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Institute of Software Engineering and Theoretical Computer Science
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Expanding the Graph Parameter Hierarchy

Duc Long Tran

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Supervisor and first reviewer: Dr. André Nichterlein
Second reviewer: Prof. Dr. Stephan Kreutzer
Co-Supervisors: Dr. Matthias Bentert

Zusammenfassung

Graphparameter werden heutzutage intensiv untersucht, da ihre Berechnung sehr komplex und ressourcenintensiv, aber dafür auch sehr verbreitet, sind. Basierend auf der Arbeit von Schröder [Sch] erweitern wir die hierarchische Struktur, die Graphparameterhierarchie, die als Werkzeug verwendet werden kann, um alternative Berechnungsmethoden zu finden. Mit dem Ziel, die Graphparameterhierarchie so effizient wie möglich zu erweitern, haben wir sie um die folgenden sechs Parameter erweitert: twin cover number, edge clique cover number, neighborhood diversity, modular-width, c-closure und twin-width. Wir haben eine Methode eingeführt, die jede Beziehung in der Hierarchie bestimmt und gleichzeitig die Menge der zu erfüllenden Bedingungen minimiert. Dabei haben wir die sechs genannten Parameter untersucht und sind auch auf einige interessante Zusammenhänge zwischen verschiedenen Arten von Parametern gestoßen.

Abstract

Graph parameters are highly investigated, since their computation is very complex and resource intensive but also very common these days. Based on the work by Schröder [Sch], we further develop the hierarchical structure called the graph parameter hierarchy, which can be used as a tool to find alternative methods of computation. With the goal of expanding the graph parameter hierarchy as efficient as possible, we extented it by the following six parameters: twin cover number, edge clique cover number, neighborhood diversity, modular-width, c-closure and twin-width. We introduced a method which determines every relation in this hierarchy while minimazing the amount of conditions needed to be fulfilled. While doing so, we investigated the six mentioned parameters and also encountered some interesting connections between different types of parameters.

Contents

Chapter 1

Introduction

Researchers like Li et al. [Li+20] have thoroughly investigated the computation of NP-hard problems, since the computation of huge amounts of data and information is very common these days. With direct solutions being very resource-heavy, alternative methods have been sought after in order to increase efficiency. One particular method is the use of parametrized algorithms, which use other parameters than the input size. In particular for graph parameters, it is very common to relate them to each other, since they are often used to approximate each other. For example, consider the chromatic number, which is NP-hard to compute. Rather than optimizing for the chromatic number, a similar parameter like degeneracy can be used, since the degeneracy can be used as an upper limit of chromatic number while being computable in linear time.

But, it can be quite unintuitive to find these relations between parameters. Relations like between chromatic number and degeneracy are well known, but there are many other parameters that could bound the chromatic number, such as the feedback vertex number. Though, with the huge amount of different graph parameters, trying to check every possible relation is not desirable. Thus, an interest in a graph parameter hierarchy has recently evolved.

With a graph parameter hierarchy, it is possible to determine close relations through transitivity, which makes finding more relations less of a guessing game. The result is a collection of graph parameters which can be used as a tool. By having such a hierarchy as a basis, it is easier to start more in-dept research, since false relations can be dismissed earlier, as well as speed up research, since results can be found easier by investigating similar related parameters. For instance, consider the boxicity of a graph. Next, we want want to know whether modular-width upper bounds boxicity. So instead of having to directly determine the relation between boxicity and modular-width, we already know that if modular-width does not upper bound to chordality, then is also does not upper bound boxicity, since boxicity upper bounds chordality.

1.1 Related Work

There have already been several instances of making a hierarchy for graph parameters. This paper could be considered as a follow-up paper on the graph parameter hierarchy developed by Schröder [Sch]. The focus therein was to provide a complete parameter hierarchy, where as many relations between the parameters as possible are determined.

On that note, complex relations themselves may have already been investigated. Bonnet et al. [Bon+21] showed many results related to the twin-width of a graph. Additionally, Gajarský et al. [GLO13] already formed a smaller parameter hierarchy around the modular-width, which can be implemented as well. Not only research on relations but also characteristics of parameters themselves are useful in this paper. For instance, Terry et al. [TWY] showed specific characteristics for the genus of a graph, which helped in completing the hierarchy.

1.2 Our Contribution

Our goal is to expand the graph parameter hierarchy build up by Schröder [Sch]. For this purpose, we studied the following six parameters: twin cover number, edge clique cover number, neighborhood diversity, modular-width, c-closure and twin-width.

We precisely locate each of their position within the graph parameter hierarchy. For this, instead of investigating every possible relation, we developed method to reduce the amount of necessary proofs. Thus, we present interesting results for each parameter. We investigate in chapter 3 the twin cover number which upper bounds distance to cluster. In chapter 4 we investigate edge clique cover number, neighborhood diversity and modular-width together. We discover that edge clique cover number only upper bounds neighborhood diversity which upper bounds boxicity while modular-width upper bounds max diameter of components. Additionally, we highlight in chapter 5 that c-closure is upper bounded by minimum feedback edge set and in chapter 6 that twin-width is upper bounded by distance to planar. As the main result, we expanded the graph parameter hierarchy by these six parameters, as seen in Figure 1.1.

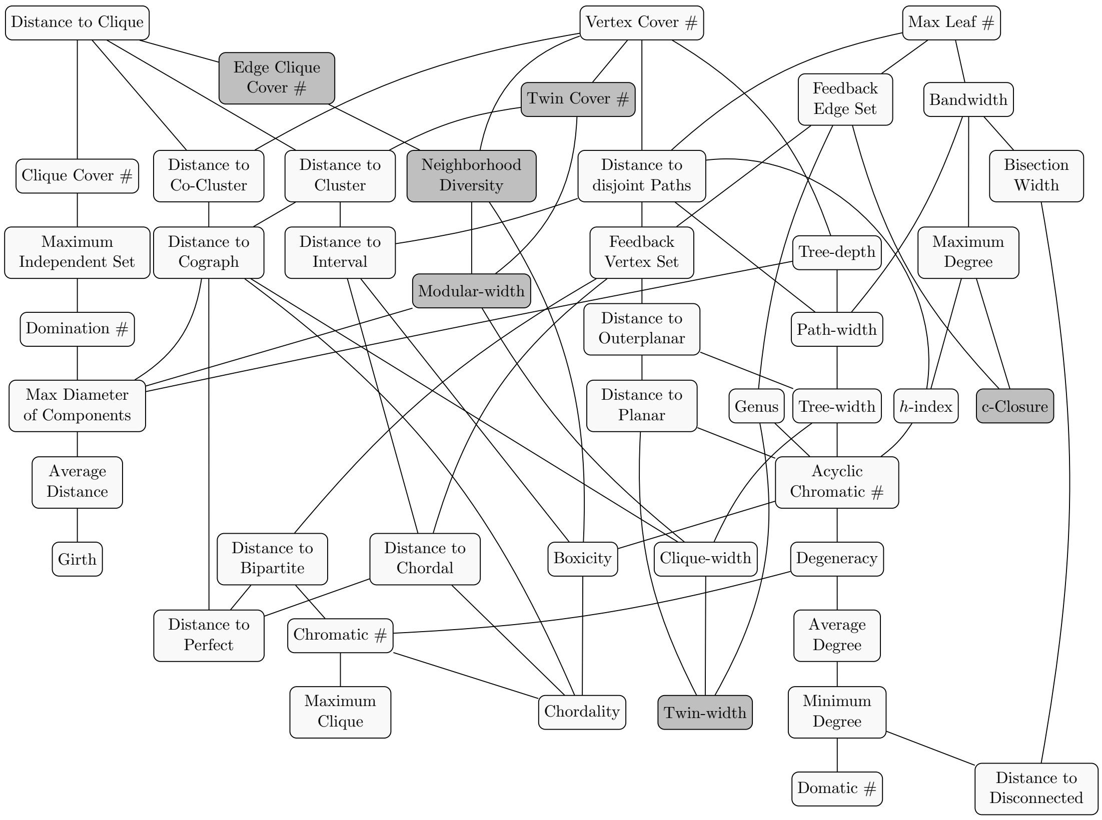

Figure 1.1: A Hasse graph displaying the graph parameter hierarchy expanded by parameters in darkgray. Note that the relation between chordality and maximum clique as well as chordality and c-closure is currently unknown.

Chapter 2

Preliminaries

In this section, we provide notations and definitions used in the following chapters.

Graph Theory. We define $G = (V, E)$ as an undirected graph where V is the set of vertices and $E \subseteq \{\{v, w\} | v, w \in V, v \neq w\}$ is the set of edges. Furthermore, we denote commonly used expressions.

$V(G)$ is the vertex set V of $G = (V, E)$.

$E(G)$ is the edge set E of $G = (V, E)$.

$G[V']$ is the subgraph induced by V' in G.

$\bar{G}$ is complement graph to G, that is, $\bar{G} = (V, \bar{E})$, $\bar{E} = \{\{v, w\} | v, w \in V\} \setminus E$.

$\text{dist}_G(v, w)$ is the distance between two vertices v, w in a graph G.

$N_G(v)$ is the (open) neighborhood of vertex v in G, that is, $\{w \in V(G) | \{v, w\} \in E(G)\}$.

$N_G(V')$ is the (open) neighborhood of a vertex set V' in G, that is, $\bigcup_{v \in V'} N_G(v) \setminus V'$

$E_{V,W}$ is the edge set between two vertex sets V and W, that is:

$$\{\{v, w\} \in E(G) | v \in V, w \in W\}$$

We call the edge set complete if $E_{V,W} = \{\{v, w\} | v \in V, w \in W\}$. Additionally, we say that two subsets $V', W' \subseteq V$ are adjacent if their vertices are pairwise adjacent and disjoint if the edge set is empty. A k-partition of a graph G is the partitioning of $V(G)$ into k mutually exclusive subsets of vertices denoted as $(V_1, \ldots, V_k)$. A subset $V' \in V(G)$ is called a module if its neighborhood $N_G(V')$ is equal to $N_G(v) \setminus V'$, for each $v \in V'$.

We define a trigraph as a graph where an edge between two vertices is either a black edge or a red edge. A contraction on a graph or a trigraph is the contraction of two vertices v, w where each shared adjacent (by a black edge) vertex $u \in N(v) \cap N(w)$ is connected by a black edge and each not shared adjacent vertices $u \in (N(v) \cup N(w)) \setminus (N(v) \cap N(w))$ is connected by a red edge. Additionally, incident red edges stay as red edges.

