**Technische Universität Berlin**
Electrical Engineering and Computer Science
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Algorithmics and Computational Complexity (AKT)

# Comparing Graph Parameters

## Dependencies and Independencies

## Bachelor Thesis

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a thesis submitted for the degree of „Bachelor of Science“ (B. Sc.)
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## Zusammenfassung

Aufbauend auf der Arbeit von [Sa19] und [Fr8] untersuchen wir Verbindungen zwischen Graphparametern, um die Arbeit an der Graphparameterhierarchie weiterzuführen. Wir tuen dies indem wir systematisch alle Parameter vergleichen für die bisher noch keine gegenseitigen Einschränkungen entdeckt wurden. Dabei ist der hauptsächliche Beitrag dieser Arbeit zur Graphparameterhierarchie, dass sie Beweise für die Unabhängigkeit fast aller dieser Parameter anbringt. Ein wesentlicher Bestandteil davon ist das Sammeln von verwandter Forschung und ihre Anwendung auf die Graphparameterhierarchie.

## Abstract

Based on the work by [Sa19] as well as [Fr8], we investigate unknown connections between graph parameters to continue the work on the graph parameter hierarchy. We do so by systematically comparing all parameters for which no bounds have been discovered yet. The main contribution of this thesis to the hierarchy lies in providing the proofs of independence for close to all of these parameters. An essential component of this work is the collection of related research and the application of this research to the hierarchy.

# Contents

# 1 Intro

With parameterized algorithms being a well-researched tool when facing ***NP***-hard problems, it is important to detect when to apply them by recognizing when they will be effective. As an example, if you tried to solve dominating set on instances with bounded `Feedback Vertex Set` it might not be initially obvious that graphs with bounded `Feedback Vertex Set` also have bounded `Treewidth`, which helps you to compute the problem efficiently as `Treewidth` is a versatile parameter in parameterized complexity since numerous problems including dominating set can be parameterized by it. This is just one of many examples where understanding connections between graph parameters can be helpful in solving problems on graphs. As a result of this [Jan13] stated that "a thorough understanding of the interplay between parameters is crucial for a proper grip on problem complexity" . Additionally, since the known ways of computing different graph parameters vary strongly in complexity with some like `Minimum Degree` being linear-time computable and others like `Minimum Vertex Cover` being ***NP***-hard to compute, the knowledge about connections between parameters can be used to find more easily computable higher or lower bounds for some parameters.

Following this notion a so-called graph parameter hierarchy was created to show which parameters are bounded by others thus giving an overview of the relations between different graph properties.

## 1.1 Related Work

Several attempts have been made to actually discover bounds between often used graph parameters in the form of a graph parameter hierarchy. It has been approached by [Sa19] who have accumulated a list of graph parameters that are used in different fields of research in computer science and started to investigate the bounds between all those parameters, numerous other researchers have independently researched bounds between graph parameters and thus contributed to the known hierarchy. Additionally, this work is based on [Fr8] which includes proofs for many individual bounds.

## 1.2 Our Contribution

This thesis builds upon previous work on the graph parameter hierarchy and tries to complete it by finding the last remaining undiscovered bounds between parameters and proving for all pairs of parameters for which no bound has been discovered, that they are indeed independent. We prove the independence of these parameters by traversing the parameter hierarchy from top to bottom and showing for every parameter that it does not upper-bound the parameters for which no bounds have been discovered. This process is facilitated by the fact that, since we can choose every upper-bounding function $f$ to be monotone, if we prove that a parameter $a$ does not upper-bound another parameter $b$, it follows that every parameter $a'$ does not upper-bound any parameter $b'$ if $a$ upper-bounds $a'$ and $b'$ upper-bounds $b$, since in that case the function $f_{b',b} \circ f_{a',b'} \circ f_{a',a}$ would be a valid upper bound between $a$ and $b$. The acummulated findings on the graph parameter hierarchy can be observed in Figure 1.1. It is a Hasse diagramm with the edges representing strict upper-bound relations. The results of the research in this paper can be more clearly observed in Table 1.1 where for every pair of parameters it is indicated whether one upper-bounds the other and if they are unbounded where to find the proof for that in this thesis.

