\ &= \sum_{c \in C'} \sum_{a_i \in A} \left(|C| - 1 - \left|\{c' \in C \mid c \succ_{t,i} c'\}\right|\right) \\ &= \frac{|C|}{2} |A| (|C| - 1) - \mathrm{score}_t^{\textsc{Borda}}(C') \\ &= \mathrm{score}_t^{\textsc{Borda}}(C \setminus C'), \text{ and} \end{aligned}$$

$$
\begin{aligned}
\mathrm{score}_{2t}^{\textsc{Copeland}^{0.5}}(C') &= \sum_{c \in C'} \left(\left|\{c' \in C \mid c >_{2t}^{\mu} c'\}\right| + \frac{1}{2} \left|\{c' \in C \setminus \{c\} \mid c =_{2t}^{\mu} c'\}\right| \right) \\
&= \sum_{c \in C'} \left|\{c' \in C \mid c' >_{t}^{\mu} c\}\right| + \frac{1}{2} \sum_{c \in C'} \left|\{c' \in C \setminus \{c\} \mid c' =_{t}^{\mu} c\}\right| \\
&= \sum_{c \in C'} \left(|C| - 1 - \left|\{c' \in C \mid c >_{t}^{\mu} c'\}\right| - \left|\{c' \in C \setminus \{c\} \mid c =_{t}^{\mu} c'\}\right|\right) \\
&\quad + \frac{1}{2} \sum_{c \in C'} \left|\{c' \in C \setminus \{c\} \mid c' =_{t}^{\mu} c\}\right| \\
&= \sum_{c \in C'} (|C| - 1) \\
&\quad - \sum_{c \in C'} \left(\left|\{c' \in C \mid c >_{t}^{\mu} c'\}\right| + \frac{1}{2} \left|\{c' \in C \setminus \{c\} \mid c' =_{t}^{\mu} c\}\right| \right) \\
&= \frac{|C|}{2}(|C| - 1) - \mathrm{score}_{t}^{\textsc{Copeland}^{0.5}}(C') \\
&= \mathrm{score}_{t}^{\textsc{Copeland}^{0.5}}(C \setminus C').
\end{aligned}
$$

($\Rightarrow$) Let $(C_1, \ldots, C_\tau)$ be a solution to I such that $|C_1| = k$. Since $\ell = 0$, we have that $C_t = C_{t'}$ for every $t, t' \in \{1, \ldots, \tau\}$. We claim that $(C'_1, \ldots, C'_{2\tau})$ with $C'_{2t-1} = C_t$ and $C'_{2t} = C \setminus C_t$ for every $t \in \{1, \ldots, \tau\}$ is a solution to I'. First, note that $|C'_t| = k$ for all $t \in \{1, \ldots, 2\tau\}$. Since $p'_{2t-1} = p_t$ for all $t \in \{1, \ldots, \tau\}$ and $(C_1, \ldots, C_\tau)$ is a solution to I, we have $\mathrm{score}_{2t-1}^{\mathrm{R}}(C'_{2t-1}) \geq x$ for all $\mathrm{R} \in \{\textsc{Borda}, \textsc{Copeland}^{0.5}\}$. Since $\mathrm{score}_{2t}^{\textsc{Borda}}(C'_{2t}) = \mathrm{score}_{t}^{\textsc{Borda}}(C \setminus C'_{2t}) = \mathrm{score}_{t}^{\textsc{Borda}}(C_t)$ and $\mathrm{score}_{2t}^{\textsc{Copeland}^{0.5}}(C'_{2t}) = \mathrm{score}_{t}^{\textsc{Copeland}^{0.5}}(C \setminus C'_{2t}) = \mathrm{score}_{t}^{\textsc{Copeland}}(C_t)$ for all $t \in \{1, \ldots, \tau\}$ and $(C_1, \ldots, C_\tau)$ is a solution to I, we have $\mathrm{score}_{2t}^{\mathrm{R}}(C'_{2t}) \geq x$ for all $\mathrm{R} \in \{\textsc{Borda}, \textsc{Copeland}^{0.5}\}$. Lastly, as $C'_t \Delta C'_{t+1} = C$ for every $t \in \{1, \ldots, 2\tau - 1\}$, the claim follows.

($\Leftarrow$) Let $(C_1, \ldots, C_{2\tau})$ be a solution to I'. First observe that, due to $\ell = |C|$, we have that $C_t \Delta C_{t+1} = C$ for every $t \in \{1, \ldots, 2\tau - 1\}$. Since $k = |C|/2$, it follows that $C_{2t} = C_{2t'}$ and $C_{2t-1} = C_{2t'-1}$ for every $t, t' \in \{1, \ldots, \tau\}$ and $|C_t| = k$ for every $t \in \{1, \ldots, 2\tau\}$. We claim that $(C'_1, \ldots, C'_\tau)$ with $C'_t = C_{2t-1}$ for every $t \in \{1, \ldots, \tau\}$ is a solution to I. Note that $|C'_t| = k$ for all $t \in \{1, \ldots, \tau\}$ and $C'_t \Delta C'_{t+1} = \emptyset$ for all $t \in \{1, \ldots, \tau - 1\}$. By construction, $\mathrm{score}_{t}^{\mathrm{R}}(C'_t) = \mathrm{score}_{2t-1}^{\mathrm{R}}(C_{2t-1})$. Since $(C_1, \ldots, C_{2\tau})$ is a solution to I', $\mathrm{score}_{t}^{\mathrm{R}}(C'_t) \geq x$ for all $t \in \{1, \ldots, \tau\}$ and $\mathrm{R} \in \{\textsc{Borda}, \textsc{Copeland}^{0.5}\}$. This proves the claim. □

Combining Lemmas 3.14 and 3.16, we get the following.

Corollary 3.17. Revolutionary Multistage Borda Voting *and* Revolutionary Multistage Copeland$^{0.5}$ Voting *are* NP*-hard even for two agents,* $\ell = 2k$*, and* $k = |C|/2$.

Theorem 3.13 now follows from Lemmas 3.2 and 3.14 and Corollary 3.17.

3.4 Chamberlin-Courant and Monroe

In this section, we discuss the classical computational hardness of two voting systems based on the idea of proportional representation, Chamberlin-Courant and Monroe. Classical multiwinner elections under these rules are known to be NP-hard. We have the following hardness results from prior works for single-stage elections:

- Utilitarian variant of Chamberlin-Courant using Borda scores is NP-hard [LB11].
- Utilitarian variants of Chamberlin-Courant and Monroe based on approval ballots are NP-hard [PRZ08].
- Utilitarian variant of Chamberlin-Courant using Borda scores and all egalitarian variants are NP-hard [BSU13].

Hence, all variants of Π Multistage Chamberlin-Courant Voting and Π Multistage Monroe Voting are NP-hard even for $\tau = 1$. Combining these results and Lemma 3.2, we get the following.

Corollary 3.18. *For all* R $\in \{\alpha$-Chamberlin-Courant, β-Chamberlin-Courant, α-Monroe, β-Monroe$\}$*,* Π Multistage R Voting *is* NP*-complete even for* $\tau = 1$*.*

Despite these results, under some domain restrictions typical single-stage Chamberlin-Courant and Monroe elections become polynomial-time solvable. Betzler, Slinko, and Uhlmann [BSU13] show polynomial-time algorithms for single-peaked profiles for all variants but ΠMγ_U-β-MV and Skowron et al. [Sko+15] show that for single-crossing preferences, the single-stage problem admits a polynomial-time algorithm for the Chamberlin–Courant rule.

To the best of our knowledge, single-crossingness and single-peakedness have not yet been defined for a sequence of profiles as it is the case in our multistage setting. One natural way to interpret these domain restrictions for a sequence of profiles is to evaluate every stage independently and call the sequence of profiles single-crossing or single-peaked if all profiles satisfy the property. We show that the problems become NP-hard even under these domain restrictions adhering to this interpretation.

We note that another interpretation can be to call a sequence of voting profiles single-crossing or single-peaked if there exists a common linear order satisfying the property for all profiles. We leave open whether the problems admit polynomial-time algorithms according to this interpretation.

3.4.1 Approval-based Variant

We begin with the variants based on approval ballots, namely α-Chamberlin-Courant and α-Monroe. By showing their NP-hardness in case of profiles with one agent, we can conclude that the problems are NP-hard even for single-peaked and single-crossing profiles, since the voting profiles with one agent are clearly single-peaked and single-crossing. First, we prove the following theorem for Chamberlin-Courant and then show the equivalence of both rules for the special case used in proof.

Theorem 3.19. Conservative Multistage R Voting *with* $\ell = 0$*, and* Revolutionary Multistage R Voting *with* $\ell = 2k$ *are* NP*-hard even for one agent,* $x = 1$*, and* R $\in$ {α-Chamberlin-Courant, α-Monroe}*.* Conservative Multistage R Voting *with* R $\in$ {α-Chamberlin-Courant, α-Monroe} *is* W[2]*-hard when parameterized by* k*, even if* $|A| = 1$*,* $\ell = 0$ *and* $x = 1$*.*

We reduce Hitting Set to CMα-CCV to show the W[2]-hardness and the NP-hardness of CMα-CCV. Then, we give a reduction from a special case of CMα-CCV, which we also prove to be NP-hard, to RMα-CCV.

Lemma 3.20. *From an instance* $(E, \mathcal{C}, k)$ *of* Hitting Set*, one can compute in polynomial time an equivalent instance* (A, C, U, k, ℓ, x) *of* Conservative Multistage α-Chamberlin-Courant Voting *with* $|A| = 1$*,* $\ell = 0$ *and* $x = 1$*.*

Proof. Let $I = (E, \mathcal{C}, k)$ be an instance of Hitting Set, and let $\mathcal{C} = \{S_1, \ldots, S_m\}$. We construct an instance $I' = (A, C, U, k, \ell, x)$ of CMα-CCV with $|A| = 1$ in polynomial time as follows.

Construction: Let the set C of candidates be equal to E, and let the set A of agents be equal to $\{a_1\}$. We construct m voting profiles as follows. For every $t \in \{1, \ldots, m\}$, set $u_t(a_1) = S_t$. Finally, set $\ell = 0$ and $x = 1$.

Correctness: We prove that I is a `yes`-instance if and only if I' is a `yes`-instance.

($\Rightarrow$) Let $X \subseteq E$ be a hitting set of size at most k. We claim that the sequence $(C_1, \ldots, C_m)$ with $C_t = X$ for every $t \in \{1, \ldots, m\}$ is a solution to I'. Firstly, observe that $C_t \Delta C_{t+1} = \emptyset$ for every $t \in \{1, \ldots, m-1\}$, and that $|C_t| \leq k$ for every $t \in \{1, \ldots, m\}$. Note that for every $t \in \{1, \ldots, m\}$ it holds that

$$\begin{aligned} \text{score}_t^{\gamma_E\text{-}\alpha\text{-CC}}(C_t) &= \begin{cases} 0, & \text{if } |\{a_i \in A \mid u_t(a_i) \cap C_t \neq \emptyset\}| < |A| = 1, \\ 1, & \text{if } |\{a_i \in A \mid u_t(a_i) \cap C_t \neq \emptyset\}| = |A| = 1, \end{cases} \\ &= \begin{cases} 0, & \text{if } u_t(a_1) \cap C_t = \emptyset, \\ 1, & \text{if } u_t(a_1) \cap C_t \neq \emptyset, \end{cases} \\ &= |\{a_i \in A \mid u_t(a_i) \cap C_t \neq \emptyset\}| \\ &= \text{score}_t^{\gamma_U\text{-}\alpha\text{-CC}}(C_t). \end{aligned} \tag{3.5}$$

Hence, since $S_t \cap X \neq \emptyset$ for every $t \in \{1, \ldots, m\}$, it holds that

$$\text{score}_t^{\gamma_E\text{-}\alpha\text{-CC}}(C_t) = \text{score}_t^{\gamma_U\text{-}\alpha\text{-CC}}(C_t) = 1 \geq x$$

for every $t \in \{1, \ldots, m\}$. This proves the claim.