Graph Parameter Hierarchy. A graph parameter is a function $f : \mathbb{G} \to \mathbb{R}$ where $\mathbb{G}$ is the set of all finite graph and which returns a real number. We say that a parameter p *upper bounds* a parameter q if there is a non-decreasing function $f_{p,q}$ such that $f_{p,q}(p(G)) \geq q(G)$ for all graphs G while we say that if p does not upper bound q, q is *unbounded* in p.

We say that a parameter p *strictly upper bounds* parameter q if p upper bounds q and p is unbounded in q. Thus, we call p an upper bound for q and q a lower bound for p. Additionally, we call parameters p and q *incomparable* if neither parameter upper bounds the other and *equal* if both of them upper bound each other.

Throughout the paper, we also make use of characteristics of the hierarchy.
For parameters a, b, c, d:

Lemma 2.1. *If a upper bounds b and b upper bounds c, then a also upper bounds c.*

Proof. We know that there are non-decreasing functions f and g such that $f(a) \geq b$ and $g(b) \geq c$. Note that $g(f(a)) \geq g(b) \geq c$ shows the claimed transitivity. □

Lemma 2.2. *If a upper bounds b, b upper bounds c, a does not upper bound d and d does not upper bound c, then b and d are incomparable.*

Proof. If b would upper bound d, then by the transitivity shown in Lemma 2.1, d upper bounds c which is a contradiction. Similar, if d upper bounds b, then a upper bounds d which is also a contradiction. Thus, b and d are incomparable. □

We introduce two hidden parameter p_0 and p_∞ with p_0 being upper bounded by any parameter and p_∞ upper bounding any parameter in case there are no known lower or upper bounds.

Reduced Hierarchy. In a hierarchy, given a parameter b, we form a sub-hierarchy where parameters incomparable by Lemma 2.2 are removed. Note that these are the parameters which are incomparable to an upper and a lower bound for b.

Local Maxima. For a parameter b with its lower bounds C, local maxima are parameters, that are incomparable to b, upper bound every $c \in C$ and have either no known upper bounds or are only upper bounded by parameters which upper bound b.

Local Minima. For a parameter b with its upper bounds A, local minima are parameters, that are incomparable to b, are upper bounded by every $a \in A$ and have either no known lower bounds or only upper bound parameters which are upper bounded by b.

Lemma 2.3. *In a reduced hierarchy and for a parameter b, any parameter d which upper bounds a local minimum is incomparable to b if d does not upper bound every lower bound for b. Analogically, if d is upper bounded by a local maximum, d is incomparable to b if d is not upper bounded by every upper bound a of b.*

Proof. With Lemma 2.1 d cannot be upper bounded by b, since a local minimum is incomparable to b. Additionally, if d does not upper bound every lower bound for b, then they contradict, since any parameter upper bounding b also upper bounds any lower bounds for b. Analogically, d cannot upper bound b if d is upper bounded by a local maximum, and is incomparable if it is not upper bounded by every upper bound for b. Thus, d is incomparable to b. □

In order to expand the parameter hierarchy by a parameter b, we first determine its upper bounds a and lower bounds c while trying to identify as many possible related parameters by transitivity (Lemma 2.1). It is left to determine incomparability to any other parameter.

We use the previous step to form a reduced hierarchy for b. Note that in this reduced hierarchy any unknown related parameter d is (1) upper bounded by a potential local maximum and upper bounds every lower bound for b, or (2) upper bounds a potential local minimum and is upper bounded by every upper bound for b. In case of (1) and (2), we know with Lemma 2.2 that d is incomparable to b while in case of either only (1) or only (2), we know with Lemma 2.3 that d is incomparable to b. Thus, by proving that every potential local extremum is indeed incomparable to b, we have determined every relation for parameter b.

In the following we provide definitions for our investigated parameters.

Twin Cover Number. An edge $\{v, w\}$ is a twin edge if vertices v and w have the same neighborhood excluding each other ($N_G(v)\backslash\{w\} = N_G(w)\backslash\{v\}$). The twin cover number $\text{tcn}(G)$ of a graph G is the size of a smallest set $V' \subseteq V(G)$ of vertices such that every edge in $E(G)$ is either a twin edge or incident to a vertex in V'.

Edge Clique Cover Number. The edge clique cover number $\text{eccn}(G)$ of a graph G is the minimum number of complete subgraphs required such that each edge is contained in at least one of them.

Neighborhood Diversity. The neighborhood diversity $\text{nd}(G)$ of a graph G is the smallest number k such that there is a k-partition $(V_1, \ldots, V_k)$ of G, where each subset V_i, $i \in [k]$ is a module and is either a complete set or an independent set.

Modular-width. The modular-width $\text{mw}(G)$ of a graph G is the smallest number h such that a k-partition $(V_1, \ldots, V_k)$ of G exists, where $k \leq h$ and each subset V_i, $i \in [k]$ is a module and either contains a single vertex or for which the modular-subgraph $G[V_i]$ has a modular-width of h. Since $G[V_i]$ might have a k-partition as well, we also call any further induced subgraph a modular-subgraph of G.

c-Closure. The c-closure $\text{cc}(G)$ of a graph G is the smallest number c such that any pair of vertices $v, w \in V(G)$ with $|N_G(v) \cap N_G(w)| \geq c$ is adjacent. That is, whenever two vertices share at least c common neighbors, then they are neighbors themselves.

Twin-width. A contraction sequence is a sequence of contractions which as a result leaves only one vertex. The width of a contraction sequence s is determined by the highest degree of red edges of a vertex in any point of the sequence. The twin-width $\mathrm{tww}(G)$ of a graph G is the smallest width of every contraction sequence of G.

We also provide definitions for the following parameters and graphs, which we will encounter throughout the paper.

Clique. A clique graph is a graph with n vertices and a complete edge set, that is, $E = \{\{v, w\} | v, w \in V, v \neq w\}$.

Cluster. A cluster graph is a graph of disjoint cliques.

Independent Set. A independent set graph is a graph with n vertices and an empty edge set.

Planar. A planar graph is a graph that can be drawn on a plane without any edge crossings.

Interval. A interval graph is a graph that can be formed from a set of intervals on a real line, where vertices are formed by intervals with each intersection bein represented by an edge.

Chordal. A chordal graph is a graph where every induced cycle in the graph has exactly three vertices.

Perfect. A perfect graph is a graph where the chromatic number of every induced subgraph is equal to the size of the largest clique of that subgraph. Note that a perfect graph cannot have an induced cycle of length 5 or more.

Bipartite. A bipartite graph is a graph whose vertices can be divided into two independent sets.

Distance to. The distance to a graph class H of a graph $G = (V, E)$ is the size of a smallest set $V_d \subseteq V$ of vertices such that $G[V \setminus V_d] \in H$.

Vertex Cover Number. The vertex cover number of a graph G is the size of a smallest set $V' \subseteq V(G)$ of vertices such that every edge in $E(G)$ is adjacent to a vertex in V'.

Clique Cover Number. The clique cover number of a graph G is the minimum amount of cliques in G needed to contain every vertex in $V(G)$.

Feedback Edge Number. The feedback edge number of a graph G is the size of a smallest set $E' \subset E(G)$ of edges such that any cycle contains an edge e, with $e \in E'$

Bisection Width. The bisection width of a graph G is the size of a smallest set $E' \subset E(G)$ of edges such that removing E' from E splits G into two equal halves of components, that is, a graph equal to a disjoint union of two equal graphs.

Maximum Degree. The maximum degree of a graph G is the highest degree of a vertex in $V(G)$.

Genus. The genus of a graph G is the smallest number γ such that G can be drawn on a sphere with γ handles without any edge crossings.

Euler Genus. The euler genus of a graph G is the smallest number k, such that G can be drawn on a sphere without crossings using $\frac{k}{2}$ handles or k crosscaps.

Domination Number. The domination number of a graph G is the size of a smallest set $V' \subseteq V(G)$ of vertices such that any vertex is either in V' or adjacent to vertex in V' (dominating set).

Domatic Number. The domatic number of a graph G is the biggest number k such that there is a k-partition $(V_1, \ldots, V_k)$ of G, where each V_i is a dominating set.

Maximum Clique. The maximum clique of a graph G is the size of the biggest clique in G.

Clique-width. The clique-width of a graph G is the minimum number of labels required to construct G, using the following 4 operations:

- Creation of a new vertex v with label i
- Disjoint union of two labeled graphs G and H
- Joining by an edge every vertex labeled i to every vertex labeled j, where $i \neq j$
- Renaming label i to label j

Boxicity. The boxicity of a graph G is the minimum amount of interval graphs required, such that their intersecten results in G.

Chordality. The chordality of a graph G is the minimum amount of chordal graphs required, such that their intersecten results in G.

Chapter 3

Twin Cover Number

In this chapter, we determine the position of the twin cover number in the graph parameter hierarchy. To do this, we first reduce the graph parameter hierarchy, by proving for its upper bounds (vertex cover number) and its lower bounds (distance to cluster). Then, we can determine every potential local extrema as seen in Figure 3.2. By proving that each of these potential local maxima (distance to clique) and potential local minima (maximum clique, domatic number, distance to disconnected and distance to co-cluster) are indeed incomparble, we have proven with Lemma 2.2 and Lemma 2.3 that there are no further upper and lower bounds for twin cover number in this hierarchy, since any undetermined parameter within a reduced parameter hierarchy is related to a local extrema.

3.1 Upper Bounds

We prove for upper bounds for twin cover number in the following.

Observation 3.1. *Vertex Cover Number* ***strictly upper bounds*** *Twin Cover Number.*

Proof. Note that a clique of size n has a twin cover number of 0 and a vertex cover number of $n-1$. Hence, twin cover number does not upper bound vertex cover.

Note that each vertex cover is by definition also a twin cover, as each edge is incident to a vertex in the vertex cover. Thus, vertex cover number strictly upper bounds twin cover number. □

3.2 Lower Bounds

We prove for lower bounds for twin cover number in the following.