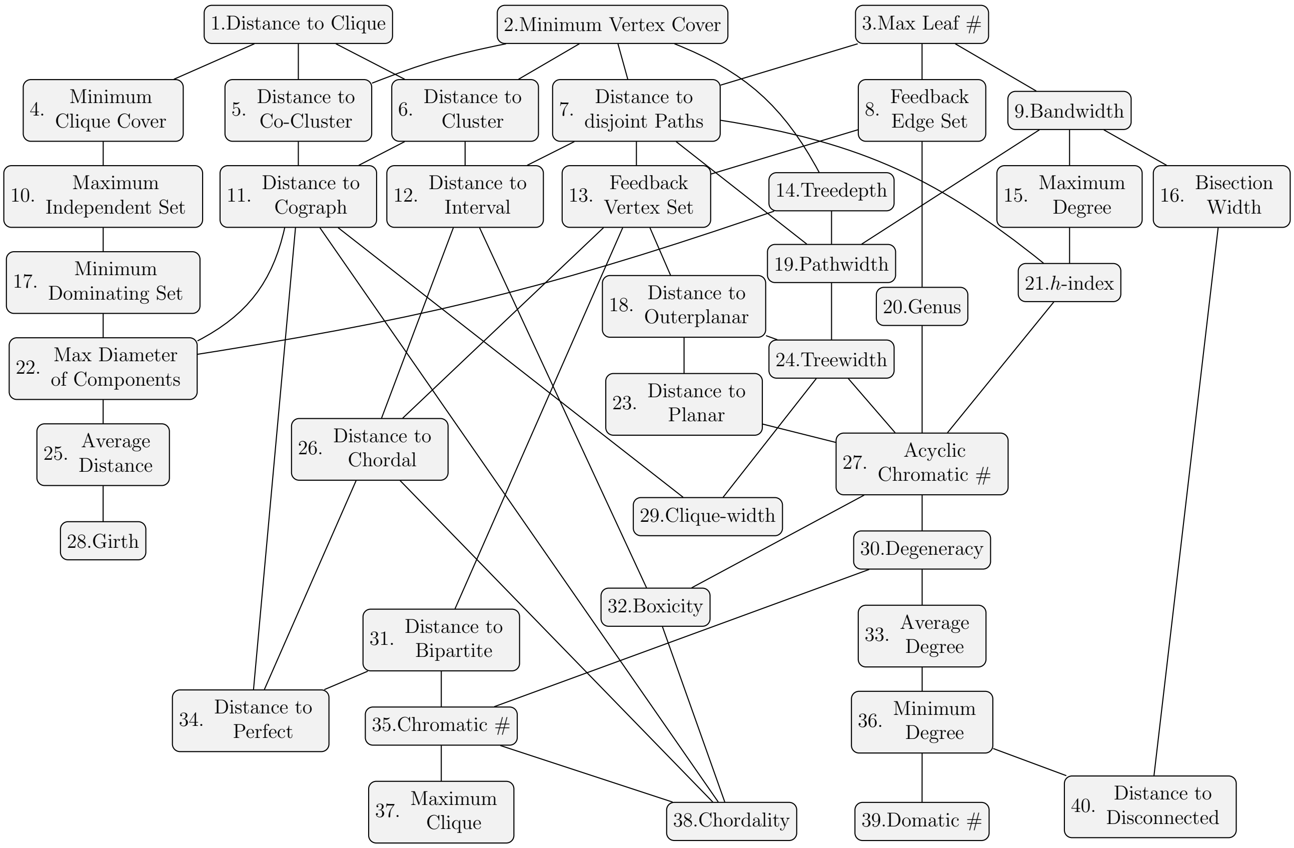



Figure 1.1: A Hasse graph displaying the graph parameter hierarchy

| | 1. | 2. | 3. | 4. | 5. | 6. | 7. | 8. | 9. | 10. | 11. | 12. | 13. | 14. | 15. | 16. | 17. | 18. | 19. | 20. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. | | 3.7 | 3.7 | | | | 3.7 | 3.7 | 3.7 | | | | 3.7 | 3.7 | 3.7 | 3.8 | | 3.7 | 3.7 | 3.7 |
| 2. | 3.5 | | 3.18 | 3.5 | | | | 3.18 | 3.20 | 3.5 | | | | | 3.19 | 3.20 | 3.5 | | | 3.18 |
| 3. | 3.14 | 3.14 | | 3.14 | 3.14 | 3.14 | | | | 3.14 | 3.14 | | | 3.14 | | | 3.14 | | | |
| 4. | | 3.7 | 3.7 | | 3.11 | 3.11 | 3.7 | 3.7 | 3.7 | | 3.11 | 3.11 | 3.7 | 3.7 | 3.7 | 3.8 | | 3.7 | 3.7 | 3.7 |
| 5. | | | 3.7 | 3.5 | | 3.12 | 3.7 | 3.7 | 3.7 | 3.5 | | 3.12 | 3.7 | 3.7 | 3.7 | 3.8 | 3.5 | 3.7 | 3.7 | 3.7 |
| 6. | | | 3.7 | 3.5 | 3.3 | | 3.7 | 3.7 | 3.7 | 3.5 | | | 3.7 | 3.7 | 3.7 | 3.8 | 3.5 | 3.7 | 3.7 | 3.7 |
| 7. | 3.14 | | | 3.14 | 3.14 | 3.14 | | 3.18 | 3.20 | 3.14 | 3.14 | | | 3.14 | 3.19 | 3.20 | 3.14 | | | 3.18 |
| 8. | 3.14 | 3.14 | | 3.14 | 3.14 | 3.14 | 3.21 | | 3.29 | 3.14 | 3.14 | 3.21 | | 3.14 | 3.22 | 3.29 | 3.14 | | 3.16 | |
| 9. | 3.14 | 3.14 | | 3.14 | 3.14 | 3.14 | 3.24 | 3.24 | | 3.14 | 3.14 | 3.24 | 3.24 | 3.14 | | | 3.14 | 3.13 | | 3.27 |
| 10. | | 3.7 | 3.7 | | 3.11 | 3.11 | 3.7 | 3.7 | 3.7 | | 3.11 | 3.11 | 3.7 | 3.7 | 3.7 | 3.8 | | 3.7 | 3.7 | 3.7 |
| 11. | | | 3.7 | 3.5 | | | 3.7 | 3.7 | 3.7 | 3.5 | | 3.12 | 3.7 | 3.7 | 3.7 | 3.8 | 3.5 | 3.7 | 3.7 | 3.7 |
| 12. | | | | 3.5 | 3.3 | | | 3.7 | 3.7 | 3.5 | 3.14 | | 3.7 | 3.7 | 3.7 | 3.8 | 3.5 | 3.7 | 3.7 | 3.7 |
| 13. | 3.14 | | | 3.14 | 3.14 | 3.14 | | | 3.29 | 3.14 | 3.14 | 3.21 | | 3.14 | 3.22 | 3.29 | 3.14 | | 3.16 | 3.18 |
| 14. | 3.5 | | 3.18 | 3.5 | 3.24 | 3.24 | 3.24 | 3.18 | 3.20 | 3.5 | 3.24 | 3.24 | 3.24 | | 3.19 | 3.20 | 3.5 | 3.13 | | 3.18 |
| 15. | 3.14 | 3.14 | | 3.14 | 3.14 | 3.14 | 3.24 | 3.24 | | 3.14 | 3.14 | 3.24 | 3.24 | 3.14 | | 3.26 | 3.14 | 3.13 | 3.26 | 3.27 |
| 16. | 3.14 | 3.14 | | 3.14 | 3.14 | 3.14 | 3.24 | 3.24 | | 3.14 | 3.14 | 3.24 | 3.24 | 3.14 | 3.28 | | 3.14 | 3.13 | 3.28 | 3.27 |
| 17. | | 3.7 | 3.7 | | 3.11 | 3.11 | 3.7 | 3.7 | 3.7 | | 3.11 | 3.11 | 3.7 | 3.7 | 3.7 | 3.8 | | 3.7 | 3.7 | 3.7 |
| 18. | 3.14 | | | 3.14 | 3.14 | 3.14 | | | 3.29 | 3.14 | 3.14 | 3.21 | | 3.14 | 3.22 | 3.29 | 3.14 | | 3.16 | 3.18 |