($\Leftarrow$) Let $(C_1, \ldots, C_\tau)$ be a solution to I'. Note that since $\ell = 0$, $C_i = C_j$ for every $i, j \in \{1, \ldots, m\}$. We claim that $X := C_1$ is a hitting set of E (note that $|X| \leq k$). Let $t \in \{1, \ldots, m\}$ be arbitrary but fixed. Since $\text{score}_t^{\gamma_E\text{-}\alpha\text{-CC}}(C_t) = \text{score}_t^{\gamma_U\text{-}\alpha\text{-CC}}(C_t) \geq 1$ we know that $u_t(a_1) \cap C_t \neq \emptyset$, implying that $S_t \cap X \neq \emptyset$. Since t was chosen arbitrarily, the claim follows. □

W[2]-hardness follows directly from the reduction. If we reduce the special variant Half Hitting Set, which is also NP-hard, then we get the following.

Corollary 3.21. Conservative Multistage α-Chamberlin-Courant Voting *is* NP*-hard even for one agent,* $\ell = 0$*,* $x = 1$*,* $k = |C|/2$*, and* W[2]*-hard when parameterized by* k*, even if* $|A| = 1$*,* $\ell = 0$ *and* $x = 1$*.*

We can reduce CMα-CCV to RMα-CCV to show the NP-hardness of RMα-CCV by using the same construction Bredereck, Fluschnik, and Kaczmarczyk [BFK20] used in their work to reduce Conservative Multistage Plurality Voting to Revolutionary Multistage Plurality Voting.

Lemma 3.22. *From an instance* (A, C, U, k, ℓ, x) *with* $\ell = 0$ *and* $k = |C|/2$ *of* Conservative Multistage α-Chamberlin-Courant Voting*, one can compute in polynomial time an equivalent instance* $(A, C', U', k', \ell', x)$ *of* Revolutionary Multistage α-Chamberlin-Courant Voting *with* $k' = |C'|/2$*,* $\ell' = 2k'$*, and* $|U'| = 2|U| + 1$*.*

Proof. Let $I = (A, C, U, k, \ell, x)$ be an instance of CMα-CCV with $\ell = 0$, $k = |C|/2$, and τ profiles. We construct instance $I' = (A, C', U', k', \ell', x)$ of RMα-CCV with $\ell' = 2k$ and $U' = \{u'_1, \ldots, u'_{2\tau}\}$ in polynomial time as follows.
Construction: Let the set C' of candidates be equal to $C \cup \{z, y\}$, where z, y are new candidates not contained in C. First, set $u'_{2t-1}(a_i) = u_t(a_i)$ and $u'_{2t}(a_i) = \{y\}$ for all $a_i \in A$ and $t \in \{1, \ldots, \tau\}$. Moreover, set $u'_{2\tau+1}(a_i) = \{z\}$ for all $a_i \in A$. Finally, set $k' = k + 1$ and $\ell' = 2k' = |C'|$. This finishes the construction.
Correctness: We prove that I is a yes-instance if and only if I' is a yes-instance. First, note that for every $t \in \{1, \ldots, \tau\}$ and $C' \subseteq C$ it holds that

$$\begin{aligned}
\text{score}_{2t}^{\gamma_E\text{-}\alpha\text{-CC}}(C') &= \begin{cases} 0, & \text{if } |\{a_i \in A \mid u'_{2t}(a_i) \cap C' \neq \emptyset\}| < |A|, \\ 1, & \text{if } |\{a_i \in A \mid u'_{2t}(a_i) \cap C' \neq \emptyset\}| = |A|, \end{cases} \\
&= \begin{cases} 0, & \text{if } |\{a_i \in A \mid \{y\} \cap C' \neq \emptyset\}| < |A|, \\ 1, & \text{if } |\{a_i \in A \mid \{y\} \cap C' \neq \emptyset\}| = |A|, \end{cases} \\
&= \left|\{y\} \cap C'\right| \\
\text{score}_{2t}^{\gamma_U\text{-}\alpha\text{-CC}}(C') &= \left|\{a_i \in A \mid u'_{2t}(a_i) \cap C' \neq \emptyset\}\right| \\
&= \left|\{a_i \in A \mid \{y\} \cap C' \neq \emptyset\}\right| \\
&= |A| \cdot \left|\{y\} \cap C'\right| \\
\text{and } \text{score}_{2\tau+1}^{\gamma_E\text{-}\alpha\text{-CC}}(C') &= \begin{cases} 0, & \text{if } \left|\{a_i \in A \mid u'_{2\tau+1}(a_i) \cap C' \neq \emptyset\}\right| < |A|, \\ 1, & \text{if } \left|\{a_i \in A \mid u'_{2\tau+1}(a_i) \cap C' \neq \emptyset\}\right| = |A|, \end{cases} \\
&= \begin{cases} 0, & \text{if } |\{a_i \in A \mid \{z\} \cap C' \neq \emptyset\}| < |A|, \\ 1, & \text{if } |\{a_i \in A \mid \{z\} \cap C' \neq \emptyset\}| = |A|, \end{cases} \\
&= \left|\{z\} \cap C'\right| \\
\text{score}_{2\tau+1}^{\gamma_U\text{-}\alpha\text{-CC}}(C') &= \left|\{a_i \in A \mid u'_{2\tau+1}(a_i) \cap C' \neq \emptyset\}\right| \\
&= \left|\{a_i \in A \mid \{z\} \cap C' \neq \emptyset\}\right| \\
&= |A| \cdot \left|\{z\} \cap C'\right|
\end{aligned}$$

($\Rightarrow$) Let $(C_1, \ldots, C_\tau)$ be a solution to I such that $|C_1| = k$. Since $\ell = 0$, we have that $C_t = C_{t'}$ for every $t, t' \in \{1, \ldots, \tau\}$. Let $X := C_1 \cup \{z\}$ and $Y := C' \setminus X$.

Note that $|X| = |Y| = |C'|/2$. We claim that $(C'_1, \ldots, C'_{2\tau+1})$ with $C'_{2t-1} = X$ for every $t \in \{1, \ldots, \tau+1\}$ and $C'_{2t} = Y$ for every $t \in \{1, \ldots, \tau\}$ is a solution to I'. First, note that $|C'_t| = k$ for all $t \in \{1, \ldots, 2\tau+1\}$. Since $u'_{2t-1}(a_i) = u_t(a_i)$ for all $a_i \in A$ and $t \in \{1, \ldots, \tau\}$, and $(C_1, \ldots, C_\tau)$ is a solution to I, we have $\text{score}^{\text{R}}_{2t-1}(C'_{2t-1}) \geq x$ for all $\text{R} \in \{\gamma_E\text{-}\alpha\text{-CC}, \gamma_U\text{-}\alpha\text{-CC}\}$. Since $y \in Y$, we have $\text{score}^{\gamma_E\text{-}\alpha\text{-CC}}_{2t}(C'_{2t}) = |\{y\} \cap C'_{2t}| = 1 \geq x$ and $\text{score}^{\gamma_U\text{-}\alpha\text{-CC}}_{2t}(C'_{2t}) = |A| \cdot |\{y\} \cap C'_{2t}| = |A| \geq x$ for all $t \in \{1, \ldots, \tau\}$. Since $z \in X$, we have $\text{score}^{\gamma_E\text{-}\alpha\text{-CC}}_{2\tau+1}(C'_{2\tau+1}) = \left|\{z\} \cap C'_{2\tau+1}\right| = 1 \geq x$ and $\text{score}^{\gamma_U\text{-}\alpha\text{-CC}}_{2\tau+1}(C'_{2\tau+1}) = |A| \cdot \left|\{z\} \cap C'_{2\tau+1}\right| = |A| \geq x$. Lastly, as $C'_t \Delta C'_{t+1} = C$ for every $t \in \{1, \ldots, 2\tau\}$, the claim follows.

($\Leftarrow$) Let $(C_1, \ldots, C_{2\tau+1})$ be a solution to I'. First observe that, due to $\ell = |C|$, we have that $C_t \Delta C_{t+1} = C$ for every $t \in \{1, \ldots, 2\tau\}$. Since $k = |C|/2$, it follows that $C_{2t} = C_{2t'}$ and $y \in C_{2t}$ for every $t, t' \in \{1, \ldots, \tau\}$, and $C_{2t-1} = C_{2t'-1}$ and $z \in C_{2t-1}$ for every $t, t' \in \{1, \ldots, \tau+1\}$ and $|C_t| = k$ for every $t \in \{1, \ldots, 2\tau\}$. We claim that $(C'_1, \ldots, C'_\tau)$ with $C'_t = C_{2t-1} \setminus \{z\}$ for every $t \in \{1, \ldots, \tau\}$ is a solution to I. Note that $|C'_t| \leq k$ for all $t \in \{1, \ldots, \tau\}$. By construction, $\text{score}^R_t(C'_t) = \text{score}^R_{2t-1}(C_{2t-1}) \geq x$ for all $t \in \{1, \ldots, \tau\}$ and $\text{R} \in \{\gamma_E\text{-}\alpha\text{-CC}, \gamma_U\text{-}\alpha\text{-CC}\}$, and as $C'_t \Delta C'_{t+1} = \emptyset$ for all $t \in \{1, \ldots, \tau-1\}$, the claim follows. □

Combining Corollary 3.21 and Lemma 3.22, we get the following.

Corollary 3.23. Revolutionary Multistage α-Chamberlin-Courant Voting *is* NP*-hard even for one agent, $\ell = 2k$, $x = 1$, and $k = |C|/2$.*

Next, we show the equivalence of ΠMα-CCV and ΠMα-MV in case of a single agent. Combining this equivalence we use with Corollary 3.23 will prove Theorem 3.19.

Lemma 3.24. Π Multistage α-Chamberlin-Courant Voting *and* Π Multistage α-Monroe Voting *are equivalent if there is only one agent.*

Proof. The only difference between these two problems is the way the scores are computed. Let $I = (A, C, U, k, \ell, x)$ with $|A| = 1$ and τ profiles be an instance of ΠMα-MV and $I' := I$ be an instance of ΠMα-CCV . We claim that $\text{score}^{\alpha\text{-M}}_t(C') = \text{score}^{\alpha\text{-CC}}_t(C')$ for all $C' \subseteq C$, $t \in \{1, \ldots, \tau\}$. Let $A = \{a_1\}$ be the set of agents. Since there is only one agent, every assignment satisfies Monroe criterion, which means $\Phi^M(C') = \Phi(C')$ for every set C' of candidates. Since

$$
\begin{aligned}
\text{score}^{\gamma_U\text{-}\alpha\text{-M}}_t(C') &= \max_{\pi \in \Phi^M(C')} |\{a_i \in A \mid \pi(a_i) \in u_t(a_i)\}| \\
&= \begin{cases} 0, & \text{if } \pi(a_1) \notin u_t(a_1) \text{ for all } \pi \in \Phi(C'), \\ 1, & \text{if } \pi(a_1) \in u_t(a_1) \text{ for some } \pi \in \Phi(C'), \end{cases} \\
&= \begin{cases} 0, & \text{if } \max_{\pi \in \Phi^M(C')} |\{a_i \in A \mid \pi(a_i) \in u_t(a_i)\}| < |A|, \\ 1, & \text{if } \max_{\pi \in \Phi^M(C')} |\{a_i \in A \mid \pi(a_i) \in u_t(a_i)\}| = |A|, \end{cases} \\
&= \text{score}^{\gamma_E\text{-}\alpha\text{-M}}_t(C')
\end{aligned}
$$

$$\text{and } \operatorname{score}_t^{\alpha\text{-M}}(C') = \begin{cases} 0, & \text{if } \pi(a_1) \notin u_t(a_1) \text{ for all } \pi \in \Phi(C'), \\ 1, & \text{if } \pi(a_1) \in u_t(a_1) \text{ for some } \pi \in \Phi(C'), \end{cases}$$
$$= \begin{cases} 0, & \text{if } u_t(a_1) \cap C' = \emptyset, \\ 1, & \text{if } u_t(a_1) \cap C' \neq \emptyset, \end{cases}$$
$$= \operatorname{score}_t^{\alpha\text{-CC}}(C') \qquad \text{(see Equation (3.5))}$$

for any time step t, the claim follows. □

Theorem 3.19 now follows from Corollaries 3.21 and 3.23 and Lemma 3.24.