Theorem 3.2. *Twin Cover Number* ***strictly upper bounds*** *Distance to Cluster.*

Proof. We show that twin cover number is not upper bounded by distance to cluster. Consider the class of graphs $G = (V, E)$ consisting of a clique $V_C \in V$ of size n where the vertex set V_1 of half of the vertices $v_i \in V_C$, $i = 1, \dots, \lfloor \frac{n}{2} \rfloor$ are adjacent to a single vertex $v_s \in V$. Let vertex set $V_2 = V \backslash (V_1 \cup \{v_s\})$ contain the other vertices of the clique

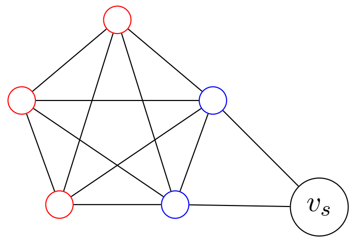

Figure 3.1: Given a clique with size $n = 5$ and vertex v_s with V_1 in blue and V_2 in red. Note that E_{V_1,V_2} is complete and that, considering v_s, none of these edges are twin edges.

V_C. It is apparent that the distance to clique and therefore the distance to cluster is 1 as removing v_s results in a clique. Note that the edge set E_{V_1,V_2} is complete with no twin edges (Figure 3.1). Thus, the twin cover set V_{tc} has to contain the entirety of V_1 or V_2 due to the fact that if a vertex $v_i \in V_1$ and $v_j \in V_2$ are both not contained in V_{tc}, then the edge $\{v_i, v_j\}$ is neither a twin edge nor covered by V_{tc}. Thus, the twin cover number is at least $min(|V_1|, |V_2|) = \lfloor \frac{n}{2} \rfloor$ and cannot be upper bounded by distance to cluster.

Next, we show that a graph H with a twin cover of size k has a distance to cluster of at most k. Note that removing the twin cover set W_{tc} with $|W_{tc}| = k$, a subgraph $H' = H[V(V) \backslash W_{tc}]$ with only twin edges is left. Given that vertices left only have incident twin edges, connected vertices are always true twins and we can conclude that every connected component of H' has to be a clique, and therefore H' forms a cluster graph. Thus, twin cover number strictly upper bounds distance to cluster. □

3.3 Incomparability

Considering each previously discovered bound, we form a reduced hierarchy of twin cover number as seen in Figure 3.2. We discover that distance to clique is the only potential local maximum. Thus, we prove that the potential local maximum is indeed incomparable to twin cover number in the following.

Observation 3.3. *Twin Cover Number is* ***incomparable*** *to Distance to Clique.*

Proof. Note that twin cover number is upper bounded by vertex cover number and therefore by Lemma 2.1 cannot upper bound distance to clique.

We have shown in the proof of Theorem 3.2 that the twin cover number is not upper bounded by distance to clique. Hence, twin cover number is incomparable to distance to clique. □

It remains to show that every potential local minimum (maximum clique, domatic number, distance to disconnected and distance to clique) is indeed incomparable to twin cover number.

Observation 3.4. *Twin Cover Number is* ***incomparable*** *to Maximum Clique, Domatic Number and Distance to Disconnected.*

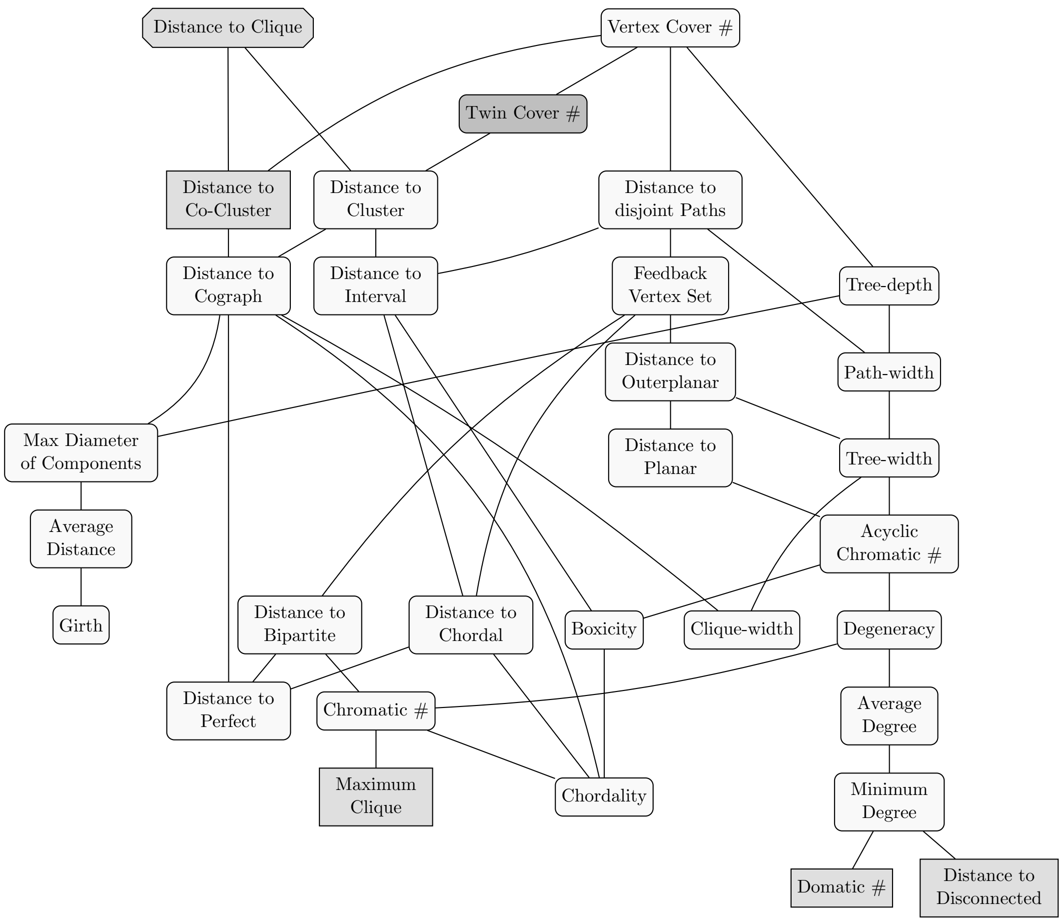

Figure 3.2: Reduced parameter hierarchy for twin cover number (darkgray). We show proven local minima (gray rectangles) and local maxima (gray hexagons).

Proof. Note that twin cover number upper bounds clique-width and therefore by Lemma 2.1 cannot be upper bounded by maximum clique, domatic number or distance to disconnected.

Next, we show that twin cover number does not upper bound maximum clique, domatic number or distance to disconnected. Consider a clique of size n. The twin cover number is 0, while by its definition maximum clique is n, domatic number is n as every vertex on its own covers every vertex and distance to disconnected is $n - 1$ as every incedent edge of a single vertex, which are $n - 1$ edges, has to be deleted to isolate a vertex. Hence, twin cover number is incomparable to maximum clique, domatic number and distance to disconnected. □

Proposition 3.5. *Twin Cover Number is* ***incomparable*** *to Distance to Co-Cluster.*

Proof. Note that the twin cover number upper bounds distance to cluster and therefore by Lemma 2.1 cannot be upper boundedby distance to co-cluster.

Next, we show that twin cover number does not upper bound distance to co-cluster. Consider a graph consisting of two disjoint cliques of size $n > 1$. The twin cover number is 0 as all edges are twin edges. Since a co-cluster graph can only be either one component or a single independent set, of two disjoint cliques of size $n > 1$ an entire clique has to be removed, thus distance to co-cluster is at least n. Hence, twin cover number is incomparable to distance to co-cluster. □

Since every potential local extremum is indeed a local extremum, with Lemma 2.2 and Lemma 2.3 we have determined every relation involving twin cover number in the graph parameter hierarchy.

Chapter 4

Edge Clique Cover Number, Neighborhood Diversity and Modular-width

In this chapter, we determine the position of multiple parameters at once (edge clique cover number, neighborhood diversity and modular-width). To do this, we determine their internal relation and then prove upper and lower bounds for each individual parameter while making use of their transitivity to safe some proofs. Since edge clique cover number upper bounds neighborhood diversisity which upper bounds modular-width, we determine upper bounds for edge clique cover number (distance to clique), then upper bounds for neighborhood diversity (vertex cover number) and upper bounds for modular-width (twin cover number). For lower bounds we determine lower bounds for modular-width (max diameter of components and clique-width), then lower bounds for neighborhood diversity (boxicity) and no lower bounds for edge clique cover number.

It remains to prove incomparability for any other parameter. We can form a reduced hierarchy for each parameter to determine each potential local extrema. We show a united reduced hierarchy in Figure 4.2. By proving that each of these potential local maxima (vertex cover number, distance to cluster, distance to co-cluster and twin cover number) and potential local minima (domination number, distance to perfect and chordality) are indeed incomparable, we have proven with Lemma 2.2 and Lemma 2.3 that there are no more upper and lower bounds for edge clique cover number, neighborhood diversity or modular-width, since any undetermined parameter within a reduced parameter hierarchy is related to a local extrema.

4.1 Internal Relation

We prove for relations between edge clique cover number, neighborhood diversity and modular-width.

Theorem 4.1. *Edge Clique Cover Number **strictly upper bounds** Neighborhood Diversity.*

Proof. Note that given two independent sets of size n whose vertices are pairwise adjacent, the edge clique cover number is n^2 while the neighborhood diversity is 2

Next, we show that given a graph $G = (V, E)$ with edge clique cover number k, the neighborhood diversity is at most 2^k. Note that the neighborhood diversity is the minimum number of sets given a l-partition into modules and cliques or independent sets. Consider a edge-clique cover set $S = C_1, \ldots, C_k$ and every atomic intersection $I_i = \bigcap S_i \setminus \bigcap \bar{S}_i, \bar{S}_i = S \setminus S_i$, where S_i is any combination of items from S. Observe that each intersection I_i is a module and a clique with vertices $v \in I_i$ having a common neighborhood $N_G(v) = N_G(I_i) = \bigcap S_i\ I_i$. Thus, every vertex $v \in C_i$ covered by the edge-clique cover set S can be partitioned into the module $I_i \in I$. For vertices $w \in V \setminus \bigcup C_i$ we know that each vertex w has to be isolated, as else incident edges would have meant coverage by the edge-clique cover, and therefore can be partitioned into additional module and an independent set. With $|I| \leq 2^k$, we conclude that G can be partitioned into at most $2^k + 1$ modules and a clique or independent set. Hence, edge-clique cover strictly upper bounds neighborhood diversity. □

Note that Gajarský, J. et al. [GLO13] showed that neighborhood diversity strictly upper bounds modular-width.