| 19. | 3.5 | | | 3.5 | 3.24 | 3.24 | | 3.18 | | 3.5 | 3.24 | 3.24 | 3.24 | | 3.19 | 3.20 | 3.5 | 3.13 | | 3.18 |
| 20. | 3.14 | 3.14 | | 3.14 | 3.14 | 3.14 | 3.21 | | 3.29 | 3.14 | 3.14 | 3.21 | 3.24 | 3.14 | 3.22 | 3.29 | 3.14 | 3.34 | 3.16 | |
| 21. | 3.14 | | | 3.14 | 3.14 | 3.14 | 3.24 | 3.24 | | 3.14 | 3.14 | 3.24 | 3.24 | 3.14 | | 3.26 | 3.14 | 3.13 | 3.26 | 3.27 |
| 22. | | 3.7 | 3.7 | | 3.11 | 3.11 | 3.7 | 3.7 | 3.7 | | 3.11 | 3.11 | 3.7 | 3.7 | 3.7 | 3.8 | | 3.7 | 3.7 | 3.7 |
| 23. | 3.14 | | | 3.14 | 3.14 | 3.14 | | | 3.29 | 3.14 | 3.14 | 3.21 | | 3.14 | 3.22 | 3.29 | 3.14 | | 3.16 | 3.18 |
| 24. | 3.14 | | | 3.14 | 3.14 | 3.14 | | | 3.29 | 3.14 | 3.14 | 3.21 | | | 3.22 | 3.29 | 3.14 | | | 3.18 |
| 25. | | 3.7 | 3.7 | | 3.11 | 3.11 | 3.7 | 3.7 | 3.7 | | 3.11 | 3.11 | 3.7 | 3.7 | 3.7 | 3.8 | | 3.7 | 3.7 | 3.7 |
| 26. | | | | 3.5 | 3.3 | | | 3.7 | 3.7 | 3.5 | 3.14 | | | 3.7 | 3.7 | 3.8 | 3.5 | 3.7 | 3.7 | 3.7 |
| 27. | 3.14 | | | 3.14 | 3.14 | 3.14 | | | 3.29 | 3.14 | 3.14 | 3.21 | | 3.14 | 3.22 | 3.29 | 3.14 | | 3.16 | 3.18 |
| 28. | | 3.7 | 3.7 | | 3.11 | 3.11 | 3.7 | 3.7 | 3.7 | | 3.11 | 3.11 | 3.7 | 3.7 | 3.7 | 3.8 | | 3.7 | 3.7 | 3.7 |
| 29. | | | | 3.14 | | | | | | 3.14 | | 3.21 | | | 3.22 | 3.29 | 3.14 | | | 3.18 |
| 30. | 3.14 | | | 3.14 | 3.14 | 3.14 | | | | 3.14 | 3.14 | 3.21 | | | | 3.29 | 3.14 | | | |
| 31. | 3.14 | | | 3.14 | 3.14 | 3.14 | | | 3.29 | 3.14 | 3.14 | 3.21 | | 3.14 | 3.22 | 3.29 | 3.14 | [CFM11] | 3.16 | 3.18 |
| 32. | | | | 3.5 | 3.3 | | | | | 3.5 | 3.14 | | | | | 3.8 | 3.5 | | | |
| 33. | 3.14 | | | 3.14 | 3.14 | 3.14 | | | | 3.14 | 3.14 | 3.21 | | | | 3.29 | 3.14 | | | |
| 34. | | | | 3.5 | | | | 3.7 | 3.7 | 3.5 | | | | 3.7 | 3.7 | 3.8 | 3.5 | 3.7 | 3.7 | 3.7 |
| 35. | 3.14 | | | 3.14 | 3.14 | 3.14 | | | | 3.14 | 3.14 | 3.21 | | | | 3.29 | 3.14 | | | |
| 36. | 3.14 | | | 3.14 | 3.14 | 3.14 | | | | 3.14 | 3.14 | 3.21 | | | | 3.29 | 3.14 | | | |
| 37. | 3.14 | | | 3.14 | 3.14 | 3.14 | | | | 3.14 | 3.14 | 3.21 | | | | 3.29 | 3.14 | | | |
| 38. | | | | 3.14 | 3.14 | 3.14 | | | | 3.14 | 3.14 | 3.21 | | | | 3.29 | 3.14 | | | |
| 39. | 3.14 | | | 3.14 | 3.14 | 3.14 | | | | 3.14 | 3.14 | 3.21 | | | | 3.29 | 3.14 | | | |
| 40. | 3.14 | | | 3.14 | 3.14 | 3.14 | | | | 3.14 | 3.14 | 3.21 | | | | 3.29 | 3.14 | | | |