3.4.2 Variant using Borda scores

In this subsection, we focus on the variants using Borda scores, namely β-Chamberlin-Courant and β-Monroe. We prove that they are NP-hard even for one agent. It follows that they remain NP-hard for single-peaked and single-crossing profiles as the approval-based variants. Since both rules are equivalent in case of a single agent, we start with proving the following theorem for Chamberlin-Courant and then show the equivalence of both rules for the special case used in proof.

Theorem 3.25. Conservative Multistage β-Chamberlin-Courant Voting *and* Conservative Multistage β-Monroe Voting *are* NP*-hard even for one agent and* $\ell = 0$*, and* W[2]*-hard when parameterized by* k*, even if* $\ell = 0$.

First, we reduce Hitting Set to CMβ-CCV to prove the W[2]-hardness and the NP-hardness of CMβ-CCV.

Lemma 3.26. Conservative Multistage β-Chamberlin-Courant Voting *is* NP*-hard even for one agent and* $\ell = 0$*, and* W[2]*-hard when parameterized by* k*, even if* $\ell = 0$.

Proof. Let $I = (E, \mathcal{C}, k)$ be an instance of Hitting Set, and let $\mathcal{C} = \{S_1, \ldots, S_m\}$. We construct an instance $I' = (A, C, P, k, \ell, x)$ of CMβ-CCV with $|A| = 1$ and $\ell = 0$ in polynomial time as follows.
Construction: Let the set C of candidates be equal to $E \cup D$, where $D := \bigcup_{i \in \{1,\ldots,m\}} D_i$, $D_i := \{d_i^1, \ldots d_i^{|E|}\}$, and the set A of agents equal to $\{a_1\}$. We construct m voting profiles as follows. For profile p_t, $t \in \{1, \ldots, m\}$, set $c \succ_{t,1} d$, $c \succ_{t,1} c'$, $d \succ_{t,1} c'$ for all $c \in S_t$, $d \in D_t$, and $c' \in C \setminus (S_t \cup D_t)$, and rank them arbitrarily among themselves. Finally, set $\ell = 0$ and $x = m|E|$. An example of the construction is given in Figure 3.4.
Correctness: We prove that I is a yes-instance if and only if I' is a yes-instance. First, note that for every $t \in \{1, \ldots, m\}$ it holds that

$$\begin{aligned} \operatorname{score}_t^{\gamma_U\text{-}\beta\text{-CC}}(C_t) &= \sum_{a_i \in A} \max_{c \in C_t} \left|\{c' \in C \mid c \succ_{t,i} c'\}\right| \\ &= \max_{c \in C_t} \left|\{c' \in C \mid c \succ_{t,1} c'\}\right| \qquad (3.6) \\ &= \min_{a_i \in A} \max_{c \in C_t} \left|\{c' \in C \mid c \succ_{t,i} c'\}\right| \\ &= \operatorname{score}_t^{\gamma_E\text{-}\beta\text{-CC}}(C_t). \end{aligned}$$

Hitting Set instance $I = (E, \mathcal{C}, k)$ with

$E = \{e, f, g, h\}$, $\mathcal{C} = \{S_1 = \{e, f, g\}, S_2 = \{h\}, S_3 = \{f, g\}\}$, and $k = 2$

Conservative Multistage β-Chamberlin-Courant Voting instance

$I' = (A, C, P, k, \ell, x)$ with $A = \{a_1\}$, $C = \{e, f, g, h\} \cup \bigcup_{j=1}^{3}(D_j = \{d_j^1, d_j^2, d_j^3, d_j^4\})$,

$k = 2$, $\ell = 0$, $x = 12$, and

P:

t	a_i	S_t	D_t	$C \setminus (S_t \cup D_t)$
1	a_1	$e \succ f \succ g \succ$	$\overrightarrow{D_1} \succ$	$h \succ \overrightarrow{D_2} \succ \overrightarrow{D_3}$
2	a_1	$h \succ$	$\overrightarrow{D_2} \succ$	$e \succ f \succ g \succ \overrightarrow{D_1} \succ \overrightarrow{D_3}$
3	a_1	$f \succ g \succ$	$\overrightarrow{D_3} \succ$	$e \succ h \succ \overrightarrow{D_1} \succ \overrightarrow{D_2}$

Figure 3.4: An instance of Hitting Set and the equivalent instance of Conservative Multistage β-Chamberlin-Courant Voting, constructed as described in the proof of Lemma 3.26. $\overrightarrow{D_j}$ denotes $d_j^1 \succ d_j^2 \succ d_j^3 \succ d_j^4$. Possible solutions are marked with red in both instances.

($\Rightarrow$) Let $X \subseteq E$ be a hitting set of size at most k. We claim that the sequence $(C_1, \dots, C_m)$ with $C_t = X$ for every $t \in \{1, \dots, m\}$ is a solution to I'. Firstly, observe that $C_t \Delta C_{t+1} = \emptyset$ for every $t \in \{1, \dots, m-1\}$, and that $|C_t| \le k$ for every $t \in \{1, \dots, m\}$. Since $C_t \cap S_t \neq \emptyset$, we have $\text{score}_t^{\gamma_U\text{-}\beta\text{-CC}}(C_t) = \text{score}_t^{\gamma_E\text{-}\beta\text{-CC}}(C_t) = \max_{c \in C_t} |\{c' \in C \mid c \succ_{t,1} c'\}| \ge |E \setminus S_t| \cup |D| \ge m\,|E| = x$ for every $t \in \{1, \dots, m\}$. This proves the claim.

($\Leftarrow$) Let $(C_1, ..., C_\tau)$ be a solution to I'. Note that since $\ell = 0$, $C_i = C_j$ for every $i, j \in \{1, \dots, m\}$. For every $t \in \{1, \dots, m\}$ such that $C_1 \cap D_t \neq \emptyset$, pick an arbitrary candidate from S_t and let Y be the set of these picked candidates. We claim that $X := (C_1 \cap E) \cup Y$ is a hitting set of E (note that $|X| \le k$). Let $t \in \{1, \dots, m\}$ be arbitrary but fixed. Since $\text{score}_t^{\gamma_U\text{-}\beta\text{-CC}}(C_t) = \text{score}_t^{\gamma_E\text{-}\beta\text{-CC}}(C_t) = \max_{c \in C_t} |\{c' \in C \mid c \succ_{t,1} c'\}| \ge m|E|$, we know that $C_t \cap (S_t \cup D_t) \neq \emptyset$. If $C_1 \cap S_t = \emptyset$, then $C_1 \cap D_t \neq \emptyset$ and Y contains a candidate from S_t, and if $C_1 \cap S_t \neq \emptyset$, $C_1 \cap E$ contains a candidate from S_t. Hence, $X \cap S_t \neq \emptyset$. Since t was chosen arbitrarily, the claim follows. □

Next, we show the equivalence of ΠMβ-CCV and ΠMβ-MV in the case of a single agent. Combining this equivalence with Lemma 3.26 will prove Theorem 3.25.

Lemma 3.27. Π Multistage β-Chamberlin-Courant Voting *and* Π Multistage β-Monroe Voting *are equivalent if there is only one agent.*

Proof. The only difference between these two problems is the way the scores are computed. Let $I = (A, C, P, k, \ell, x)$ with $|A| = 1$ and τ profiles be an instance of ΠMβ-MV and $I' := I$ be an instance of ΠMβ-CCV . We claim that $\text{score}_t^{\beta\text{-M}}(C') = \text{score}_t^{\beta\text{-CC}}(C')$ for all $C' \subseteq C$, $t \in \{1, \dots, \tau\}$. Let $A = \{a_1\}$ be the set of agents. Since there is only one agent, every assignment satisfies Monroe criterion, which means $\Phi^M(C') = \Phi(C')$ for every set C' of candidates. Since

$$\begin{aligned}
\operatorname{score}_t^{\gamma_U\text{-}\beta\text{-M}}(C') &= \max_{\pi\in\Phi(C')} \sum_{a_i\in A} |\{c \in C \mid \pi(a_i) \succ_{t,i} c\}| \\
&= \max_{\pi\in\Phi(C')} |\{c \in C \mid \pi(a_1) \succ_{t,1} c\}| \\
&= \max_{\pi\in\Phi(C')} \min_{a_i\in A} |\{c \in C \mid \pi(a_i) \succ_{t,i} c\}| \\
&= \operatorname{score}_t^{\gamma_E\text{-}\beta\text{-M}}(C') \\
\text{and } \operatorname{score}_t^{\beta\text{-M}}(C') &= \max_{\pi\in\Phi(C')} |\{c \in C \mid \pi(a_1) \succ_{t,1} c\}| \\
&= \max_{c\in C_t} \left|\{c' \in C \mid c \succ_{t,1} c'\}\right| \\
&= \operatorname{score}_t^{\beta\text{-CC}}(C') \qquad \text{(see Equation (3.6))}
\end{aligned}$$

for any time step t, the claim follows. □

Theorem 3.19 now follows from Lemmas 3.26 and 3.27.

Chapter 4

Size k of a Committee

In this chapter, we study the parameterized complexity of our problems regarding the parameter k, the desired maximum size of any elected committee. We show that all problems studied in this thesis are contained in XP when parameterized by k.

Theorem 4.1. Π Multistage R Voting *for both* $\Pi \equiv$ Conservative *and* $\Pi \equiv$ Revolutionary *admits*

(i) an $\mathcal{O}(\tau \cdot m^{2k+2}n)$ *algorithm if* R $\in$ {Veto, Bloc, Borda, Copeland$^\alpha$, α-Chamberlin-Courant, β-Chamberlin-Courant}*,*

(ii) an $\mathcal{O}(\tau \cdot m^{2k} \cdot poly(n,m))$ *algorithm if* R $\in$ {α-Monroe, β-Monroe}*,*

and hence is contained in XP *when parameterized by* k*.*

Bredereck, Fluschnik, and Kaczmarczyk [BFK20] prove that CMPV and RMPV are contained in XP when parameterized by k. To this end, they compute an auxiliary directed graph in XP-running time in which they then check for the existence of an s-t path witnessing a `yes`-instance. This idea can be also applied to the problems discussed in this work.