4.2 Upper Bounds

Since determining upper bounds also determines upper bounds for further lower bounds, we start with highest, that is, the edge clique cover number.

4.2.1 Edge Clique Cover Number

We prove for upper bounds for edge clique cover number.

Proposition 4.2. *Distance to Clique* ***strictly upper bounds*** *Edge Clique Cover Number.*

Proof. Note that given two cliques of size n, the distance to clique is n, while edge clique cover number is 2. Hence, edge clique cover number does not upper bound distance to clique.

Next, we show that given a graph $G = (V, E)$ with distance to clique k has an edge clique cover number of at most $\binom{k}{2} + k + 1$. For the distance set V_d, the amount of cliques needed to cover all edges in $E(G[V_d])$ is at most the number of possible edges $\binom{k}{2}$. In addition, for each vertex $v \in V_d$, incident edges towards vertices $u \in V_{cl} = V \setminus V_d$ can be covered with a single clique. Finally, it is left to cover edges of the clique $E(G[V_{cl}])$ which can be covered by one last clique. Thus, the edge clique cover number is at most $\binom{k}{2} + k + 1$. Hence, distance to clique strictly upper bounds edge clique cover number. □

4.2.2 Neighborhood Diversity

We prove for upper bounds for neighborhood diversity.

Proposition 4.3. *Vertex Cover Number* ***strictly upper bounds*** *Neighborhood Diversity.*

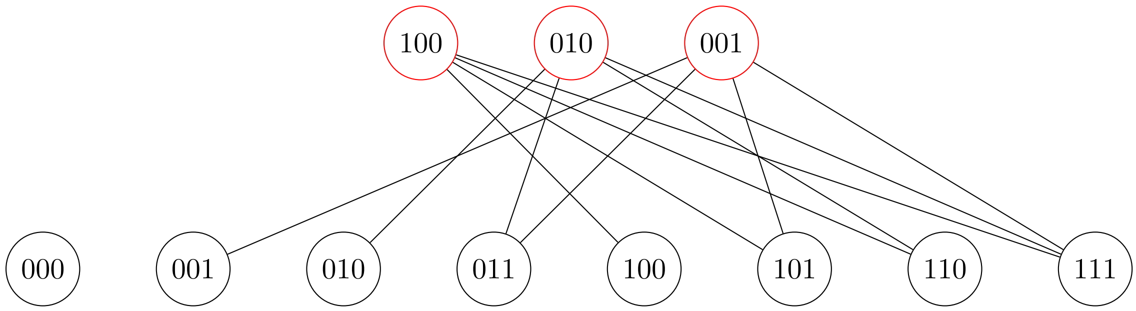

Figure 4.1: Given a vertex cover of size $n = 3$ (red), with a permutaion of $2^3 = 8$ different possible neighborsets as shown by their binary value.

Proof. Note that given a clique of size n, the neighborhood diversity is 1, while vertex cover number is $n-1$. Hence, neighborhood diversity does not upper bound vertex cover number.

Next, we show that a graph $G = (V, E)$ with a vertex cover of size k has a neighborhood diversity of at most $k + 2^k$. Given a vertex cover $V_{vc} \subseteq V$ means that the vertices in $V' = V \backslash V_{vc}$ form an independent set which furthermore can be split into multiple independent sets. Observe that there are 2^k subsets of V_{vc} and that vertices $v \in V'$ with the same neighborhood $N_G(v) \subset V_{vc}$ can form a module and an independent set. Thus, vertices in V' can be partitioned into at most 2^k different modules (see for example Figure 4.1), while an additional k modules for each single vertex in V_{vc} can be added, summarizing to a neighborhood diversity of at most $k + 2^k$. Hence, vertex cover strictly upper bounds neighborhood diversity. □

4.2.3 Modular-width

For upper bounds for modular-width, Gajarský, J. et al. [GLO13] showed that twin cover number strictly upper bounds modular-width.

4.3 Lower Bounds

Since determining lower bounds also determines lower bounds for further upper bounds, we start with the lowest, that is, the modular-width.

4.3.1 Modular-width

We first prove for some charateristics of modular-width.

Lemma 4.4. *The Modular-width of a path P with length $n > 3$ is n.*

Proof. Note that given any graph G with $V(G) \geq 2$, $\text{mw}(G) \geq 2$. For any set S of $2 \leq k < n$ vertices in $V(P)$, there is at least one vertex $v \in N_P(S)$ with $S \nsubseteq N_P(v)$. Thus, a n-partition of subsets of size 1 is necessary, that is, the modular-width is n. □

Lemma 4.5. *Given any graph G, $mw(G) = mw(\bar{G})$.*

Proof. Let the $\mathrm{mw}(G) = h \geq k$ and let there be a k-partition $(V_1, \ldots, V_k)$ such that for each $i \in [k]$ it holds that V_i is a module in G and $G[V_i]$ has a modular-width of at most h. Observe that the edge set E_{V_i,V_j} is either complete or empty in order for V_i and V_j to be modules. Consider the same k-partition for $\bar{G}$. Since V_i and V_j are modules, E_{V_i,V_j} is either complete or empty, therefore $\bar{E}_{V_i,V_j}$ is also either complete or empty, meaning V_i and V_j are still modules. Analogically, each further k-partition for $G[V_i]$ is also applicable for each further induced $\bar{G}[V_i]$. Hence, $\mathrm{mw}(G) = \mathrm{mw}(\bar{G})$. □

We prove for lower bounds for modular-width.

Proposition 4.6. *Modular-width* ***strictly upper bounds*** *Clique-width.*

Proof. We know that given a path P of length $n > 3$, as shown in Lemma 4.4, the modular width is n. Note that the clique-width of a path is 3 and therefore clique-width cannot upper bound modular-width.

Courcelle et al. [CMR00] showed that given any graph G of clique-width $cw(G) = max(cw(H))$, where H is a representative graph of a modular-subgraph of G, that is, a graph where each module is represented by a vertex adjacent to other vertices if their modules are adjacent as well. We know that $cw(H)$ is at most $|V(H)|$ and that $|V(H)|$ is at most $\mathrm{mw}(G)$. Hence, $\mathrm{mw}(G) \geq cw(G)$. Thus, modular-width strictly upper bounds clique-width. □

Theorem 4.7. *Modular-width* ***strictly upper bounds*** *Max Diameter of Components.*

Proof. We show that max diameter of components cannot upper bound modular-width. Consider the complementary graph $\bar{P}$ of a path of length $n > 4$. Observe that any two vertices v, w have a non-empty shared neighborhood $N_{\bar{P}}(v) \cap N_{\bar{P}}(w) \neq \emptyset$. Thus, the max diameter of components is at most 2. We have proven in Lemma 4.4 that the modular-width of a path of length $n > 3$ is n. Additionally, it is stated in Lemma 4.5 that the modular-width of a complement graph is equal to the original graph. Hence, the modular-width of $\bar{P}$ is also n.

Next, we show that given a graph G with modular-width h, the max diameter of components is at most h. Let d be the maximum diameter of any connected component in G and let v, w be two vertices with $\mathrm{dist}_G(v, w) = d$. Let $(V_1, \ldots, V_k)$, $k \leq h$ be a k-partition of G or a modular-subgraph of G such tha V_i is a module, with $G[V_i]$ having a modular-width of at most h. Note that vertices v and w must be in the same connected component and must at some point be in different modules of a modular-subgeraph G'. Since the distance between vertices between each adjacent module is 1, we construct a representative graph H of G', where modules $V_i, V_j \subseteq V(G')$, $i \neq j$ are represented by a vertices $v'_i, v'_j \in V(H)$, with $\{v'_i, v'_j\} \in E(H)$, if V_i and V_j are adjecent. Observe how for vertices $v_i \in V_i$ and $v_j \in V_j$, $\mathrm{dist}_G(v_i, v_j) = \mathrm{dist}_H(v'_i, v'_j)$. Thus, since the max diameter of components of H is at most $|H|$, with $|H| \leq h$, the max diameter of components of G is at most h. Hence, modular-width strictly upper bounds max diameter of components. □

4.3.2 Neighborhood Diversity

We first prove for some charateristics of neighborhood diversity.

Lemma 4.8. *The Neighborhood Diversity of a Path P with length $n > 3$ is n.*

Proof. Note that $V(P)$ is neither an independent set nor a clique. For any set S of $2 \leq k < n$ vertices, there is at least one vertex $v \in N_P(S)$ with $S \nsubseteq N_P(v)$. Thus, a n-partition of subsets of size 1 is necessary, meaning, the neighborhood diversity is n. □

We prove for lower bounds for neighborhood diversity.

Theorem 4.9. *Neighborhood Diversity **strictly upper bounds** Boxicity.*

Proof. Note that given a path of length $n > 3$. The boxicity is 1 while Lemma 4.8 states that the neighborhood diversity is n.

Next, we show that given a graph $G = (V, E)$ with neighborhood diversity of k, its boxicity is at most $k + k^2$. Recall that the boxicity of a graph G is the minimum amount of interval graphs needed to intersect to form G. We construct interval graphs $G_i = (V, E_i)$ such that for each pair $u, v \in V$ it holds that $\{u, v\} \in E$ if and only if $\{u, v\} \in E_i$ for all i. Note that each such interval graph has to contain all edges E and for each pair of vertices u, v with $\{u, v\} \notin E$ there needs to be at least one interval graph G_i wtih $\{v, w\} \notin E_i$. We can now construct interval graphs for certain modules and pairs of modules using the following methods.

With a k-partition $(V_1, \ldots, V_k)$ where each subset is a module and a clique or an independent set, we may construct an interval graph $G_i = (V, E_i)$ for each module. If V_i is an independent set, then all vertices in $V \backslash V_i$ form a clique, while all vertices $v \in V_i$ form an independent set which are adjacent to $V \backslash V_i$. If V_i is a clique, then we do not require an additional interval graph.