| | 21. | 22. | 23. | 24. | 25. | 26. | 27. | 28. | 29. | 30. | 31. | 32. | 33. | 34. | 35. | 36. | 37. | 38. | 39. | 40. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. | 3.7 | | 3.7 | 3.7 | | | 3.7 | | | 3.7 | 3.7 | | 3.7 | | 3.7 | 3.7 | 3.7 | | 3.7 | 3.8 |
| 2. | | | | | | | | | | | | | | | | | | | | |
| 3. | | 3.14 | | | 3.14 | | | 3.14 | | | | | | | | | | | | |
| 4. | 3.7 | | 3.7 | 3.7 | | 3.11 | 3.7 | | 3.9 | 3.7 | 3.7 | 3.11 | 3.7 | 3.6 | 3.7 | 3.7 | 3.7 | 3.11 | 3.7 | 3.8 |
| 5. | 3.7 | | 3.7 | 3.7 | | 3.12 | 3.7 | | | 3.7 | 3.7 | 3.4 | 3.7 | | 3.7 | 3.7 | 3.7 | | 3.7 | 3.8 |
| 6. | 3.7 | | 3.7 | 3.7 | | | 3.7 | | | 3.7 | 3.7 | | 3.7 | | 3.7 | 3.7 | 3.7 | | 3.7 | 3.8 |
| 7. | | 3.14 | | | 3.14 | | | 3.14 | | | | | | | | | | | | |
| 8. | 3.22 | 3.14 | | | 3.14 | | | 3.14 | | | | | | | | | | | | |
| 9. | | 3.14 | 3.13 | | 3.14 | 3.24 | | 3.14 | | | 3.24 | | | 3.24 | | | | | | |
| 10. | 3.7 | | 3.7 | 3.7 | | 3.11 | 3.7 | | 3.9 | 3.7 | 3.7 | 3.11 | 3.7 | 3.6 | 3.7 | 3.7 | 3.7 | 3.11 | 3.7 | 3.8 |
| 11. | 3.7 | | 3.7 | 3.7 | | 3.12 | 3.7 | | | 3.7 | 3.7 | 3.4 | 3.7 | | 3.7 | 3.7 | 3.7 | | 3.7 | 3.8 |
| 12. | 3.7 | 3.14 | 3.7 | 3.7 | 3.14 | | 3.7 | 3.14 | [GR99] | 3.7 | 3.7 | | 3.7 | | 3.7 | 3.7 | 3.7 | | 3.7 | 3.8 |
| 13. | 3.22 | 3.14 | | | 3.14 | | | 3.14 | | | | | | | | | | | | |
| 14. | 3.22 | | 3.13 | | | 3.24 | | | | | 3.24 | | | 3.24 | | | | | | |
| 15. | | 3.14 | 3.13 | 3.26 | 3.14 | 3.24 | | 3.14 | 3.26 | | 3.24 | | | 3.24 | | | | | | |
| 16. | 3.28 | 3.14 | 3.13 | 3.28 | 3.14 | 3.24 | 3.28 | 3.14 | 3.32 | 3.28 | 3.24 | 3.31 | 3.28 | 3.24 | 3.31 | 3.28 | 3.33 | 3.31 | 3.28 | |
| 17. | 3.7 | | 3.7 | 3.7 | | 3.11 | 3.7 | | 3.9 | 3.7 | 3.7 | 3.11 | 3.7 | 3.6 | 3.7 | 3.7 | 3.7 | 3.11 | 3.7 | 3.8 |
| 18. | 3.22 | 3.14 | | | 3.14 | 3.23 | | 3.14 | | | 3.23 | | | 3.23 | | | | | | |
| 19. | 3.22 | 3.14 | 3.13 | | 3.14 | 3.24 | | 3.14 | | | 3.24 | | | 3.24 | | | | | | |
| 20. | 3.22 | 3.14 | 3.34 | 3.17 | 3.14 | 3.24 | | 3.14 | 3.17 | | 3.24 | | | 3.24 | | | | | | |
| 21. | | 3.14 | 3.13 | 3.26 | 3.14 | 3.24 | | 3.14 | 3.26 | | 3.24 | | | 3.24 | | | | | | |
| 22. | 3.7 | | 3.7 | 3.7 | | 3.11 | 3.7 | | 3.9 | 3.7 | 3.7 | 3.11 | 3.7 | 3.6 | 3.7 | 3.7 | 3.7 | 3.11 | 3.7 | 3.8 |
| 23. | 3.22 | 3.14 | | 3.17 | 3.14 | 3.23 | | 3.14 | 3.17 | | 3.23 | | | 3.23 | | | | | | |
| 24. | 3.22 | 3.14 | 3.13 | | 3.14 | 3.23 | | 3.14 | | | 3.23 | | | 3.23 | | | | | | |
| 25. | 3.7 | | 3.7 | 3.7 | | 3.11 | 3.7 | | 3.9 | 3.7 | 3.7 | 3.11 | 3.7 | 3.6 | 3.7 | 3.7 | 3.7 | 3.11 | 3.7 | 3.8 |
| 26. | 3.7 | 3.14 | 3.7 | 3.7 | 3.14 | | 3.7 | 3.14 | [GR99] | 3.7 | 3.7 | 3.25 | 3.7 | | 3.7 | 3.7 | 3.7 | | 3.7 | 3.8 |
| 27. | 3.22 | 3.14 | | | 3.14 | 3.23 | | 3.14 | 3.17 | | 3.23 | | | 3.23 | | | | | | |
| 28. | 3.7 | | 3.7 | 3.7 | | 3.11 | 3.7 | | 3.9 | 3.7 | 3.7 | 3.11 | 3.7 | 3.6 | 3.7 | 3.7 | 3.7 | 3.11 | 3.7 | 3.8 |
| 29. | 3.22 | 3.14 | 3.13 | | 3.14 | 3.23 | 3.11 | 3.14 | | 3.11 | 3.23 | 3.4 | 3.7 | 3.23 | 3.11 | 3.7 | 3.7 | 3.11 | 3.7 | 3.8 |
| 30. | | 3.14 | | | 3.14 | 3.23 | | 3.14 | 3.17 | | 3.23 | | | 3.23 | | | | | | |
| 31. | 3.22 | 3.14 | [CFM11] | [CFM11] | 3.14 | 3.12 | [CFM11] | 3.14 | 3.26 | 3.12 | | [CFM11] | 3.12 | | | 3.12 | | | 3.12 | 3.12 |
| 32. | | 3.14 | | | 3.14 | 3.24 | | 3.14 | [GR99] | 3.7 | 3.7 | | 3.7 | 3.24 | 3.7 | 3.7 | 3.7 | | 3.7 | 3.8 |
| 33. | | 3.14 | | | 3.14 | 3.23 | | 3.14 | 3.17 | | 3.23 | 3.36 | | 3.23 | 3.35 | | 3.35 | 3.36 | | |
| 34. | 3.7 | 3.14 | 3.7 | 3.7 | 3.14 | | 3.7 | 3.14 | [GR99] | 3.7 | 3.7 | 3.25 | 3.7 | | 3.7 | 3.7 | 3.7 | 3.11 | 3.7 | 3.8 |
| 35. | | 3.14 | | | 3.14 | 3.23 | | 3.14 | 3.17 | | | [CFM11] | 3.12 | 3.23 | | 3.12 | | | 3.12 | 3.12 |
| 36. | | 3.14 | | | 3.14 | 3.23 | | 3.14 | 3.17 | | 3.23 | 3.36 | | 3.23 | 3.35 | | 3.35 | 3.36 | | |
| 37. | | 3.14 | | | 3.14 | 3.23 | | 3.14 | 3.17 | | | [CFM11] | 3.12 | 3.23 | | 3.12 | | | 3.12 | 3.12 |
| 38. | | 3.14 | | | 3.14 | 3.23 | | 3.14 | 3.17 | | | | 3.12 | 3.23 | | 3.12 | 3.7 | | 3.12 | 3.12 |
| 39. | | 3.14 | | | 3.14 | 3.23 | | 3.14 | 3.17 | | 3.23 | 3.36 | | 3.23 | 3.35 | | 3.35 | 3.36 | | 3.30 |
| 40. | | 3.14 | | | 3.14 | 3.23 | | 3.14 | 3.17 | | 3.23 | 3.31 | | 3.23 | 3.35 | | 3.35 | 3.36 | 3.28 | |