Lemma 4.2. *For all* R $\in$ {Veto, Bloc, Borda, Copeland$^\alpha$, α-Chamberlin-Courant, β-Chamberlin-Courant, α-Monroe, β-Monroe}*,* Π Multistage R Voting *admits an* $\mathcal{O}(\tau \cdot m^{2k}) \cdot T_{\mathrm{R}}(n,m)$ *algorithm, where* $T_{\mathrm{R}}(n,m)$ *is the running time of the algorithm that computes* $\mathrm{score}_t^{\mathrm{R}}(C)$ *of committee* C *at any time step* t*.*

Proof. Let $I = (A, C, U, k, \ell, x)$ be an instance of CMRV (or RMRV) with n agents, m candidates, τ voting profiles, and R $\in$ {Veto, Bloc, Borda, Copeland$^\alpha$, α-Chamberlin-Courant, β-Chamberlin-Courant, α-Monroe, β-Monroe}. We construct a directed graph $D = (V, A)$ in XP-running time as follows.
Construction: Set the vertex set $V = \{s, z\} \uplus V^1 \uplus \ldots \uplus V^\tau$, where

$$V^t = \{v_X^t \mid X \subseteq C, |X| \leq k, \mathrm{score}_t^{\mathrm{R}}(X) \geq x\}$$

and arc set $A = \{(s,v) \mid v \in V^1\} \uplus \{(v,z) \mid v \in V^\tau\} \uplus A^1 \uplus, \ldots, \uplus A^{\tau-1}$, where

$$A^t = \begin{cases} \{(v_X^t, v_Y^{t+1}) \mid |X \Delta Y| \leq \ell\}, & \text{if } \Pi = \textsc{Conservative}, \\ \{(v_X^t, v_Y^{t+1}) \mid |X \Delta Y| \geq \ell\}, & \text{if } \Pi = \textsc{Revolutionary}. \end{cases}$$

Conservative Multistage Bloc Voting instance $I = (A, C, U, k, \ell, x)$ with $A = \{a_1, a_2, a_3\}$, $C = \{e, f, g, h\}$, $k = 2$, $\ell = 2$, $x = 4$, and

U :

a_i	$u_1(a_i)$	$u_2(a_i)$	$u_3(a_i)$
a_1	$\{e, f\}$	$\{e, h\}$	$\{f, g\}$
a_2	$\{e, g\}$	$\{e, f\}$	$\{f, g\}$
a_3	$\{f, g\}$	$\{g, h\}$	$\{g, h\}$

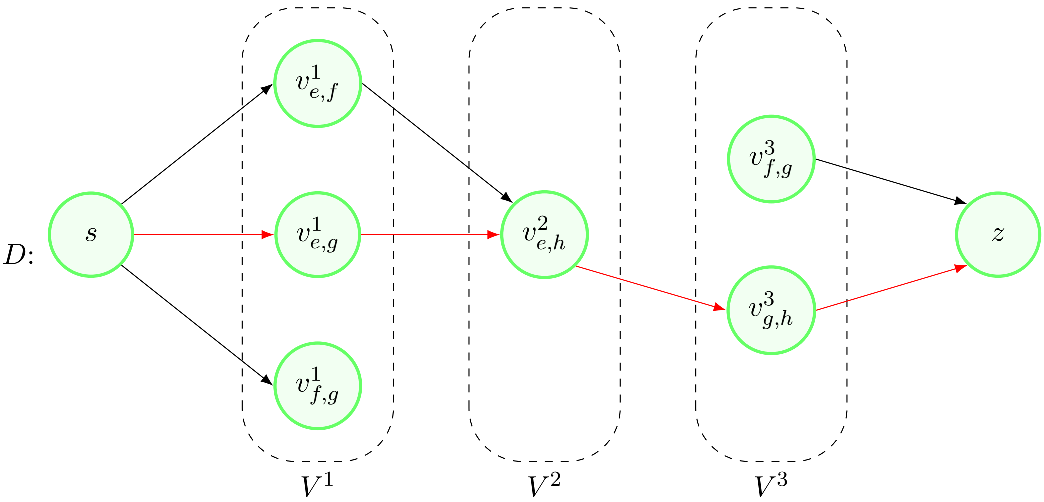


Figure 4.1: An instance of Conservative Multistage Bloc Voting and the corresponding directed graph D constructed as described in the proof of Lemma 4.2. For the sake of readability, we denote $v^t_{\{x,y\}}$ by $v^t_{x,y}$. A possible solution to the given Conservative Multistage Bloc Voting instance and the corresponding s-z path in D are marked with red.

An example of the construction is given in Figure 4.1.

Running time: Note that there are at most $\tau \cdot m^k + 2$ vertices and at most $(\tau - 1) \cdot m^{2k} + 2m^k$ arcs. Hence D can be constructed in $\mathcal{O}(\tau \cdot m^{2k}) \cdot T_{\mathrm{R}}(n, m)$ time. Note that we can check for an s-z path in D in time linear in the size of D.

Correctness: We prove that I is a **yes**-instance if and only if D admits an s-z path.

($\Rightarrow$) Let $(C_1, \dots, C_\tau)$ be a solution to I. Observe that $|C_t| \le k$ and $\text{score}^{\mathrm{R}}_t(C_t) \ge x$ for all $t \in \{1, \dots, \tau\}$. Hence, $v^t_{C_t}$ exists for all $t \in \{1, \dots, \tau\}$. We claim that $P = (s, v^1_{C_1}, \dots, v^\tau_{C_\tau}, z)$ is an s-z path in D. Clearly, the arcs $(s, v^1_{C_1})$ and $(v^\tau_{C_\tau}, z)$ are in A. For each $t \in \{1, \dots, \tau - 1\}$, the arc $(v^t_{C_t}, v^{t+1}_{C_{t+1}})$ exists since $|C_t \Delta C_{t+1}| \le \ell$ ($\ge \ell$ in the revolutionary case). Hence, P is an s-z path in D.

($\Leftarrow$) Let $P = (s, v^1_{C_1}, \dots, v^\tau_{C_\tau}, z)$ be an s-z path in D. We claim that $(C_1, \dots, C_\tau)$ is a solution to I. First, observe that $|C_t| \le k$ and $\text{score}^{\mathrm{R}}_t(C_t) \ge x$ for all $t \in \{1, \dots, \tau\}$. Moreover, we have that $|C_t \Delta C_{t+1}| \le \ell$ ($\ge \ell$ in the revolutionary case) for each $t \in \{1, \dots, \tau - 1\}$. The claim thus follows. □

Theorem 4.1 now follows from Lemma 4.2 and Corollary 3.1. Since $k \leq m$, it follows that all problem variants are fixed-parameter tractable when parameterized by m.

Corollary 4.3. Π Multistage R Voting *for both* $\Pi \equiv$ Conservative *and* $\Pi \equiv$ Revolutionary *admits*

(i) an $\mathcal{O}(\tau \cdot m^{2m+2}n)$ *algorithm if* R $\in$ {Veto, Bloc, Borda, Copeland$^{\alpha}$, α-Chamberlin-Courant, β-Chamberlin-Courant},

(ii) an $\mathcal{O}(\tau \cdot m^{2m} \cdot poly(n, m))$ *algorithm if* R $\in$ {α-Monroe, β-Monroe}

and hence is contained in FPT *when parameterized by* m.

Next, we make some observations and use some prior results to show that Π Multistage R Voting for R $\in$ {Veto, Borda, Copeland$^{0.5}$, Copeland^{1}, β-Chamberlin-Courant} is contained in XP with respect to parameter x as well.

Observation 4.4. *Let* $I = (A, C, P, k, \ell, x)$ *be an instance of* Π Multistage Borda Voting. *For every committee* $C' \subseteq C$ *it holds that* $\text{score}_t^{\text{Borda}}(C') \geq \frac{|C'|^2 - |C'|}{2} \cdot |A|$.

Proof. A committee $C' \subseteq C$ gets a score of at least $\frac{|C'|^2-|C'|}{2}$ from every agent, even if the members of the committee are the least preferred candidates by the agent. □

Observation 4.5. *Let* $I = (A, C, P, k, \ell, x)$ *be an instance of* Π Multistage Copeland$^{\alpha}$ Voting *with* $\alpha \in \{0.5, 1\}$. *For every committee* $C' \subseteq C$ *it holds that* $\text{score}_t^{\text{Copeland}^{\alpha}}(C') \geq \frac{|C'|^2-|C'|}{2}$.

Proof. A committee $C' \subseteq C$ gets a score of at least 1 from every pairwise majority comparison between every two candidates in the committee, regardless of the preferences of the agents and there are $\frac{|C'|^2-|C'|}{2}$ pairwise majority comparisons between the candidates of C'. □

Observation 4.6. *Let* $I = (A, C, P, k, \ell, x)$ *be an instance of* Π Multistage β-Chamberlin-Courant Voting. *For every committee* $C' \subseteq C$ *it holds that* $\text{score}_t^{\beta\text{-CC}}(C') \geq |C'| - 1$.

Proof. The score of a committee $C' \subseteq C$ is at least $|C'|-1$, even if there is only one agent and the members of the committee are the least preferred candidates by the agent. □

Let $I = (A, C, P, k, \ell, x)$ be an instance of Π Multistage R Voting with R $\in$ {Veto, Borda, Copeland$^{0.5}$, Copeland^{1}, β-Chamberlin-Courant}. From Observations 3.6 and 4.4 to 4.6 it follows that x can be lower-bounded by a polynomial function $g(k)$ of k, which means that I is a `yes`-instance if $x < g(k)$ and we can assume that $x \geq g(k)$. Together with Theorem 4.1, we thus get the following.

Corollary 4.7. *For* R $\in$ {Veto, Borda, Copeland$^{0.5}$, Copeland^{1}, β-Chamberlin-Courant}, Π Multistage R Voting *parameterized by* x *is contained in* XP.

We gave an XP-algorithm with respect to parameter k and also achieved results for parameters m and x as a side-product. In the following two sections we give parameterized reductions to the variants based on Bloc, Borda and Copeland rules to prove their $W[2]$-hardness when parameterized by k.

4.1 Bloc

We give a parameterized reduction from Hitting Set to CMBlV to show W[2]-hardness with respect to k. This reduction also proves para-NP-hardness of CMBlV when parameterized by x.

Theorem 4.8. Conservative Multistage Bloc Voting *is* W[2]*-hard when parameterized by k, even if $\ell = 0$ and $x = 1$.*

Proof. Let $I = (E, \mathcal{C}, k)$ be an instance of Hitting Set, and let $\mathcal{C} = \{S_1, \ldots, S_m\}$ and $S_t = \{s_t^1, \ldots, s_t^{|S_t|}\}$ for all $t \in \{1, \ldots, m\}$. We construct an instance $I' = (A, C, U, k, \ell, x)$ of CMBlV in polynomial time as follows.
Construction: Let the set C of candidates be equal to $E \cup D$, where $D := \bigcup_{t \in \{1,\ldots,m\}} D_i$, $D_t := \{d_t^1, \ldots, d_t^k\}$ and the set A of agents equal to $\{a_1, \ldots, a_{\lceil |E|/k \rceil}\}$. Next construct m voting profiles as follows. For every profile u_t, $t \in \{1, \ldots, m\}$, set

$$u_t(a_i) = \{s_t^{k(i-1)+1}, \ldots, s_t^{ki}\} \qquad \text{if } i \in \{1, \ldots, \lceil |S_t|/k \rceil - 1\},$$
$$u_t(a_i) = \{s_t^{k(i-1)+1}, \ldots, s_t^{|S_t|}\} \cup \bigcup_{j=1}^{ki-|S_t|} \{d_t^j\} \qquad \text{if } i = \lceil |S_t|/k \rceil, \text{ and}$$
$$u_t(a_i) = D_t \qquad \text{if } i \in \{\lceil |S_t|/k \rceil + 1, \ldots, \lceil |E|/k \rceil\}.$$

Finally, set $\ell = 0$ and $x = 1$. An example of the construction is given in Figure 4.2.
Correctness: We prove that I is a `yes`-instance if and only if I' is a `yes`-instance. First, note that for every $t \in \{1, \ldots, m\}$ it holds that

$$\begin{aligned}
\text{score}_t^{\text{Bloc}}(C_t) &= \sum_{a_i \in A} |u_t(a_i) \cap C_t| \\
&= |u_t(a_1) \cap C_t| + \ldots + \left|u_t(a_{\lceil |S_t|/k \rceil - 1}) \cap C_t\right| + \left|u_t(a_{\lceil |S_t|/k \rceil}) \cap C_t\right| \\
&\quad + \left|u_t(a_{\lceil |S_t|/k \rceil + 1}) \cap C_t\right| + \ldots + \left|u_t(a_{\lceil |E|/k \rceil}) \cap C_t\right| \\
&= \left| \bigcup_{i=1}^{\lceil |S_t|/k \rceil} u_t(a_i) \cap C_t \right| + \sum_{i=\lceil |S_t|/k \rceil + 1}^{\lceil |E|/k \rceil} |D_t \cap C_t| \\
&= \left| \left(S_t \cup \bigcup_{j=1}^{k\lceil |S_t|/k \rceil - |S_t|} \{d_t^j\} \right) \cap C_t \right| + \sum_{i=\lceil |S_t|/k \rceil + 1}^{\lceil |E|/k \rceil} |D_t \cap C_t| \\
&= |S_t \cap C_t| + \left| \bigcup_{j=1}^{k\lceil |S_t|/k \rceil - |S_t|} \{d_t^j\} \cap C_t \right| + (\lceil |E|/k \rceil - \lceil |S_t|/k \rceil) |D_t \cap C_t| .
\end{aligned}$$

($\Rightarrow$) Let $X \subseteq E$ be a hitting set of size at most k. We claim that the sequence $(C_1, \ldots, C_m)$ with $C_t = X$ for every $t \in \{1, \ldots, m\}$ is a solution to I'. Firstly, observe that $C_t \Delta C_{t+1} = \emptyset$ for every $t \in \{1, \ldots, m-1\}$, and that $|C_t| \leq k$ for every $t \in \{1, \ldots, m\}$. Since $S_t \cap X \neq \emptyset$ for every $t \in \{1, \ldots, m\}$, we have $\text{score}_t^{\text{Bloc}}(C_t) \geq 1 = x$ for every $t \in \{1, \ldots, m\}$. This proves the claim.