Next, we construct an interval graph $G_{ij} = (V, E_{ij})$ for each pair V_i, V_j that is non-adjacent, meaning the edge set E_{V_i,V_j} is empty. In this case, each vertex set $V_{oth} V \backslash (V_i \cup V_j)$, V_i and V_j forms a clique while V_i and V_j will be adjacent to V_{oth}. As both cliques V_i and V_j can independently (without overlapping each other) intersect with V_{oth}, G_{ij} will be an interval graph.

Observe that any edge in G is contained in all constructed interval graphs. It remains to show that each pair of non-adjacent vertices is non-adjacent in at least one interval graph. Consider any pair $v, w \in V$ with $\{v, w\} \notin E$. If v and w belong to the same module V_i then this module is an independent set and by construction $\{v, w\} \notin E_i$. If v and w belong to different modules V_i and V_j, then these two modules are non-adjacent and by construction $\{v, w\} \notin E_{ij}$. Hence, the intersection of each G_i and each G_{ij}, which summarizes to a maximum of $k + k^2$ interval graphs, does form G. Thus, neighborhood diversity strictly upper bounds boxicity. □

4.4 Incomparability

Considering each previously discovered bounds, we form a reduced hierarchy for each parameter. We show a united reduced hierarchy in Figure 4.2.

Since each local maxima is determined by the upper bounds for the lower bounds for a parameter we start with the lowest, that is, the modular-width.

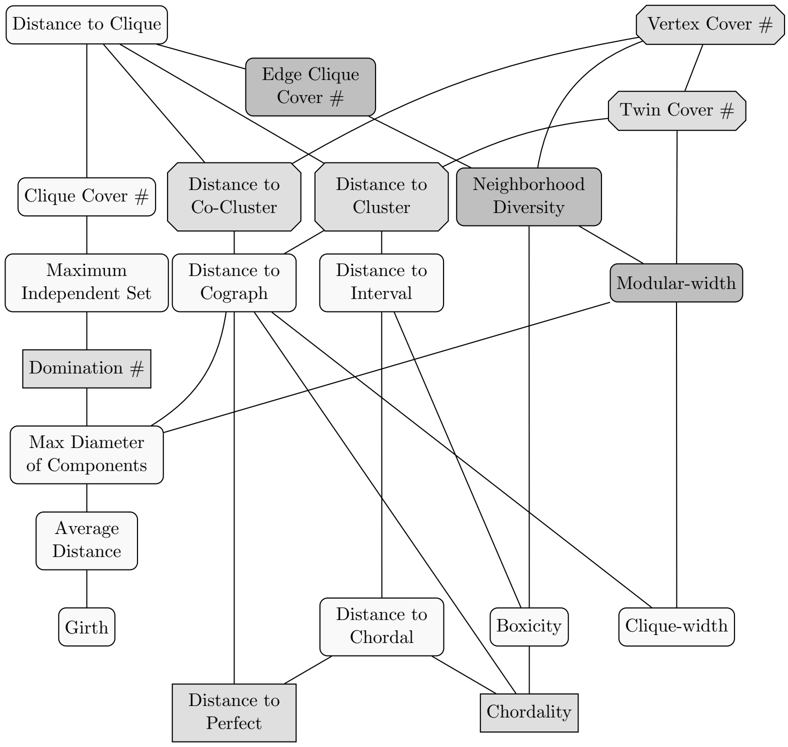

Figure 4.2: United reduced parameter hierarchy for edge clique cover number, neighborhood diversity and modular-width (darkgray). We show proven local minima (gray rectangles) and local maxima (gray hexagons).

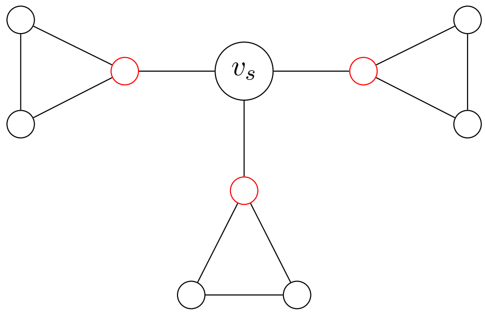

Figure 4.3: Three cliques of size $m = 3$ with one vertex v_i (red) in each clique being adjacent to v_s. Observe that each vertex v_i as well as vertex v_s can only form a module on their own.

4.4.1 Local Maxima

We prove for potential local maxima (distance to cluster and distance to co-cluster) of modular-width.

Proposition 4.10. *Modular-width is* ***incomparable*** *to Distance to Cluster.*

Proof. Note that modular-width is upper bounded by neighborhood diversity and therefore cannot upper bound distance to cluster since distance to cluster and neighborhood diversity are incomparable.

Next, we show that modular-width is not upper bounded by distance to cluster. Consider the class of graph $G = (V, E)$ with n cliques $V_i \in V$ of size $m > 1$ and where a single vertex of each clique $v_{i,1} \in V_i$ is adjacent to a single vertex $v_s \in V$ (Figure 4.3). The distance to cluster is 1 since removing v_s results in a cluster graph. While v_s has to form its own module, as else a common neighborhood is not possible, each clique will have to be partitioned into 2 modules $V_{i,1}$, with $\{v \in V_{i,1} | \{v, v_s\} \in E\}$, and $V_{i,2}$, with $\{w \in V_{i,2} | \{w, v_s\} \notin E\}$, meaning that the modular-width is $2n + 1$. Hence, modular-width is incomparable to distance to cluster. □

Proposition 4.11. *Modular-width is* ***incomparable*** *to Distance to Co-Cluster.*

Proof. We have already shown in Proposition 4.10 that modular-width is incomparable to distance to cluster. Observe that distance to cluster of G is equal to distance to co-cluster of $\bar{G}$. At the same time, we have already proven that modular-width of $\bar{G}$ is equal to modular-width of G. Hence, modular-width is incomparable to distance to co-cluster. □

Thus with Lemma 2.3, we know that modular-width does not have any further known upper bounds. We now proceed with the only potential local maximum (twin cover) of the neighborhood diversity.

Proposition 4.12. *Neighborhood Diversity is* ***incomparable*** *to Twin Cover Number.*

Proof. Note that the neighborhood diversity upper bounds clique-width and therefore by Lemma 2.1 cannot be upper bounded by distance to perfect.

Next, we show that neighborhood diversity does not upper bound twin cover number. Consider n disjoint cliques of size $m > 1$. Since each clique has to be its own module and a clique, the neighborhood diversity is atleast n while the twin cover number is 0 since each edge is a twin edge. Hence, neighborhood diversity is incomparable to twin cover number. □

Thus with Lemma 2.3, we know that neighborhood diversity does not have any further known upper bounds. We now proceed with the only potential local maxima (vertex cover number) of the edge clique cover number.

Proposition 4.13. *Edge Clique Cover Number is* ***incomparable*** *to Vertex Cover Number.*

Proof. Note that the edge clique cover number is upper bounded by distance to clique and therefore by Lemma 2.1 cannot upper bound vertex cover number.

Next, we show that edge clique cover number is not upper bounded by vertex cover number. Consider a star with n leaves. It is apparent that vertex cover number is 1 while edge clique cover number is n. Hence, edge clique cover number is incomparable to vertex cover number. □

Thus with Lemma 2.3, we know that edge clique cover number does not have any further known upper bounds.

Since each local minima is determined by the lower bounds for the upper bounds for a parameter we start with the highest, that is, the edge clique cover number.

4.4.2 Local Minima

We prove for potential local minima (domination number and distance to perfect) of edge clique cover number.

Proposition 4.14. *Edge Clique Cover Number is* ***incomparable*** *to Domination Number.*

Proof. Note that the edge-clique cover number upper bounds clique-width and therefore by Lemma 2.1 cannot be upper bounded by domination number.

Next, we show that edge clique cover number does not upper bound domination number. Consider an independent set of size n. It is apparent that edge clique cover number is 0 while domination number is n. Hence, edge clique cover number is incomparable to domination number. □

Proposition 4.15. *Edge Clique Cover Number is* ***incomparable*** *to Distance to Perfect.*

Proof. Note that the edge clique cover number upper bounds clique-width and therefore by Lemma 2.1 cannot be upper bounded by distance to perfect.

Next, we show that edge clique cover number does not upper bound distance to perfect. Consider the graph class of five cliques $V_1, \ldots, V_5$, each of size n, which form a cycle by having vertex sets V_i and V_{i+1} ($V_6 = V_1$) be adjacent. The edge clique cover

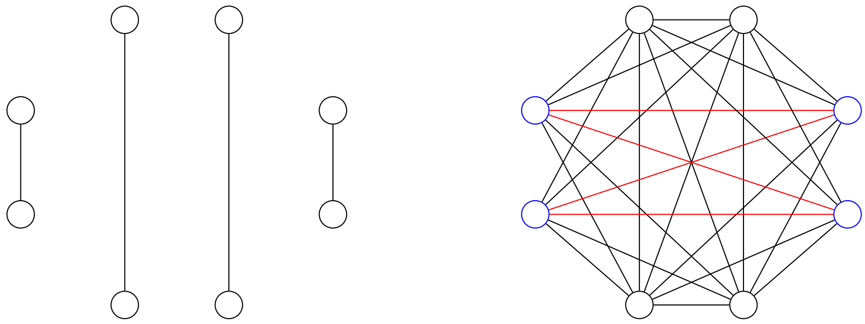

Figure 4.4: Observe how the modular-width of graph $\bar{G}$ (left) is 2, since each connected component is of size 2; Also observe that the instance of the graph with two missing edges (right) between pairs v_i, w_i and v_j, w_j (blue) is not a chordal graph, since there is an induced cycle $v_i w_i v_j w_j$ (red)

number is 5 by forming a clique around each $V_i \cup V_{i+1}$ to cover the edge set $E_{V_i,V_{i+1}}$ as well while given that a perfect graph cannot contain an induced cycle of length 5, distance to perfect has to be at least n since if there is at least one vertex from each of the five cliques left, then these five vertices form a cycle of length 5. Hence, edge clique cover number is incomparable to distance to perfect. □

Thus with Lemma 2.3, we know that edge clique cover number does not have any further known lower bounds. Since neighborhood diversity has no potential local minima, we proceed with the only potential local minima (chordality) of the neighborhood diversity.

Theorem 4.16. *Modular-width is* ***incomparable*** *to Chordality.*

Proof. Note that modular-width upper bounds clique-width and therefore by Lemma 2.1 cannot be upper bounded by chordality.