Table 1.1: Table of all bounds between parameters and their proofs.

- Green cells indicate that the row parameter strictly upper bounds the column parameter.
- Yellow cells indicate that the column parameter strictly upper bounds the row parameter.
- Red cells indicate the row parameter being unbounded by the column parameter.
- Blue cells indicate that it is yet unknown whether the row parameter is bounded by the column parameter.

# 2 Preliminaries

Here we will explain the notation we will use as well as the definition of some expressions. By a *graph parameter* we refer to a function $f : G \to \mathbb{R}$ which maps a graph to a real number. We say that a parameter $p$ *upper-bounds* another parameter $q$ if there exists a function $f_{p,q} : \mathbb{R} \to \mathbb{R}$ such that $f_{p,q}(p(G)) \geq q(G)$ for all graphs $G$ and we say that $p$ *strictly upper-bounds* $q$, if $p$ upper-bounds $q$ and $q$ does not upper-bound $p$. Furthermore, we say that a parameter $p$ is *unbounded* by another parameter $q$, if $q$ does not upper-bound $p$, and we call $p$ and $q$ *independent*, if neither parameter upper-bounds the other.

All parameters are used as defined on undirected graphs $G = (V, E)$ in [Sa19].

This thesis uses some regularly used notations in graph theory. We denote the complement graph $G' = (V, (V \times V) \setminus E)$ to a graph $G = (V, E)$ as $\bar{G}$. We also use specific names to denote well-known graph classes.

- $K_n$ denotes the clique graph of size $n$,
- $K_{n,m}$ denotes the complete bipartite graph with $n$ vertices in one partition and $m$ vertices in the other,
- $C_n$ denotes the circle graph with $n$ vertices, and
- $P_n$ denotes a path with $n$ vertices

Furthermore we use the function $N_G(v)$ to denote the set of neighbors of the vertex $v$ in the graph $G$ and the function $deg_G(v)$ to denote the degree of $v$ in the graph $G$. To make the graph classes used in this paper easier to understand we also define the following operators which can be used on both graphs and graph classes.