HITTING SET instance $I = (E, \mathcal{C}, k)$ with

$E = \{e, f, g, h\}$, $\mathcal{C} = \{S_1 = \{e, f, g\}, S_2 = \{h\}, S_3 = \{f, g\}\}$, and $k = 2$

CONSERVATIVE MULTISTAGE BLOC VOTING instance $I' = (A, C, U, k, \ell, x)$ with

$A = \{a_1, a_2\}$, $C = \{e, f, g, h\} \cup \bigcup_{j=1}^{3} \{d_j^1, d_j^2\}$, $k = 2$, $\ell = 0$, $x = 1$, and

U :

a_i	$u_1(a_i)$	$u_2(a_i)$	$u_3(a_i)$
a_1	$\{e, f\}$	$\{h, d_2^1\}$	$\{f, g\}$
a_2	$\{g, d_1^1\}$	$\{d_2^1, d_2^2\}$	$\{d_3^1, d_3^2\}$

Figure 4.2: An instance of HITTING SET and the equivalent instance of CONSERVATIVE MULTISTAGE BLOC VOTING, constructed as described in the proof of Theorem 4.8. Possible solutions are marked with red in both instances.

($\Leftarrow$) Let $(C_1, ..., C_\tau)$ be a solution to I'. Note that since $\ell = 0$, $C_i = C_j$ for every $i, j \in \{1, \dots, m\}$. For every $t \in \{1, \dots, m\}$ such that $C_1 \cap D_t \neq \emptyset$, pick an arbitrary candidate from S_t and let Y be the set of these picked candidates. We claim that $X := (C_1 \cap E) \cup Y$ is a hitting set of E (note that $|X| \leq k$). Let $t \in \{1, \dots, m\}$ be arbitrary but fixed. Since $\text{score}_t^{\text{BLOC}}(C_t) \geq 1$ we know that $(S_t \cup D_t) \cap C_t \neq \emptyset$. If $C_1 \cap S_t = \emptyset$, then $C_1 \cap D_t \neq \emptyset$ and Y contains a candidate from S_t, and if $C_1 \cap S_t \neq \emptyset$, $C_1 \cap E$ contains a candidate from S_t. Hence, $X \cap S_t \neq \emptyset$. Since t was chosen arbitrarily, the claim follows. □

4.2 Borda and Copeland

We give a parameterized reduction from HITTING SET to CMBoV and CMC$^\alpha$V to prove their W[2]-hardness when parameterized by k.

Theorem 4.9. CONSERVATIVE MULTISTAGE COPELAND$^\alpha$ VOTING *and* CONSERVATIVE MULTISTAGE BORDA VOTING *are* W[2]*-hard when parameterized by k, even for two agents and $\ell = 0$.*

Proof. Let $I = (E, \mathcal{C}, k)$ be an instance of HITTING SET, and let $\mathcal{C} = \{S_1, \dots, S_m\}$. We construct an instance $I' = (A, C, P, k, \ell, x)$ of CMRV with R $\in$ {BORDA, COPELAND$^\alpha$} in polynomial time as follows.
Construction: Let the set C of candidates be equal to $E \cup D$, where $D := \bigcup_{t \in \{1,\dots,m\}} D_i$, and $D_t := \{d_t^1, \dots, d_t^{|E|}\}$ for all $t \in \{1, \dots, m\}$, and let the set A of agents be equal to $\{a_1, a_2\}$. We construct m voting profiles as follows. Let $M_t := S_t \cup \{d_t^1, \dots, d_t^{|E|-|S_t|}\}$ and $M_t' := (E \cup D_t) \setminus M_t$. For profile p_t, $t \in \{1, \dots, m\}$, set $c \succ_{t,i} e$, $c \succ_{t,i} d$, $e \succ_{t,i} d$ for every $c \in M_t$, $e \in M_t'$, $d \in D \setminus D_t$ and $i \in \{1, 2\}$. Rank candidates $\{c, c'\} \subseteq M$, $M \in \{M_t, M_t', D \setminus D_t\}$, in any order among themselves such that $c \succ_{t,1} c'$ if and only

if $c \succ_{t,2} c'$. Finally, set $\ell = 0$ and

$$x = \begin{cases} (2\,|E|\,m - |E| - 1)k + 1, & \text{if R} = \text{Borda}, \\ (|E|\,m + (\alpha - 1)\,|E| - \alpha)\,k + 1, & \text{if R} = \text{Copeland}^{\alpha}. \end{cases}$$

This finishes the construction. An example of the construction is given in Figure 4.3.
Correctness: We prove that I is a yes-instance if and only if I' is a yes-instance. First, note that for every $t \in \{1, \ldots, m\}$ and $C' \subseteq C$

$$\begin{aligned} \text{score}_t^{\text{Borda}}(C') &= \sum_{c \in C'} \sum_{a_i \in A} \left|\{c' \in C \mid c \succ_{t,i} c'\}\right| \\ &= \sum_{c \in C'} \left(\left|\{c' \in C \mid c \succ_{t,1} c'\}\right| + \left|\{c' \in C \mid c \succ_{t,2} c'\}\right|\right) \\ &= \sum_{c \in C' \cap M_t} (|M_t \setminus \{c\}| + 2\,|C \setminus M_t|) + \sum_{c \in C' \cap M'_t} \left(\left|M'_t \setminus \{c\}\right| + 2\,|D \setminus D_t|\right) \\ &\quad + \sum_{c \in C' \cap (D \setminus D_t)} |(D \setminus D_t) \setminus \{c\}| \\ &= \left|C' \cap M_t\right| (2\,|E|\,m + |E| - 1) + \left|C' \cap M'_t\right| (2\,|E|\,m - |E| - 1) \\ &\quad + \left|C' \cap (D \setminus D_t)\right| (|E|\,m - |E| - 1) \end{aligned}$$

and
$$\begin{aligned} \text{score}_t^{\text{Copeland}^{\alpha}}(C') &= \sum_{c \in C'} \left(\left|\{c' \in C \mid c >^{\mu}_t c'\}\right| + \alpha\left|\{c' \in C \setminus \{c\} \mid c' =^{\mu}_t c\}\right|\right) \\ &= \sum_{c \in C' \cap M_t} (|C \setminus M_t| + \alpha\,|M_t \setminus \{c\}|) + \sum_{c \in C' \cap M'_t} \left(|D \setminus D_t| + \alpha\left|M'_t \setminus \{c\}\right|\right) \\ &\quad + \sum_{c \in C' \cap (D \setminus D_t)} (|\emptyset| + \alpha\,|(D \setminus D_t) \setminus \{c\}|) \\ &= \left|C' \cap M_t\right| (|E|\,m + \alpha\,|E| - \alpha) + \left|C' \cap M'_t\right| (|E|\,m + (\alpha - 1)\,|E| - \alpha) \\ &\quad + \left|C' \cap (D \setminus D_t)\right| (\alpha\,|E|\,m - \alpha\,|E| - \alpha)\,. \end{aligned}$$

($\Rightarrow$) Let $X \subseteq E$ be a hitting set of size k. We claim that the sequence $(C_1, \ldots, C_m)$ with $C_t = X$ for every $t \in \{1, \ldots, m\}$ is a solution to I'. Firstly, observe that $C_t \Delta C_{t+1} = \emptyset$ for every $t \in \{1, \ldots, m-1\}$, and that $|C_t| \leq k$ for every $t \in \{1, \ldots, m\}$. Since $S_t \cap X \neq \emptyset$, and hence $M_t \cap X \neq \emptyset$, for all $t \in \{1, \ldots, m\}$, we have

$$\text{score}_t^{\text{Borda}}(C_t) \geq (2\,|E|\,m + |E| - 1) + (k - 1)(2\,|E|\,m - |E| - 1) \geq x, \text{ and}$$

$$\text{score}_t^{\text{Copeland}^{\alpha}}(C_t) \geq (|E|\,m + \alpha\,|E| - \alpha) + (k - 1)\,(|E|\,m + (\alpha - 1)\,|E| - \alpha) \geq x$$

for every $t \in \{1, \ldots, m\}$. This proves the claim.

($\Leftarrow$) Let $(C_1, ..., C_\tau)$ be a solution to I'. Note that since $\ell = 0$, $C_i = C_j$ for every $i, j \in \{1, \ldots, m\}$. For every $t \in \{1, \ldots, m\}$ such that $C_1 \cap D_t \neq \emptyset$, pick an arbitrary candidate from S_t and let Y be the set of these picked candidates. We claim that $X := (C_1 \cap E) \cup Y$ is a hitting set of E (note that $|X| \leq k$). Let $t \in \{1, \ldots, m\}$ be arbitrary but fixed. Since $\text{score}_t^{\text{Borda}}(C_t) \geq k \cdot (2\,|E|\,m - |E| - 1) + 1$ and $\text{score}_t^{\text{Copeland}^{\alpha}}(C_t) \geq$

HITTING SET instance $I = (E, \mathcal{C}, k)$ with

$E = \{e, f, g, h\}$, $\mathcal{C} = \{S_1 = \{e, f, g\}, S_2 = \{h\}, S_3 = \{f, g\}\}$, and $k = 2$

CONSERVATIVE MULTISTAGE R VOTING instance $I' = (A, C, P, k, \ell, x)$ with

$A = \{a_1, a_2\}$, $C = \{e, f, g, h\} \cup \bigcup_{j=1}^{3}(D_j = \{d_j^1, d_j^2, d_j^3, d_j^4\})$, $k = 2$, $\ell = 0$,

$x = \begin{cases} 39, & \text{if R} = \text{BORDA}, \\ 17 + 6\alpha, & \text{if R} = \text{COPELAND}^\alpha, \end{cases}$ and

P :

t	a_i	M_t	M'_t	$D \setminus D_t$
1	a_1	$e \succ f \succ g \succ d_1^1 \succ$	$h \succ d_1^2 \succ d_1^3 \succ d_1^4 \succ$	$\overrightarrow{D_2} \succ \overrightarrow{D_3}$
	a_2	$d_1^1 \succ g \succ f \succ e \succ$	$d_1^4 \succ d_1^3 \succ d_1^2 \succ h \succ$	$\overleftarrow{D_3} \succ \overleftarrow{D_2}$
2	a_1	$h \succ d_2^1 \succ d_2^2 \succ d_2^3 \succ$	$e \succ f \succ g \succ d_2^4 \succ$	$\overrightarrow{D_1} \succ \overrightarrow{D_3}$
	a_2	$d_2^3 \succ d_2^2 \succ d_2^1 \succ h \succ$	$d_2^4 \succ g \succ f \succ e \succ$	$\overleftarrow{D_3} \succ \overleftarrow{D_1}$
3	a_1	$f \succ g \succ d_3^1 \succ d_3^2 \succ$	$e \succ h \succ d_3^3 \succ d_3^4 \succ$	$\overrightarrow{D_1} \succ \overrightarrow{D_2}$
	a_2	$d_3^2 \succ d_3^1 \succ g \succ f \succ$	$d_3^4 \succ d_3^3 \succ h \succ e \succ$	$\overleftarrow{D_2} \succ \overleftarrow{D_1}$

Figure 4.3: An instance of HITTING SET and the equivalent instances of CONSERVATIVE MULTISTAGE BORDA VOTING and CONSERVATIVE MULTISTAGE COPELAND$^\alpha$ VOTING, constructed as described in the proof of Theorem 4.9. $\overrightarrow{D_j}$ denotes $d_j^1 \succ d_j^2 \succ d_j^3 \succ d_j^4$ and $\overleftarrow{D_j}$ denotes $d_j^4 \succ d_j^3 \succ d_j^2 \succ d_j^1$. Possible solutions are marked with red in both instances.