Next, we show that modular-width cannot upper bound chordality. Consider the graph class of a clique of size $2n$ with a perfect matching removed, that is, $G_n = (V, E) = (\{v_i, w_i | i \in [n]\}, \{\{v_i, v_j\}, \{v_i, w_j\}, \{w_i, w_j\} | i \neq j \in [n]\})$. Since the complement graph $\bar{G}_n$ is a perfect matching (Figure 4.4, left), its modular width is 2, and by Lemma 4.5, G_n also has a modular-width of 2.

It remains to show that the chordality of G is n. Recall that the chordality is the minimum number of chordal graphs needed whose intersection is G_n. Note that each such chordal graph has to contain all edges $E_i \subseteq E$ and for each pair of vertices u, v with $\{u, v\} \notin E$ there needs to be at least one chordal graph which does not contain $\{v, w\}$.

Assume toward contradiction that there are two pairs v_i, w_i and v_j, w_j with $i \neq j$ such that a chordal graph contains neither $\{v_i, w_i\}$ nor $\{v_j, w_j\}$. Then, it contains the induced cycle $v_i w_i v_j w_j$ (Figure 4.4, right), a contradiction to being a chordal graph. Hence, for each pair v_i, w_i a seperate chordal graph is needed and the chordality is n. Thus, modular-width is incomparable to chordality. □

Thus with Lemma 2.3, we know that modular-width does not have any further known lower bounds.

Since every potential local extremum is indeed a local extremum, with Lemma 2.2 and Lemma 2.3, we have determined every relation involving edge clique cover number, neighborhood diversity and modular-width in the graph parameter hierarchy.

Chapter 5

c-Closure

In this chapter, we determine the position of the c-closure in the graph parameter hierarchy. Consider, that the relation between c-closure and chordality is currently unknown. Thus, we ignore it in this section. We prove for the upper bounds for c-closure (maximum degree and feedback edge set), since there are no known lower bound for c-closure. Hence, we make use of p_0, that is the parameter that is upper bounded by any other parameter. Note that since p_0 is currently the only lower bound for c-closure, the original hierarchy is equal to the reduced hierarchy of c-closure. Thus, we can determine every potential local extrema as seen in Figure 5.1. By proving that each of these potential local maxima (vertex cover number, distance to clique, bisection width and genus), and potential local minima (distance to disconnected, domatic number, maximum clique and boxicity) are indeed incomparable, we have proven with Lemma 2.2 and Lemma 2.3 that there are no more upper and lower bounds for c-closure, since any undetermined parameter within a reduced parameter hierarchy is related to a local extrema.

5.1 Upper Bounds

We prove for upper bound for c-closure in the following.

Proposition 5.1. *Maximum Degree* ***strictly upper bounds*** *c-Closure.*

Proof. Note that given a clique of size n, the c-closure is 0, while the maximum degree is $n-1$. Hence, c-closure does not upper bound maximum degree.

Next, we show that a graph G with a maximum degree of k has a c-closure of at most k. Note that c-closure only restricts vertices with at least c neighbors. Thus, if the maximum degree is k, then the c-closure is at most $k+1$. Thus, maximum degree strictly upper bounds c-closure. □

Theorem 5.2. *Feedback Edge Number* ***strictly upper bounds*** *c-Closure.*

Proof. Note that given a clique of size n, the c-closure is 0, while the feedback edge number is at least $n-2$, since any three vertices can form a cycle of which at least one edge has to be in the feedback edge. Hence, c-closure does not upper bound feedback edge number set.

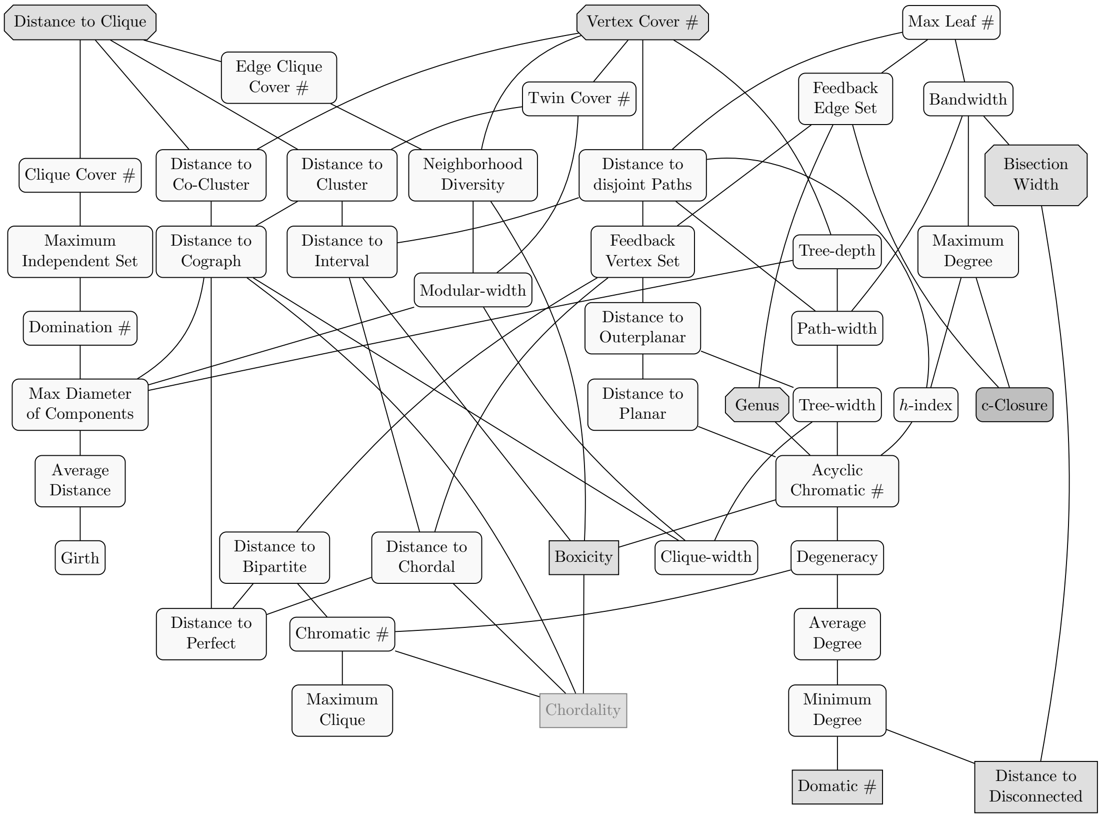

Figure 5.1: "Reduced" parameter hierarchy for c-closure (darkgray). Since c-closure has no known lower bounds, it is equal to the original hierarchy. We show proven local minima (gray rectangles) and local maxima (gray hexagons). Note that the relation between c-closure and chordality is currently unknown, thus we ignore it in this case.

Next, we show that a graph G with a feedback edge number of k has a c-closure of at most $k+2$. Recall, that a feedback edge set F is a set containing edges such that every cycle of G contains at least one edge of the feedback edge set. Note that a graph with c-closure of $c \geq 1$ contains two vertices v_a and v_b with $|N_G(v_a) \cap N_G(v_b)| = c-1$. Thus, there are atleast $c-1$ different paths between v_a and v_b, therefore $\binom{c-1}{2}$ different cycles, since any two of these paths form a cycle. We know that each cycle has to contain an edge within F and if of two paths $p1$, $p2$ there is no edge in F, then there is a cycle formed by $p1$ and $p2$ that is not covered by F. Thus, the feedback edge number is atleast $c-3$, meaning that given a feedback edge number of k, the c-closure can be at most $k+3$. Hence, feedback edge number strictly upper bounds c-closure. □

5.2 Incomparability

Since there are no lower bounds for c-closure the reduced hierarchy is equal to the original hierarchy as seen in Figure 5.1. We discover, that vertex cover number, distance to clique, bisection width and genus are the potential local maxima. Thus, we prove that the potential local maxima are indeed incomparable to c-closure in the following.

Proposition 5.3. *c-Closure is **incomparable** to Vertex Cover Number.*

Proof. Note that c-closure is upper bounded by maximum degree and therefore by Lemma 2.1 cannot upper bound vertex cover number.

Next, we show that vertex cover number does not upper bound c-closure. Consider the graph class G_n of an independent set V_{IS} of size n adjacent to two vertices v_a and v_b which are adjacent to the independent set. The vertex cover number is 2 while the c-closure is at least $n+1$ since v_a and v_b have a shared neighborhood $N_{G_n}(v_a) \cap N_{G_n}(v_b) = V_{IS}$ of size n, while being non-adjacent. Hence, vertex cover number is incomparable to c-closure. □

Proposition 5.4. *c-Closure is **incomparable** to Distance to Clique.*

Proof. Note that c-closure is upper bounded by maximum degree and therefore by Lemma 2.1 cannot upper bound distance to clique.

Next, we show that distance to clique does not upper bound c-closure. Consider the graph class of a clique V_{CL} of size n adjacent to two vertices v_a and v_b. The distance to clique is 1, as either removing v_a or v_b induces a clique while c-closure is $n+1$, since v_a and v_b have a shared neighborhood $N_{G_n}(v_a) = N_{G_n}(v_b) = V_{VL}$ of size n, while being non-adjacent. Hence, distance to clique is incomparable to c-closure. □

Proposition 5.5. *c-Closure is **incomparable** to Bisection Width.*

Proof. Note that c-closure is upper bounded by maximum degree and therefore by Lemma 2.1 cannot upper bound distance to clique.

Next, we show that distance to clique does not upper bound c-closure. Since we showed in Proposition 5.4 that a single connected component in graph G has an unbounded c-closure, we know that a disjoint union of two equal graphs also has an unbounded c-closure while the bisection width is therefore 0. Hence, bisection width is incomparable to c-closure. □

Proposition 5.6. *c-Closure is **incomparable** to Genus.*

Proof. Note that c-closure is upper bounded by maximum degree and therefore by Lemma 2.1 cannot upper bound distance to clique.

Next, we show that distance to clique does not upper bound c-closure. Consider the graph class G_n of an independent set V_{IS} of size n adjacent to two disconnected vertices v_a and v_b. It is apparent, that G_n is a planar graph, since on a plane, we can draw an infinit amount of independent paths between two vertices v_a and v_b. Hence, the genus is 0. The c-closure of G_n is at least $n+1$, since to v_a and v_b having a shared neighborhood $N_{G_n}(v_a) \cap N_{G_n}(v_b) = V_{IS}$ of size n, while being non-adjacent. Hence, distance to clique is incomparable to c-closure. □

It remains to show that every potential local minimum (distance to disconnected, domatic number, maximum clique and boxicity) is indeed incomparable to c-closure in the following.