- $\cdot : \mathbb{N} \times \mathbb{G} \to \mathbb{G}$, such that $n \cdot G$ maps $G$ to a graph which contains $n$ disconnected copies of $G$
- $+ : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$, such that $G_1 + G_2$ maps to a graph which contains disconnected copies of $G_1$ and $G_2$
- $- : \mathbb{G} \times \mathbb{V} \to \mathbb{G}$, such that $G - V'$ maps to the induced subgraph of $G = (V, E)$ on the vertex set $V \setminus V'$

Furthermore, we denominate the parameters that are upper-bounding a parameter $p$ in Figure 1.1 as "the parameters above $p$" and the parameters that are upper-bounded by $p$ as "the parameters below $p$". Both of these sets also include the parameter $p$ itself.

Moreover, we denominate the parameters that are below a parameter $p_1$ and are also above another parameter $p_2$ as "the parameters between $p_1$ and $p_2$".

# 3 Exploration of further connections

In this section we will present the result of our research. First we will go through the bounds which had not yet been included by the prior work on the graph parameter hierarchy. Then we will go through the graph to prove the unboundedness for the remaining pairs of parameters by grouping the hierarchy into different parameter groups and proving the unboundedness inside each group of parameters as well as the fact that none of the parameters in the group upper-bound any additional parameters outside of the group.

## 3.1 Dependencies

The bounds which we discovered in our work on the graph parameter hierarchy are that `Treedepth` strictly upper-bounds `Maximum Diameter of Components` and that `Distance to disjoint Paths` strictly upper-bounds `h-index`. We also included that `Chromatic Number` upper-bounds `Chordality`, the proof for which can be found in [MS93]. The strictness of that bound follows from Proposition 3.7, since it shows that `Distance to Clique` does not upper-bound `Maximum Clique`.

**Proposition 3.1.** *`Treedepth` strictly upper-bounds `Maximum Diameter of Components`*

*Proof.* We denote the treedepth of a graph $G$ as $td(G)$ and the maximum diameter of components of $G$ as $md(G)$. We will show that for each graph $G$ it holds that $md(G) \leq 2^{td(G)} - 2$. We assume towards a contradiction that there exists a graph $G = (V, E)$ such that $md(G) > 2^{td(G)} - 2$. Let $m = md(G)$ and $t = td(G)$. We prove by induction that $P_m$ has `Treedepth` at least $\lceil log_2(m+1) \rceil$.
Base case: $td(P_1) = 1 = \lceil log_2(2) \rceil = \lceil log_2(md(P_1) + 2) \rceil$.
Induction hypothesis: For some $n \in \mathbb{N}$, it holds that every $P_j$ with $j \leq n$ has treedepth at least $\lceil log_2(j+1) \rceil$.
Induction step: Following the recursive definition of `Treedepth`, the graph $P_{n+1}$ has treedepth $1 + min_{v \in V} td(P_{n+1} - v)$. It holds for every vertex $v \in V$, that $P_{n+1} - v$ contains $P_k$ with $k \geq \lceil n/2 \rceil$ as a subgraph since deleting a vertex from $P_{n+1}$ will divide $P_{n+1}$ into at most two subgraphs $P_h$ und $P_i$ with $h + i = n$. It follows from the induction hypothesis that $min_{v \in V} td(P_{n+1} - v) = td(P_{\lceil n/2 \rceil}) \geq \lceil log_2((\lceil n/2 \rceil + 1) \rceil$. Thus, $P_{n+1}$ has treedepth at least

$$1 + \lceil log_2(\lceil n/2 \rceil + 1) \rceil = \lceil 1 + log_2(\lceil n/2 \rceil + 1) \rceil = \lceil log_2(2 \cdot \lceil n/2 \rceil + 2) \rceil \geq \lceil log_2(n+2) \rceil.$$

Since $G$ has maximum diameter of components $m$, it has $P_{m+1}$ as a subgraph. This means that $G$ has treedepth at least $\lceil log_2(m+2) \rceil$. It follows that $G$ has maximum diameter of components at most $2^t - 2$, a contradiction to the assumption that $m > 2^t - 2$. `Maximum Diameter of Components` does not upper-bound `Distance to disjoint Paths` since Proposition 3.7 proves that `Distance to Clique` does not upper-bound `Domatic Number`. □

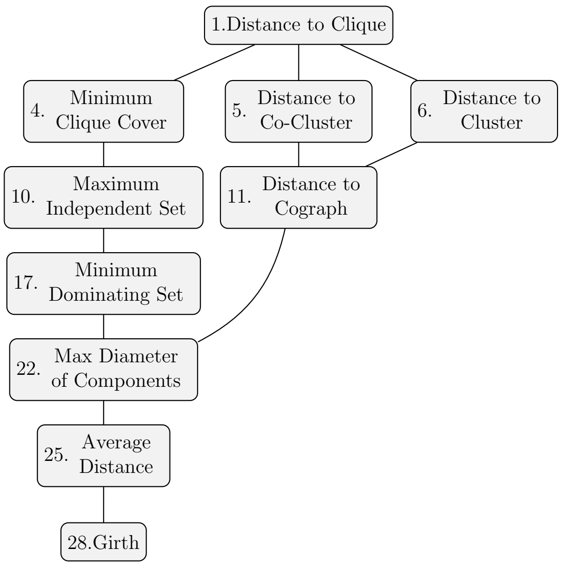
**Proposition 3.2.** *`Distance to disjoint Paths` strictly upper-bounds `h-index`*