$k \cdot (|E|\, m + (\alpha - 1)\, |E| - \alpha) + 1$, we know that $C_t \cap M_t = C_1 \cap M_t \neq \emptyset$. If $C_1 \cap S_t = \emptyset$, then $C_1 \cap D_t \neq \emptyset$ and Y contains a candidate from S_t, and if $C_1 \cap S_t \neq \emptyset$, $C_1 \cap E$ contains a candidate from S_t. Hence, $X \cap S_t \neq \emptyset$. Since t was chosen arbitrarily, the claim follows. □

Chapter 5

Bound ℓ on the size of symmetric differences

In Chapter 3, we have seen as a side-product of our general hardness results that the conservative variants of all problems are NP-hard for constant bound ℓ on the size of symmetric differences. In this chapter, we study the complexity of the revolutionary variants with respect to the parameter ℓ. We use a similar algorithm as in the work of Bredereck, Fluschnik, and Kaczmarczyk [BFK20] to prove that Revolutionary Multistage R Voting is contained in XP when parameterized by ℓ for R $\in$ {Veto, Bloc, Borda, Copeland$^\alpha$}.

Theorem 5.1. Revolutionary Multistage R Voting *can be solved in* $\mathcal{O}(\tau \cdot m^{4\ell+2} n)$ *time for all* R $\in$ {Veto, Bloc, Borda, Copeland$^\alpha$} *and hence is contained in* XP *when parameterized by* ℓ.

The XP-algorithm uses an auxiliary graph, namely, a so-called *in-out* graph [BFK20]. In order to prove Theorem 5.1 and to define *in-out* graphs we start with some observations. First, we can clearly assume that $\ell \leq 2k$. Second, every possible solution $\mathcal{C} = \{C_1, \dots, C_\tau\}$ to the revolutionary variant of our problems has a common structure. Since $|C_i \Delta C_{i+1}| \geq \ell$ for each $i \in \{1, \dots, \tau - 1\}$, there are $X_i \subseteq C_i \setminus C_{i+1}$ and $Y_i \subseteq C_{i+1} \setminus C_i$ such that $X_i \cap Y_i = \emptyset$ and $|X_i \cup Y_i| \geq \ell$.

The construction of an *in-out* graph is based on the latter observation. We define a vertex set for each pair of consecutive stages containing a vertex for each possible (X_i, Y_i) pair. Thus, every vertex indicates which candidates from a prior committee are excluded in the following one and which candidates that are not contained in the prior committee are included in the following one. Edges connect the vertices that can follow each other considering the differences between committees they represent. Formally, an *in-out* graph is defined as follows:

Definition 5.2. The *in-out* graph of an instance $I = (A, C, U, k, \ell, x)$ of Revolutionary Multistage R Voting is a directed graph $D_I = (V, E)$ with vertex set

$$V = V^1 \cup \dots \cup V^{\tau-1} \cup \{s, z\} \text{ where}$$
$$V^i = \{v^i_{X,Y} \mid X, Y \subseteq C,\ |X|, |Y| \leq \ell,\ X \cap Y = \emptyset,\ |X \cup Y| \geq \ell\},$$

REVOLUTIONARY MULTISTAGE BLOC VOTING instance $I = (A, C, U, k, \ell, x)$ with $A = \{a_1, a_2, a_3\}$, $C = \{e, f, g, h\}$, $k = 2$, $\ell = 3$, $x = 4$, and

U :

a_i	$u_1(a_i)$	$u_2(a_i)$	$u_3(a_i)$
a_1	$\{e, f\}$	$\{e, h\}$	$\{f, g\}$
a_2	$\{e, g\}$	$\{e, f\}$	$\{f, g\}$
a_3	$\{f, g\}$	$\{g, h\}$	$\{g, h\}$

D_I:

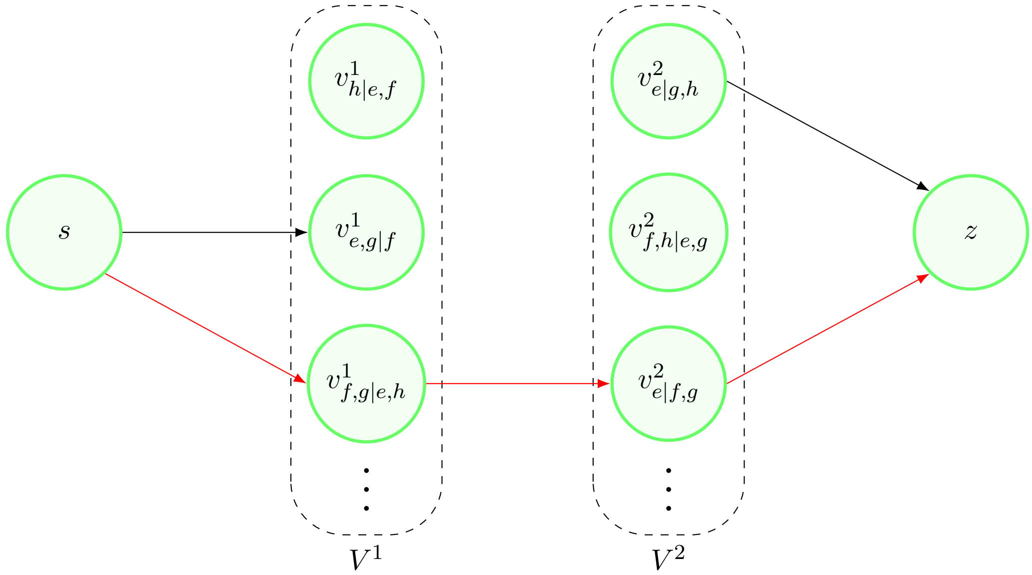


Figure 5.1: An instance of REVOLUTIONARY MULTISTAGE BLOC VOTING and its in-out graph D_I constructed according to Definition 5.2. For the sake of readability, we denote $v^t_{\{x,\dots\},\{y,\dots\}}$ by $v^t_{x,\dots|y,\dots}$. A possible solution to the given REVOLUTIONARY MULTISTAGE BLOC VOTING instance and the corresponding s-z path in D_I are marked with red.

and arc set

$$
\begin{aligned}
E &= E^1 \cup \dots \cup E^\tau \text{ where}\\
E^1 &= \{(s, v^1_{X,Y}) \mid C' \subseteq C \setminus Y,\ X \subseteq C',\ |C'| \leq k,\ \text{score}^{\text{R}}_1(C') \geq x\},\\
E^i &= \{(v^{i-1}_{X,Y}, v^i_{X',Y'}) \mid C' \subseteq C \setminus (X \cup Y'),\ Y, X' \subseteq C',\ |C'| \leq k,\ \text{score}^{\text{R}}_i(C') \geq x\}\\
&\quad \text{for all } i \in \{2, \dots, \tau - 1\}, \text{ and}\\
E^\tau &= \{(v^{\tau-1}_{X,Y}, z) \mid C' \subseteq C \setminus X,\ Y \subseteq C',\ |C'| \leq k,\ \text{score}^{\text{R}}_\tau(C') \geq x\}.
\end{aligned}
$$

An example of an *in-out* graph is given in Figure 5.1.

Lemma 5.3. *Given an instance* $I = (A, C, U, k, \ell, x)$ *of* REVOLUTIONARY MULTISTAGE

R Voting *with n agents, m candidates, τ voting profiles, and* $\mathrm{R} \in \{$Veto, Bloc, Borda, Copeland$^\alpha\}$*, the in-out graph D_I of I can be computed in $\mathcal{O}(\tau \cdot m^{4\ell+2}n)$ time.*

Proof. The number of candidate subsets of a maximum size of ℓ is at most m^ℓ. Hence, we can compute each V_i in $\mathcal{O}(m^{2\ell})$ time by brute-forcing every pair of candidate subsets each of size at most ℓ. Thus, we can compute V in $\mathcal{O}(\tau \cdot m^{2\ell})$ time. Every arc set E^i contains at most $m^{4\ell}$ arcs and since the computation of the scores needs $\mathcal{O}(m^2 n)$ time, each arc can be computed in $\mathcal{O}(m^2 n)$ time, resulting in an overall running time in $\mathcal{O}(\tau \cdot m^{4\ell+2}n)$. □

Next, we prove that we can decide an instance I of Revolutionary Multistage R Voting for $\mathrm{R} \in \{$Veto, Bloc, Borda, Copeland$^\alpha\}$ by deciding whether the in-out graph D_I contains an s-z path.

Lemma 5.4. *Let I be an instance of* Revolutionary Multistage R Voting *and D_I its in-out graph. Then, there is an s-z path in D_I if and only if I is a* `yes`*-instance.*

Proof. Let $I = (A, C, U, k, \ell, x)$ be an instance of RMRV and D_I the in-out graph for I.

($\Rightarrow$) By construction, we know that if there is an s-z path in D_I, then it is of the form $P = (s, v^1_{X_1,Y_1}, \ldots, v^{\tau-1}_{X_{\tau-1},Y_{\tau-1}}, z)$. Since arcs $(s, v^1_{X_1,Y_1})$ and $(v^{\tau-1}_{X_{\tau-1},Y_{\tau-1}}, z)$ exist, there is a committee C_1 containing X_1 but disjoint from Y_1 with $\mathrm{score}^{\mathrm{R}}_1(C_1) \geq x$ and $|C_1| \leq k$, and there is a committee C_τ containing $Y_{\tau-1}$ but disjoint from $X_{\tau-1}$ with $\mathrm{score}^{\mathrm{R}}_\tau(C_\tau) \geq x$ and $|C_\tau| \leq k$. Since the arc $(v^i_{X_i,Y_i}, v^{i+1}_{X_{i+1},Y_{i+1}})$ exists for each $i \in \{1, \ldots, \tau-2\}$, there is a committee C_{i+1} containing $Y_i \cup X_{i+1}$ but disjoint from $X_i \cup Y_{i+1}$ with $\mathrm{score}^{\mathrm{R}}_{i+1}(C_{i+1}) \geq x$ and $|C_{i+1}| \leq k$ for all $i \in \{1, \ldots, \tau-2\}$. Observe that $C_i \Delta C_{i+1} \supseteq X_i \cup Y_i$ and thus, $|C_i \Delta C_{i+1}| \geq \ell$ for all $i \in \{1, \ldots, \tau-1\}$. Hence, we can find a solution $(C_1, \ldots, C_\tau)$ to I if there is an s-z path in D_I.

($\Leftarrow$) Let $(C_1, \ldots, C_\tau)$ be a solution to I. Since $|C_i \Delta C_{i+1}| \geq \ell$ for all $i \in \{1, \ldots, \tau-1\}$, there are $X_i \subseteq C_i \setminus C_{i+1}$ and $Y_i \subseteq C_{i+1} \setminus C_i$ with $|X_i \cap Y_i| \geq \ell$ and $|X_i|, |Y_i| \leq \ell$, and vertex $v^i_{X_i,Y_i}$ exist for all $i \in \{1, \ldots, \tau-1\}$. We claim that $P = (s, v^1_{X_1,Y_1}, \ldots, v^{\tau-1}_{X_{\tau-1},Y_{\tau-1}}, z)$ forms an s-z path in D_I. Note that $|C_i| \leq k$ and $\mathrm{score}^{\mathrm{R}}_i(C_i) \geq x$ for all $i \in \{1, \ldots, \tau\}$. Since C_1 contains X_1 and is disjoint from Y_1, arc $(s, v^1_{X_1,Y_1})$ exists and since C_τ contains $Y_{\tau-1}$ and is disjoint from $X_{\tau-1}$, arc $(v^{\tau-1}_{X_{\tau-1},Y_{\tau-1}}, z)$ exists. Observe that every C_i with $i \in \{2, \ldots, \tau-1\}$ contains X_i and Y_{i-1} and is disjoint from X_{i-1} and Y_i. Hence, arc $(v^{i-1}_{X_{i-1},Y_{i-1}}, v^i_{X_i,Y_i})$ exists for all $i \in \{2, \ldots, \tau-1\}$. This proves the claim. □

Given Lemmas 5.3 and 5.4, we are set to prove Theorem 5.1.