Observation 5.7. *c-Closure is **incomparable** to Distance to Disconnected, Domatic Number and Maximum Clique.*

Proof. Note that maximum clique, domatic number and distance to disconnected are upper bounded by vertex cover number and therefore by Lemma 2.1 cannot upper bound c-closure.

Next, we show that c-closure does not upper bound maximum clique, domatic number or distance to disconnected. Consider a clique of size n. The c-closure is 0 as all vertices are connected, while by its definition maximum clique is n, domatic number is also n as every vertex on its own is a domatic partition and distance to disconnected is $n - 1$ as every edge of a single vertex, which are $n - 1$ edges, has to be deleted to isolate a vertex. Hence, c-closure is incomparable to maximum clique, domatic number and distance to disconnected. □

Proposition 5.8. *c-Closure is* ***incomparable*** *to Boxicity.*

Proof. Note that boxicity is upper bounded by vertex cover number and therefore by Lemma 2.1 cannot upper bound c-closure.

Next, we show that c-closure does not upper bound boxicity. Consider a fully subdivided clique of size n, that is, a clique K_n where any edge $\{u, w\} \in E(K_n)$ is replaced by an additional vertex v and edges $\{u, v\}$, $\{v, w\}$. Given that the graph is subdivided, c-closure is 2, since any two vertices can only share either a subdived vertex v, a common endpoint of the original clique u/w, or no neighbors. Additionaly, as shown by Chandran et al. [CMS11] the boxicity of a fully subdivided clique of size n is at least $\frac{(\log_2 \log_2 n)+1}{2}$. Hence, boxicity is incomparable to c-closure. □

Since every potential local extremum is indeed a local extremum, with Lemma 2.2 and Lemma 2.3 we have determined every relation involving c-closure in the graph parameter hierarchy.

Chapter 6

Twin-width

In this chapter, we determine the position of the twin-width in the graph parameter hierarchy. We prove for the upper bounds for twin-width (clique-width, genus and distance to planar), since there are no known lower bound for twin-width. Hence, we make use of p_0, that is the parameter that is upper bounded by any other parameter. Note that, since p_0 is the only lower bound for twin-width, the original hierarchy is equal to the reduced hierarchy of twin-width. Thus, we can determine every potential local extrema as seen in Figure 6.2. Since clique-width has no other lower bounds, we know that there are no potential local minima for twin-width, thus no possible lower bounds. By proving that each of these potential local maxima (distance to interval, distance to bipartite, clique cover number, maximum degree and bisection width) are indeed incomparable, we have proven with Lemma 2.3 that there are no more upper bounds for twin-width, since any undetermined parameter within a reduced parameter hierarchy is related to a local extrema.

We prove some characteristics of twin-width in the following.

Lemma 6.1. *Given any graph G, $tww(G) = tww(\bar{G})$, where $\bar{G}$ is the complement of G.*

Proof. Let graph G have a sequence s with width $k = \text{tww}(G)$. Consider that when contracting two vertices v, w a red edge towards vertex u only appears if $\{v, u\} \notin E(G) \wedge \{w, u\} \in E(G)$ or $\{v, u\} \in E(G) \wedge \{w, u\} \notin E(G)$, or if one of edges $\{v, u\}$, $\{w, u\}$ is already a red edge. Thus every contraction in $\bar{G}$ results in as many red edges in G. Thus, every red edge during s in G between vertices is present in s on $\bar{G}$, meaning that the width of sequence s on graph $\bar{G}$ is k as well. Thus, $\text{tww}(\bar{G}) \leq k$ and since G is also the complement of $\bar{G}$, $\text{tww}(G) = \text{tww}(\bar{G})$. □

6.1 Upper Bounds

We prove for the upper bounds for twin-width in the following.

Proposition 6.2. *Clique-width* ***strictly upper bounds*** *Twin-width.*

Proof. Consider a grid graph $G_n = (V, E)$ of size $n \times n$ with $v_{x,y} \in V$, $x, y \in [n]$, being adjacent to $v_{x\pm1,y} \in V$ and $v_{x,y\pm1} \in V$. Consider the sequence s which contracts each vertex of the left-most row $x' < n$ with its adjacent vertex in row $x'+1$, that is vertices $v_{x',y}$ with

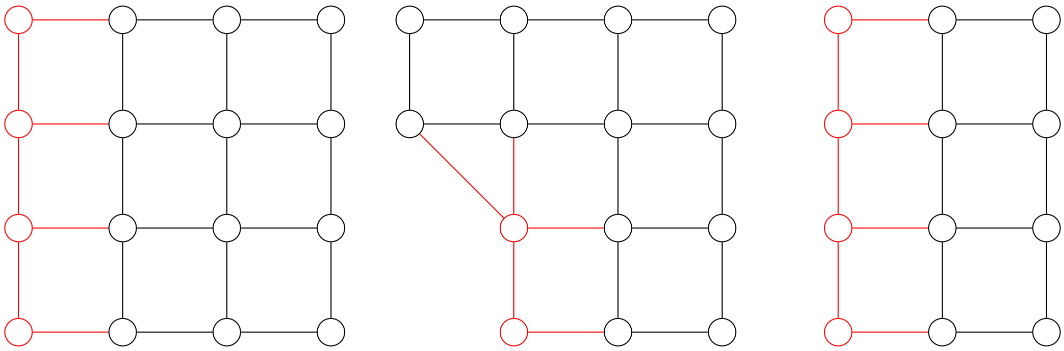

Figure 6.1: Observe how a grid trigraph of size $n \times m$ (a) can be contracted with a width of at most 4 (b) to a smaller grid trigraph of size $n-1 \times m$ (c). Note that the width of a sequence on a trigraph is greater or equal that of its equivalent normal graph.

$v_{x'+1,y}$ to $v'_{x'+1,y}$. Since $|N_{G_n}(v'_{x'+1,y})| \leq 4$ ($N_{G_n}(v'_{x'+1,y}) \subseteq \{v_{x,y+1}, v_{x+1,y\pm1}, v_{x+2,y}\}$) and since the red degree of any other not contracted vertex $v_{x,y}$ is at most 2 (Figure 6.1), the width of s is at most 4 and it remains a path trigraph of length n. With twin-width of a path being at most 2, graph G_n has a twin-width of at most 4. Dawar and Sankaran [DS] showed that grid graphs (including square grid graphs) have unbounded clique-width. Thus, twin-width cannot upper bound clique-width.

Next, Bonnet et al. [Bon+22] showed that every graph G with clique-width k has a twin-width of at most $2^{k+1}-1$. Hence, clique-width strictly upper bounds twin-width. □

Proposition 6.3. *Genus* ***strictly upper bounds*** *Twin-width.*

Proof. Terry et al. [TWY] showed that a clique of size $n \geq 3$ has a genus of $\lceil \frac{(n-3)(n-4)}{12} \rceil$ while the twin-width is 0. Hence, twin-width does not upper bound genus.

Recall that for a graph G the euler genus is the smallest number k, such that G can be drawn on a sphere without crossings using $\frac{k}{2}$ handles or k crosscaps. Since genus is the smallest number γ such that G can be drawn on a sphere without crossings with γ handles, we know that $k \leq 2\gamma$.

Bonnet et al. [BKW22] showed that for every graph G with euler genus k, the twin-width of G is at most $205k + 583 \leq 510\gamma + 583$. Hence, genus strictly upper bounds twin-width. □

Theorem 6.4. *Distance to Planar* ***strictly upper bounds*** *Twin-width.*

Proof. Note that a clique of size n has a twin-width of 0 and a distance to planar of $n-4$. Hence, twin-width does not upper bound distance to planar.

Next, we show that a graph $G = (V, E)$ with a distance to planar k, has a twin-width of at most $9 \cdot 2^k + 2^k - 1$. Let the distance set be $V_d \subseteq V$ and the planar set be $V_p = V \setminus V_d$. Note that the subgraph $G[V_p]$ is a planar graph and therefore has a twin-width of at most 9 as shown by Hliněný [Hli].

Consider the sequence s for $G[V_p]$ with width of at most 9. We construct a new sequence s' similar to s which only contracts vertices $v \in V_p$ with equal neighborhood in V_d in order to prevent any red edges between V_d and V_p.

Let there be a label $\lambda(v)$ and a type $t(v')$ for each $v \in V_p$. Initially let $\lambda(v) = v$ and $t(v) = N_G(v) \cap V_d$. We construct the sequence s'. If "s contracts vertices v and w into u", then we contract each vertices x and y into z if and only if $t(x) = t(y)$, $\lambda(x) = v$ and $\lambda(y) = w$. We set for vertecis z, $\lambda(z) = u$, and change for each left vertex u', with $\lambda(u') = v$ or $\lambda(u') = w$ into $\lambda(u') = u$.

Observe how within sequence s' there are no red edges between V_p and V_d and that with $|V_d| = k$ there are at most 2^k different types. Note that for each label λ there at most 2^k vertices since each unique type points to at most a single vertex with label λ. Thus during s', the width is at most $2^k - 1$ within vertices of the same label. We also know that each vertex v is adjacent to a vertex w only if in s the corresponding vertices of $\lambda(v)$ and $\lambda(w)$ are adjacent. Thus, the degree d_v of any vertices $v \in V_p$ during s is increased to at most $2^k \cdot d_v + 2^k - 1$. Note that since each contraction in s may result in multiple matching contractions in s', the degree of a contracted vertex during s' may be at most doubled. We can also say that the red degree increases at the same rate since in s' red edges cannot appear between vertices of different labels, if the corresponding vertices in s do not have a red edge. Hence, with width s being at most 9, the sequence s' has a width of at most $2 * (2^k \cdot 9 + 2^k - 1)$ and it remains a trigraph with $2^k + k$ vertices. Note that the twin-width of a trigraph with n vertices is at most $n - 1$ since that is the maximum possbile degree. Thus, the twin-width of G is at most $2 \cdot (2^k \cdot 9 + 2^k - 1)$ and distance to planar strictly upper bounds twin-width. □

6.2 Incomparability

Since there are no lower bounds for twin-width the reduced hierarchy is equal to the original hierarchy as seen in Figure 5.1. We discover, that distance to interval, distance to bipartite, clique cover number, maximum degree and bisection width are the potential local maximum. Thus, we prove that the potential local maxima are indeed incomparable to twin-width in the following.