*Proof.* We will show that for each graph $G$ it holds that `h-index`$(G) \leq$ `Distance to disjoint Paths`$(G) + 2$. Assume towards a contradiction that there exists a graph $G = (V, E)$ such that `h-index`$(G) >$ `Distance to disjoint Paths`$(G) + 2$. Let $p =$ `Distance to disjoint Paths`$(G)$ and $h =$ `h-index`$(G)$. By definition there exists a set $S \subseteq V$ with $|S| = p$ such that $G' := G - S$ only consists of disjoint paths. Since $G$ contains $h$ vertices $v$ such that $d_G(v) \geq h$ and $h > p$, there exists a vertex $v \in V'$ such that $d_G(v) \geq h$. Since $d_G(v) \geq h$, it follows that $d_{G'}(v) \geq h - p$, as at most $p$ of the $h$ neighbors of $v$ in $G$ could have been in $S$ and every other neighbor of $v$ in $G$ remains a neighbor of $v$ in $G'$. Since $h > p + 2$ and thus $h - p > 2$, if follows that $d_{G'}(v) > 2$. Since a collection of disjoint paths cannot contain a vertex with degree higher than 2, it follows that `Distance to disjoint Paths`$(G) \neq p$, a contradiction. `h-index` does not upper-bound `Distance to disjoint Paths` since Proposition 3.24 proves that `Bandwidth` does not upper-bound `Distance to Perfect`. □

## 3.2 Independencies

In this segment, we will traverse specific subgraphs of the graph parameter hierarchy. We will start by proving the independence of all independent parameters in the subgraph. For each pair of independent parameters in different subgraphs the proof that the first parameter does not upper-bound the second parameter can be found in the section of the subgraph containing the first parameter and the proof that the second parameter does not upper-bound the first parameter can be found in the section of the subgraph containing the second parameter. Together these proofs will show the independence of the pair. We will often prove the absence of several bounds in a single proposition. In those cases we will reference the proposition at each point in the thesis for which they are relevant.

### 3.2.1 Parameters between Distance to Clique and Girth




This section contains the subgraph of the graph parameter hierarchy between `Distance to Clique` and `Girth`. It consists of three different paths between `Distance to Clique` and `Girth` going through `Minimum Clique Cover`, `Distance to Co-Cluster`, and `Distance to Cluster` respectively.
`Distance to Co-Cluster` and `Distance to Cluster` are independent because `Distance to Co-Cluster` does not upper-bound `Boxicity` which is below `Distance to Cluster`, and because `Distance to Cluster` does not upper-bound `Distance to Co-Cluster` which we will prove directly.

**Proposition 3.3.** *`Distance to Cluster` does not upper-bound `Distance to Co-Cluster`.*

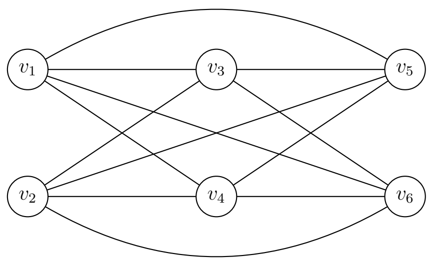



Figure 3.1: The graph $N_6$ as an example for the constructions in Proposition 3.4 and 3.31

*Proof.* Consider the graph class of $n \cdot P_2$ for $n \in \mathbb{N}$. Every induced subgraph of $n \cdot P_2$, which contains at least three vertices and at least one edge, is not a co-cluster Graph, since the complement of that graph would contain two vertices that are not connected but have a connection to every other vertex in the graph. Thus, at least one vertex of each $P_2$ has to be deleted to create a co-cluster graph meaning `Distance to Co-Cluster` grows linearly in $n$ while $n \cdot P_2$ is a cluster graph meaning it has distance to cluster zero. Since `Distance to Cluster` is bounded in this graph class and `Distance to Co-Cluster` is unbounded, `Distance to Cluster` does not upper-bound `Distance to Co-Cluster`. ☐

**Proposition 3.4.** *`Distance to Co-Cluster` does not upper-bound `Boxicity`.*

*Proof.* Consider the graph class of all cliques from which a perfect matching has been removed $N_n$ for $n = 2k$ and $k \in \mathbb{N}$ as pictured in Figure 3.1. Since $N_n$ contains no universal vertices and has `Minimum Degree` $n - 2$, it follows from [MS93] that the `Boxicity` of $N_n$ grows at least linearly in $n$. The class $N_n$ is a subclass of all co-cluster graphs because its cograph only contains the deleted matching and is thus $k \cdot K_2$ meaning it has `Distance to Co-Cluster` zero. Since `Distance to Co-Cluster` is bounded in this graph class and `Boxicity` is unbounded, `Distance to Co-Cluster` does not upper-bound `Boxicity`. ☐

To fully prove all independencies within the parameters between `Distance to Clique` and `Girth` we still have to prove the independencies of the parameters between `Minimum Clique Cover` and `Minimum Dominating Set` to `Distance to Co-Cluster`, `Distance to Cluster`, and `Distance to Cograph`. One side of these independencies follows from the fact that these three parameters do not upper-bound any parameter above `Minimum Dominating Set`, since `Minimum Vertex Cover` does not upper-bound `Minimum Dominating Set` while the other side of these independencies follows from the fact that `Minimum Clique Cover` does not upper-bound `Distance to Perfect` which is below `Distance to Cograph`, `Distance to Co-Cluster`, and `Distance to Cluster`.