Proof. Let $I = (A, C, U, k, \ell, x)$ be an instance of Revolutionary Multistage R Voting with n agents, m candidates, τ voting profiles, and $\mathrm{R} \in \{$Veto, Bloc, Borda, Copeland$^\alpha\}$. We construct the in-out graph D_I of I in $\mathcal{O}(\tau \cdot m^{4\ell+2}n)$ time and then check for an s-z path in D_I in time linear in the size of D_I. If an s-z path is found, then I is a `yes`-instance, and if no such s-z path is found, then I is a `no`-instance. □

Unlike the other rules, problems based on Chamberlin-Courant and Monroe rules are NP-hard even for constant ℓ. To show their para-NP-hardness with respect to ℓ we use their NP-hardness even for single-stage elections, i.e., $\tau = 1$.

Proposition 5.5. Revolutionary Multistage R Voting *with* $\ell = 0$ *is* NP*-hard for all* R $\in$ {α-Chamberlin-Courant, β-Chamberlin-Courant, α-Monroe, β-Monroe}*.*

Proof. Assume that all instances of Revolutionary Multistage R Voting with $\ell = 0$ and R $\in$ {α-Chamberlin-Courant, β-Chamberlin-Courant, α-Monroe, β-Monroe} can be decided in polynomial time. Then, particularly the instances with $\ell = 0$ and $\tau = 1$ are polynomial-time solvable. Since the instances with $\tau = 1$ that only differ in ℓ are equivalent amongst themselves, it implies that all instances with $\tau = 1$ are decidable in polynomial time. This contradicts the NP-hardness of the Chamberlin-Courant and Monroe rules for $\tau = 1$. Hence, the proposition follows. □

Chapter 6

Number τ of stages

In this chapter, we focus on the parameterized complexity with respect to the parameter τ. In Section 3.4, we reference some prior studies showing the NP-hardness of the Chamberlin-Courant and Monroe rules even for a constant number of stages, whereas both variants of our multistage problem based on other rules are contained in XP regarding the number of stages. To prove the following theorem, we give a dynamic programming algorithm similar to the one used to prove Theorem 5.1 by Bredereck, Fluschnik, and Kaczmarczyk [BFK20].

Theorem 6.1. *For* R $\in$ {Veto, Bloc, Borda, Copeland$^\alpha$}*,* Π Multistage R Voting *parameterized by* τ *is contained in* XP*.*

First, we define the boolean dynamic programming table needed for our algorithm.

Definition 6.2. The *back-to-back* table of an instance $I = (A, C, U, k, \ell, x)$ with $n = |A|$ and $C = \{c_1, \dots, c_m\}$ of Π Multistage R Voting with R $\in$ {Veto, Bloc, Borda, Copeland$^\alpha$} is a boolean table

$$T[i, k_1, \dots, k_\tau, d_1, \dots, d_{\tau-1}, s_1, \dots, s_\tau]$$

with $0 \le i \le m$, $0 \le k_t \le k$, $0 \le d_t \le 2k$, and $0 \le s_t \le z$, where

$$z := \begin{cases} n \cdot (m-1), & \text{if R=Veto,} \\ k \cdot n, & \text{if R=Bloc,} \\ \binom{m}{2} \cdot n = \frac{m^2-m}{2} \cdot n, & \text{if R=Borda,} \\ 2 \cdot \binom{m}{2} = m^2 - m, & \text{if R=Copeland}^\alpha. \end{cases}$$

We interpret the table as follows. An entry $T[i, k_1, \dots, k_\tau, d_1, \dots, d_{\tau-1}, s_1, \dots, s_\tau]$ in the table is true if and only if there are committees $C_1, \dots, C_\tau \subseteq \bigcup_{j=1}^{i} \{c_j\}$ with $|C_t| = k_t$ for all $t \in \{1, \dots, \tau\}$, $|C_t \Delta C_{t+1}| = d_t$ for all $t \in \{1, \dots, \tau - 1\}$, and

$$\text{score}_t^{\text{R}}(C_t) = \begin{cases} s_t, & \text{if R} \in \{\text{Veto, Bloc, Borda, Copeland}^0\text{, Copeland}^1\}, \\ s_t/2, & \text{if R} = \text{Copeland}^{0.5}, \end{cases}$$

for all $t \in \{1, \dots, \tau\}$.

Lemma 6.3. *Given an instance* $I = (A, C, U, k, \ell, x)$ *of* Π MULTISTAGE R VOTING *with* n *agents,* m *candidates,* τ *voting profiles, and* R ∈ {VETO, BLOC, BORDA, COPELAND$^\alpha$}*, the back-to-back table* $T[i, k_1, \dots, k_\tau, d_1, \dots, d_{\tau-1}, s_1, \dots, s_\tau]$ *of* I *can be computed in* $\mathcal{O}(k^{2\tau} \cdot z^\tau \cdot m^2 \cdot n)$ *time, where*

$$z = \begin{cases} n \cdot (m-1), & \textit{if } \text{R=VETO} \\ k \cdot n, & \textit{if } \text{R=BLOC} \\ \frac{m^2-m}{2} \cdot n, & \textit{if } \text{R=BORDA} \\ m^2 - m, & \textit{if } \text{R=COPELAND}^\alpha. \end{cases}$$

Proof. We note that the table is of size $\mathcal{O}(m \cdot k^{2\tau} \cdot z^\tau)$. Next, we show how to fill the table and compute the entries efficiently.

Construction: First, we set $T[0, \dots, 0]$ to true and all other entries $T[0, \dots]$ to false. For $i > 0$, we compute all entries $T[i, \dots]$ as follows, assuming all entries $T[i-1, \dots]$ have already been computed. For every "true"-entry $T[i-1, k_1, \dots, k_\tau, d_1, \dots, d_{\tau-1}, s_1, \dots, s_\tau]$ and subset $F \subseteq \{1, \dots, \tau\}$, which is interpreted as the subset of stages in which the ith candidate is selected to the committee, we set $T[i, k'_1, \dots, k'_\tau, d'_1, \dots, d'_{\tau-1}, s'_1, \dots, s'_\tau]$, if it exist, to true if

$$k'_t = \begin{cases} k_t + 1, & \text{if } t \in F, \\ k_t, & \text{otherwise}, \end{cases}$$

$$d'_t = \begin{cases} d_t + 1, & \text{if } t \in F \text{ xor } t+1 \in F, \\ d_t, & \text{otherwise}, \end{cases}$$

$$\text{and } s'_t = \begin{cases} s_t + \text{score}_t^{\text{R}}(\{c_i\}), & \text{if } t \in F \text{ and R} \neq \text{COPELAND}^{0.5}, \\ s_t + 2 \cdot \text{score}_t^{\text{R}}(\{c_i\}), & \text{if } t \in F \text{ and R} = \text{COPELAND}^{0.5}, \\ s_t, & \text{otherwise}, \end{cases}$$

for all $t \in \{1, \dots, \tau\}$. Then, set all other entries $T[i, \dots]$ to false.

Running time: There are 2^τ subsets of stages and at most $k^{2\tau} \cdot z^\tau$ "true"-entries $T[i-1, \dots]$, and the score of a single candidate can be computed in $\mathcal{O}(m \cdot n)$ time. Thus, we can compute the "true"-entries for each i in $\mathcal{O}(k^{2\tau} \cdot z^\tau \cdot m \cdot n)$ time. The "false"-entries cost another $\mathcal{O}(k^{2\tau} \cdot z^\tau)$ time for each i. Hence, the whole table can be computed in $\mathcal{O}(k^{2\tau} \cdot z^\tau \cdot m^2 \cdot n)$ time.

Correctness: We claim that each entry $T[i, k_1, \dots, k_\tau, d_1, \dots, d_{\tau-1}, s_1, \dots, s_\tau]$ in the table is set to true if and only if there are committees $C_1, \dots, C_\tau \subseteq \bigcup_{j=1}^{i} \{c_j\}$ with $|C_t| = k_t$ for all $t \in \{1, \dots, \tau\}$, $|C_t \Delta C_{t+1}| = d_t$ for all $t \in \{1, \dots, \tau - 1\}$, and

$$\text{score}_t^{\text{R}}(C_t) = \begin{cases} s_t, & \text{if R} \in \{\text{VETO, BLOC, BORDA, COPELAND}^0\text{, COPELAND}^1\}, \\ s_t/2, & \text{if R} = \text{COPELAND}^{0.5}, \end{cases}$$

for all $t \in \{1, \dots, \tau\}$.

We prove our claim via induction and note that our claim is correct for all initial entries with $i = 0$. Now, assume that for some $i \in \{0, \dots, m-1\}$ it holds that $T[i, k_1, \ldots, k_\tau, d_1, \ldots, d_{\tau-1}, s_1, \ldots, s_\tau]$ is set to true if and only if there are committees $C_1, \ldots, C_\tau \subseteq \bigcup_{j=1}^{i} \{c_j\}$ with $|C_t| = k_t$ for all $t \in \{1, \ldots, \tau\}$, $|C_t \Delta C_{t+1}| = d_t$ for all $t \in \{1, \ldots, \tau - 1\}$, and

$$\text{score}_t^{\text{R}}(C_t) = \begin{cases} s_t, & \text{if R} \in \{\text{Veto, Bloc, Borda, Copeland}^0\text{, Copeland}^1\}, \\ s_t/2, & \text{if R} = \text{Copeland}^{0.5}, \end{cases}$$

for all $t \in \{1, \ldots, \tau\}$.

Next, we show that each table entry $T[i+1, k_1, \ldots, k_\tau, d_1, \ldots, d_{\tau-1}, s_1, \ldots, s_\tau]$ is also set to true if and only if there are committees $C_1, \ldots, C_\tau \subseteq \bigcup_{j=1}^{i+1} \{c_j\}$ with $|C_t| = k_t$ for all $t \in \{1, \ldots, \tau\}$, $|C_t \Delta C_{t+1}| = d_t$ for all $t \in \{1, \ldots, \tau - 1\}$, and

$$\text{score}_t^{\text{R}}(C_t) = \begin{cases} s_t, & \text{if R} \in \{\text{Veto, Bloc, Borda, Copeland}^0\text{, Copeland}^1\}, \\ s_t/2, & \text{if R} = \text{Copeland}^{0.5}, \end{cases}$$

for all $t \in \{1, \ldots, \tau\}$.