Observation 6.5. *Twin-width is* ***incomparable*** *to Distance to Interval.*

Proof. Note that twin-width is upper bounded by clique-width and therefore by Lemma 2.1 cannot upper bound distance to interval.

Next, we show that twin-width is not upper bounded by distance to interval. Bonnet et al. [Bon+21] showed that interval graphs have unbounded twin-width, meaning that there is an interval graph with twin width k and since it is an interval graph, a distance to interval of 0. Hence, twin-width is incomparable to distance to interval. □

Proposition 6.6. *Twin-width is* ***incomparable*** *to Distance to Bipartite.*

Proof. Note that twin-width is upper bounded by clique-width and therefore by Lemma 2.1 cannot upper bound distance to bipartite.

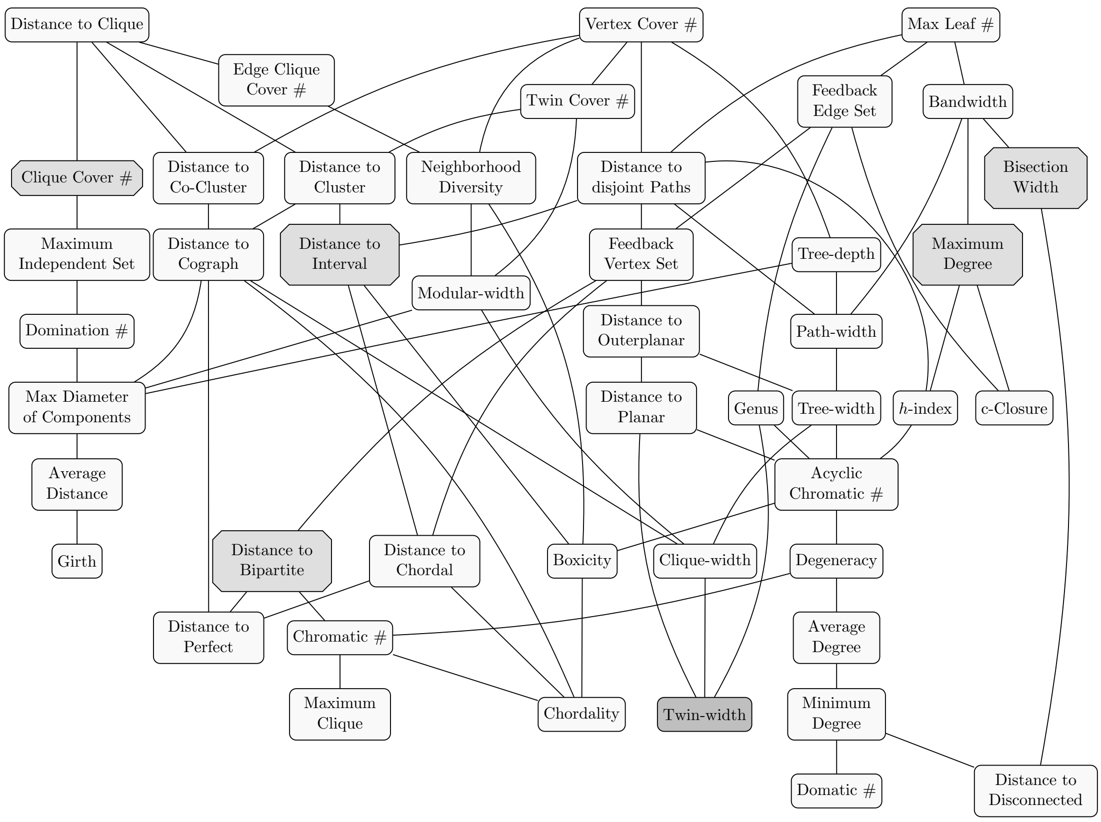

Figure 6.2: "Reduced" parameter hierarchy for twin-width (darkgray). Since twin-width has no known lower bounds, it is equal to the original hierarchy. We show proven local maxima (gray hexagons).

Next, we show that twin-width is not upper bounded by distance to bipartite. Consider a d-dimensional hypercube graph, that is a graph G constructed by duplicating itself d-times starting form a single vertex while also connecting each vertex with its duplicate. It is known that any hypercube graph is bipartite, since each duplicate can be colored with the opposite color, thus distance to bipartite is 0. Observe that the twin-width of a graph is at least the lowest possible red degree after the first contraction which is at least $2d-4$, since the coordinates differ in at least one dimemsion except for the first two dimensions. Hence, twin-width is incomparable to distance to bipartite. □

Proposition 6.7. *Twin-width is* ***incomparable*** *to Clique Cover Number.*

Proof. Note that twin-width is upper bounded by clique-width and therefore by Lemma 2.1 cannot upper bound minimum clique cover.

Next, we show that twin-width is not upper bounded by minimum clique cover. Consider the complement $\bar{G}$ of a d-dimensional hypercube graph G as seen in Proposition 6.6, that is a graph $\bar{G}$ with a minimum clique cover of at most 2, since G is bipartite. Note that Lemma 6.1 states that the twin-width of the complement graph $\bar{G}$ is equal to the twin-width of graph G. Thus, the twin-width of $\bar{G}$ is $2d-4$ as well. Hence, twin-width is incomparable to minimum clique cover. □

Proposition 6.8. *Twin-width is* ***incomparable*** *to Maximum Degree.*

Proof. Note that twin-width is upper bounded by clique-width and therefore by Lemma 2.1 cannot upper bound maximum degree.

Bonnet et al. [Bon+21] showed that the twin-width of cubic graphs is unbounded, meaning that there is a cubic graph with twin width k while per definition cubic graphs have a maximum degree of 3. Hence, twin-width is incomparable to maximum degree. □

Observation 6.9. *Twin-width is* ***incomparable*** *to Bisection Width.*

Proof. Note that twin-width is upper bounded by clique-width and therefore by Lemma 2.1 cannot upper bound bisection width.

Next, we show that twin-width is not upper bounded by bisection width. Consider two equal connected components with twin-width n. It is known that the bisection width of two equal connected components is 0 while the twin-width is the highest twin-width of all connected components. Thus, the twin-width is n. Hence, twin-width is incomparable to bisection width. □

Since there are no potential local minima, there are currently no known lower bounds for twin-width. Since every potential local maxima is indeed a local maxima and with Lemma 2.3, we have determined every relation involving twin-width in the graph parameter hierarchy.

Chapter 7

Additional Relations

We consider a gap left behind in the hierarchy by Schröder [Sch] between degeneracy and boxicity.

Proposition 7.1. *Degeneracy is* ***incomparable*** *to Boxicity.*

Proof. Note that boxicity is upper bounded by distance to clique and therefore by Lemma 2.1 cannot upper bound degeneracy.

Next, we show that degeneracy does not upper bound boxicity. Consider the graph class G_n of a fully subdivided clique of size n. Let $V_1 \in V(G_n)$ be the vertices of the original clique and $V_2 = V \backslash V_1$. Note that degeneracy is the smallest number k such that any subgraph $G[V']$ contains a vertex with degree at most k. Observe how for any subgraph $G[V']$, V' either contains a vertex $v \in V_2$, in which case degree of v is at most 2, or does not contain a vertex $v \in V_2$, in which case only vertices $w \in V_1$ are contained with degree 0, since $N_{G_n}(w) \subseteq V_2$. Thus, the degeneracy of G_n is 2. Additionaly, as shown by L. Sunil Chandran et al. [CMS11] the boxicity of a fully subdivided clique of size n is at least $\frac{(\log_2 \log_2 n)+1}{2}$. Hence, degeneracy is incomparable to boxicity. $\square$

The gap left between chordality and maximum clique remains open.

Chapter 8

Conclusion

In this thesis, we extented the graph parameter hierarchy by the following six parameters: twin cover number, edge clique cover number, neighborhood diversity, modular-width, c-closure and twin-width. With the goal of reducing the number of necessary proofs as much as possible, we determined for all parameters almost every relation possible by comparing them with selceted parameters, such that the resulting relations effectively cover any other parameter as well. For this, we introduced a method using local extrema and proved that their determined existence covers any parameter and, as a result, expanded the graph parameter hierarchy, as seen in Figure 1.1. The only two questions left are whether chordality and c-closure as well as chordality and maximum clique are incomparable or not.

Since most of the added relations were previously unknown, we discovered for each parameter some interesting relations such as twin cover number strictly upper bounding distance to cluster, since twin cover was primarly studied in relation with parameters between tree-width/neighborhood diversity and clique-width. The edge clique cover number is also surprisingly only directly related to distance to clique and neighborhood diversity. While neighborhood diversity and modular-width are very similar, it is interesting to see their difference by having only neighborhood diversity upper bounding boxicity. We found out, that the c-closure is upper bounded by the feedback edge number set, and even though planar graphs have a bounded twin-width, the twin-width being upper bounded by distance to planar was a lot more complex than initially thought.

There are many possible methods to further improve and maintain the graph parameter hierarchy. For instance, it should be considered that adding new parameters along with current research is important to keep it up to date. Simirlarly already published research like from Bonnet et al. [Bon+21] would also contribute to the hierarchy.

Since each expansion of the parameter hierarchy relies on the previously found relations, it is also important to try to keep the hierarchy complete as each potential relation may increase the efficiency of inserting even more parameters, similar to what Schröder [Sch] did.

While the graph parameter hierarchy itself contains a lot of different parameters, it is also of note that most parameters have specific edge cases in which they are easily bounded. By forcing additional constraints, we may achieve a different type of hierarchy. For instance, consider a graph parameter hierarchy on only connected graphs for which parameters like edge clique cover number and clique cover number have a bounding relation.

Finally, we should note the complexity of each relation. Even though we managed to construct relations between most of the parameters, we ignored the improtance of scale, as whether there was a linear or exponential bound did not impact the hierarchy. It would be interesting to study which bounds are linear, polynomial or exponential. Ideally, these classifications are acompanied by tight examples, that is, if we show that $f_{p,q}(p(G)) \geq q(G)$ for all graphs G, then there should be a graph G' with $q(G') = f_{p,q}(p(G))$.

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