($\Rightarrow$) Let $T[i+1, k_1^*, \ldots, k_\tau^*, d_1^*, \ldots, d_{\tau-1}^*, s_1^*, \ldots, s_\tau^*]$ be arbitrary but fixed and set to true. This implies that there is a subset $F^* \subseteq \{1, \ldots, \tau\}$ and a table entry $T[i, k_1', \ldots, k_\tau', d_1', \ldots, d_{\tau-1}', s_1', \ldots, s_\tau']$ set to true with

$$k_t^* = \begin{cases} k_t' + 1, & \text{if } t \in F^*, \\ k_t', & \text{otherwise}, \end{cases}$$

$$d_t^* = \begin{cases} d_t' + 1, & \text{if } t \in F^* \text{ xor } t+1 \in F^*, \\ d_t' & \text{otherwise}, \end{cases}$$

$$\text{and } s_t^* = \begin{cases} s_t' + \text{score}_t^{\text{R}}(\{c_{i+1}\}), & \text{if } t \in F^* \text{ and R} \neq \text{Copeland}^{0.5}, \\ s_t' + 2 \cdot \text{score}_t^{\text{R}}(\{c_{i+1}\}), & \text{if } t \in F^* \text{ and R} = \text{Copeland}^{0.5}, \\ s_t', & \text{otherwise}, \end{cases}$$

for all $t \in \{1, \ldots, \tau\}$. Thus, there are committees $C_1', \ldots, C_\tau' \subseteq \bigcup_{j=1}^{i} \{c_j\}$ with $|C_t'| = k_t'$ for all $t \in \{1, \ldots, \tau\}$, $\left|C_t' \Delta C_{t+1}'\right| = d_t'$ for all $t \in \{1, \ldots, \tau - 1\}$, and

$$\text{score}_t^{\text{R}}(C_t') = \begin{cases} s_t', & \text{if R} \in \{\text{Veto, Bloc, Borda, Copeland}^0\text{, Copeland}^1\}, \\ s_t'/2, & \text{if R} = \text{Copeland}^{0.5}, \end{cases}$$

for all $t \in \{1, \ldots, \tau\}$. We can easily observe that for the committees $C_1^*, \ldots, C_\tau^* \subseteq \bigcup_{j=1}^{i+1} \{c_j\}$ where

$$C_t^* = \begin{cases} C_t' \cup \{c_{i+1}\}, & \text{if } t \in F^*, \\ C_t', & \text{otherwise}, \end{cases}$$

it holds that $|C_t^*| = k_t^*$ for all $t \in \{1, \ldots, \tau\}$, $\left|C_t^* \Delta C_{t+1}^*\right| = d_t^*$ for all $t \in \{1, \ldots, \tau - 1\}$, and

$$\text{score}_t^{\text{R}}(C_t^*) = \begin{cases} s_t^*, & \text{if R} \in \{\text{Veto, Bloc, Borda, Copeland}^0\text{, Copeland}^1\}, \\ s_t^*/2, & \text{if R} = \text{Copeland}^{0.5}, \end{cases}$$

for all $t \in \{1, \dots, \tau\}$.

($\Leftarrow$) Let $C_1^*, \dots, C_\tau^* \subseteq \bigcup_{j=1}^{i+1}\{c_j\}$ be committees with $|C_t^*| \le k$ for all $t \in \{1, \dots, \tau\}$. Let $k_t^* := |C_t^*|$ for all $t \in \{1, \dots, \tau\}$, $d_t^* := \left|C_t^* \Delta C_{t+1}^*\right|$ for all $t \in \{1, \dots, \tau-1\}$, and

$$s_t^* = \begin{cases} \text{score}_t^{\text{R}}(C_t^*), & \text{if R} \in \{\text{Veto, Bloc, Borda, Copeland}^0\text{, Copeland}^1\}, \\ 2 \cdot \text{score}_t^{\text{R}}(C_t^*), & \text{if R} = \text{Copeland}^{0.5}, \end{cases}$$

for all $t \in \{1, \dots, \tau\}$. We claim that $T[i+1, k_1^*, \dots, k_\tau^*, d_1^*, \dots, d_{\tau-1}^*, s_1^*, \dots, s_\tau^*]$ is set to true. Let $F = \{t \in \{1, \dots, \tau\} \mid c_{i+1} \in C_t^*\}$. Note that there are committees $C_1', \dots, C_\tau' \subseteq \bigcup_{j=1}^{i}\{c_j\}$ where

$$C_t' = \begin{cases} C_t^* \setminus \{c_{i+1}\}, & \text{if } t \in F, \\ C_t^*, & \text{otherwise,} \end{cases}$$

$$\text{with } \left|C_t'\right| = k_t' = \begin{cases} k_t^* - 1, & \text{if } t \in F, \\ k_t^*, & \text{otherwise,} \end{cases}$$

$$\left|C_t' \Delta C_{t+1}'\right| = d_t' = \begin{cases} d_t^* - 1, & \text{if } t \in F \text{ xor } t+1 \in F, \\ d_t', & \text{otherwise,} \end{cases}$$

$$\text{score}_t^{\text{R}}(C_t') = \begin{cases} s_t', & \text{if R} \in \{\text{Veto, Bloc, Borda, Copeland}^0\text{, Copeland}^1\}, \\ s_t'/2, & \text{if R} = \text{Copeland}^{0.5}, \end{cases}$$

$$\text{and } s_t' = \begin{cases} s_t^* - \text{score}_t^{\text{R}}(\{c_{i+1}\}), & \text{if } t \in F \text{ and R} \neq \text{Copeland}^{0.5}, \\ s_t^* - 2 \cdot \text{score}_t^{\text{R}}(\{c_{i+1}\}), & \text{if } t \in F \text{ and R} = \text{Copeland}^{0.5}, \\ s_t^*, & \text{otherwise,} \end{cases}$$

for all $t \in \{1, \dots, \tau\}$. Thus, $T[i, k_1', \dots, k_\tau', d_1', \dots, d_{\tau-1}', s_1', \dots, s_\tau']$ was by inductive assumption set to true. Using the subset $F = \{t \mid c_{i+1} \in C_t^*\}$, our algorithm would set $T[i+1, k_1^*, \dots, k_\tau^*, d_1^*, \dots, d_{\tau-1}^*, s_1^*, \dots, s_\tau^*]$ to true and the claim follows. ☐

Given Lemma 6.3, we are set to prove Theorem 6.1.

Proof. Let $I = (A, C, U, k, \ell, x)$ be an instance of Π Multistage R Voting and T its *back-to-back* table. As a result of the definition of the *back-to-back* tables, I has a solution if and only if $T[m, k_1', \dots, k_\tau', d_1', \dots, d_{\tau-1}', s_1', \dots, s_\tau']$ is true for some combination of $k_1', \dots, k_\tau'$, $d_1', \dots, d_{\tau-1}', s_1', \dots, s_\tau'$ with $k_t' \le k$ for all $t \in \{1, \dots, \tau\}$, $d_t' \le \ell$ in the conservative variant ($d_t' \ge \ell$ in the revolutionary variant) for all $t \in \{1, \dots, \tau-1\}$, and $s_t' \ge x$ ($s_t' \ge 2x$ if R = Copeland$^{0.5}$) for all $t \in \{1, \dots, \tau\}$. We construct the *back-to-back* table T of I in $\mathcal{O}(k^{2\tau} \cdot z^\tau \cdot m^2 \cdot n)$ time, where

$$z = \begin{cases} n \cdot (m-1), & \text{if R=Veto} \\ k \cdot n, & \text{if R=Bloc} \\ \frac{m^2-m}{2} \cdot n, & \text{if R=Borda} \\ m^2 - m, & \text{if R=Copeland}^\alpha. \end{cases}$$

and then check for a "true"-entry mentioned above in time linear in the size of T. If a "true"-entry is found, then I is a `yes`-instance, and if no such "true"-entry is found, then I is a `no`-instance. □

Chapter 7

Conclusion

In this thesis, we investigated multistage committee elections for six well-known voting rules, namely, Veto, Bloc, Borda, Copeland, Chamberlin-Courant and Monroe in both the conservative variant and the revolutionary variant. We also distinguished between the variants of Copeland that differ in scoring in the event of a tie, and approval-based and Borda-based variants of Chamberlin-Courant and Monroe.

While setting the classical hardness, we showed that all but four variants are NP-hard even for small number of agents n, which means that they are para-NP-hard with respect to n. We left open whether the other four are also para-NP-hard with respect to n but we strongly believe they are. We also left open for some variants of Copeland and Monroe whether they remain NP-hard for a constant x. Since the score computation is directly dependent to the used voting rule and makes the main difference between the rules, we observed a non-surprising discrepancy in the parameterized complexity results with respect to desired minimum score x. We gave an XP-algorithm to prove that multi-stage committee elections under all but Chamberlin-Courant and Monroe rules become polynomial time solvable for a constant number of stages τ. We left open whether they are also fixed-parameter tractable regarding τ. Moreover, providing an XP-algorithm, we showed that all variants become polynomial-time solvable if the maximum size k of the committees is constant. We also proved W-hardness regarding the parameter k for all conservative variants while we leave open whether the revolutionary variants are W-hard or fixed-parameter tractable.

Apart from some parameterized complexity results that we left open, the parameterized complexity regarding the combinations of multiple parameters could also be investigated. As a result of fixed-parameter tractability with respect to the number of candidates m, we know that all problem variants are fixed-parameter tractable with respect to $m+n$, $m+\tau$ and $m+x$. We believe that it can be observed that some problem variants are fixed-parameter tractable regarding some other combinations (e.g., $n+\tau$) and the existence of polynomial kernels could be an another question.

Although we investigated multistage committee elections for quite a variety of multiwinner rules, there are still some other famous multiwinner voting rules, such as Single Transferable Vote, that could be adapted to multistage setting, and would be the subject of further research. Single Transferable Vote, for example, needs possibly a different problem formulation than ours, since it is not clear how to measure the committee quality.

Another relevant research direction might be the domain restrictions in the multistage setting. As we briefly mentioned in Section 3.4, some NP-hard single-stage multiwinner elections become polynomial-time solvable under some domain restrictions. The reinterpretation of these existing domain restrictions and the possibility of defining new restrictions for the multistage setting can be considered. For example, single-peakedness can be adapted to the multistage setting as a property which is satisfied if all ballots at all time steps are single-peaked with respect to a common linear order of candidates. We can also allow some small changes in the linear order over the time. It would be interesting to investigate whether multistage committee elections become easier under such domain restrictions.

Finally, some other problems from computational social choice, such as singlewinner elections can also be investigated in the multistage setting. The bound on symmetric differences in revolutionary and conservative variants can be replaced by single-candidate similarity concepts for singlewinner elections.

Appendix A

Algorithms

Algorithm A.1: Veto (Bloc) Score Algorithm

Input: A, C, C', u

Output: Veto (Bloc) score of committee $C' \subseteq C$ according to voting profile u of agents A

score $\leftarrow 0$;
forall $a \in A,\ c \in u(a),\ c' \in C'$ **do**
  if $c = c'$ **then**
    score $\leftarrow$ *score* $+ 1$;
return *score*

Algorithm A.2: Borda Score Algorithm

Input: A, C, C', p

Output: Borda score of committee $C' \subseteq C$ according to voting profile p of agents A

score $\leftarrow 0$;
forall $a_i \in A,\ c \in C,\ c' \in C'$ **do**
  if $c' \succ_i c$ **then**
    score $\leftarrow$ *score* $+ 1$;
return *score*

Algorithm A.3: Copeland$^\alpha$ Score Algorithm

Input: A, C, C', p

Output: Copeland$^\alpha$ score of committee $C' \subseteq C$ according to voting profile p of agents A

```
score ← 0;
forall c ∈ C, c′ ∈ C′ do
    net ← 0;
    forall a_i ∈ A do
        if c′ ≻_i c then
            net ← net + 1;
        else if c ≻_i c′ then
            net ← net − 1;
    if net > 0 then
        score ← score + 1;
    else if net = 0 then
        score ← score + α
return score
```

Algorithm A.4: γ_t-α-Chamberlin-Courant Score Algorithm

Input: A, C, C', u

Output: γ_t-α-Chamberlin-Courant score of committee $C' \subseteq C$ according to voting profile u of agents A

```
score ← 0;
forall a ∈ A do
    sat ← 0;
    forall c′ ∈ C′, c ∈ u(a) do
        if c′ = c then
            sat ← 1;
    if t = U then
        score ← score + sat;
    else if t = E ∧ sat < score then
        score ← sat;
return score
```

Algorithm A.5: γ_t-β-Chamberlin-Courant Score Algorithm

Input: A, C, C', p

Output: γ_t-β-Chamberlin-Courant score of committee $C' \subseteq C$ according to voting profile p of agents A

```
score ← 0;
forall a_i ∈ A do
    min ← 0;
    forall c' ∈ C' do
        pos ← 0;
        forall c ∈ C do
            if c ≻_i c' then
                pos ← pos + 1;
        if pos < min then
            min ← pos;
    sat ← len(C) − min;
    if t = U then
        score ← score + sat;
    else if t = E ∧ sat < score then
        score ← sat;
return score
```

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