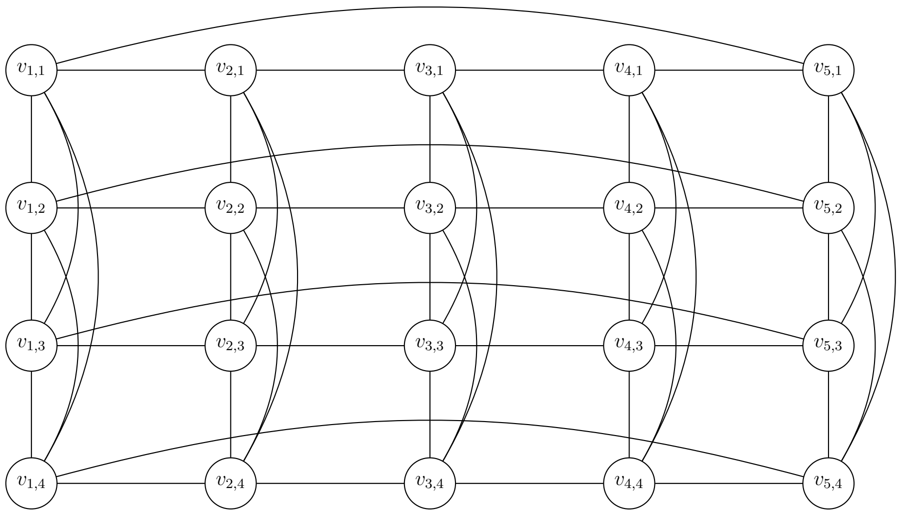



Figure 3.2: The graph $B_4$ as an example for the construction in Proposition 3.6

**Proposition 3.5.** *`Minimum Vertex Cover` does not upper-bound `Minimum Dominating Set`.*

*Proof.* Consider the graph class of all edgeless graphs $I_n$ for $n \in \mathbb{N}$. Each dominating set has to contain all vertices and hence the size of the minimum dominating set grows linearly in $n$ but the size of the `Minimum Vertex Cover` is zero for $I_n$ as the graphs do not contain any edges. Since `Minimum Vertex Cover` is bounded in this graph class and `Minimum Dominating Set` is unbounded, `Minimum Vertex Cover` does not upper-bound `Minimum Dominating Set`. □

**Proposition 3.6.** *`Minimum Clique Cover` does not upper-bound `Distance to Perfect`.*

*Proof.* Consider the graph class of all graphs $B_n = (V, E)$ such that $V = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ and $V_i = \{v_{i,j} | 1 \leq j \leq n\}$ and $E = \{\{v_{i,j}, v_{(i \bmod 5)+1,j}\} | 1 \leq i \leq 5 \wedge 1 \leq j \leq n\} \cup \{\{v_{i,j}, v_{i,k}\} | 1 \leq i \leq 5 \wedge 1 \leq j, k \leq n \wedge j \neq k\}$ for $n \in \mathbb{N}$(See Figure 3.2 as an example of this construction). Further, consider the $n$ disjunct induced subgraphs $B_{i,n}$ of $B_n$ from the vertex sets $\{v_{i,j} | 1 \leq j \leq n\}$. Each $B_{i,n}$ is a $C_5$ and because of that it is a non-perfect graph. Thus, the `Distance to Perfect` for $B_n$ grows at least linearly in $n$ as at least one of the vertices of each $B_{i,n}$ has to be deleted to obtain a perfect graph while the `Minimum Clique Cover` for $B_n$ is at most five because each $V_i$ forms a clique. Since `Minimum Clique Cover` is bounded in this graph class and `Distance to Perfect` is unbounded, `Minimum Clique Cover` does not upper-bound `Distance to Perfect`. □

We have now proven all independencies among the parameters between `Distance to Clique` and `Girth`.

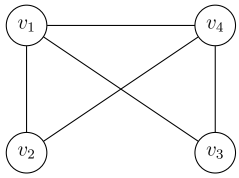



Figure 3.3: The graph $Q_4$ as an example for the construction in Proposition 3.8

In the next step, we will prove that no parameter between `Distance to Clique` and `Girth` upper-bounds a parameter that isn't below it in the hierarchy.
Since `Distance to Clique` does not upper-bound `Maximum Clique`, `Domatic Number`, or `Distance to Disconnected`, no parameter above any of these three can be upper-bounded by any parameters below `Distance to Clique`.

**Proposition 3.7.** *`Distance to Clique` does not upper-bound `Maximum Clique` or `Domatic Number`.*

*Proof.* Consider the graph class of all cliques $K_n$ for $n \in \mathbb{N}$. Since every vertex is a dominating set, the domatic number grows linearly in $n$ and the size of a maximal clique grows linearly in $n$ but the `Distance to Clique` is zero for $K_n$. Since `Distance to Clique` is bounded in this graph class and `Maximum Clique` and `Domatic Number` are unbounded, `Distance to Clique` does not upper-bound `Maximum Clique` or `Domatic Number`. □

**Proposition 3.8.** *`Distance to Clique` does not upper-bound `Distance to Disconnected`.*

*Proof.* Consider the graph class of all cliques with one missing edge $Q_n$ for $n \in \mathbb{N}$. The `Distance to Disconnected` grows linearly in $n$ but the `Distance to Clique` is one for $Q_n$. Since `Distance to Clique` is bounded in this graph class and `Distance to Disconnected` is unbounded, `Distance to Clique` does not upper-bound `Distance to Disconnected`. □

To prove that `Minimum Clique Cover` only upper-bounds the parameters between itself and `Girth` we have to prove that it does not upper-bound any other parameters below `Distance to Clique`. This follows from the fact that `Minimum Clique Cover` does not upper-bound `Distance to Perfect`, `Chordality` or `Cliquewidth` who are below every vertex that is below `Distance to Clique` without being below `Girth`. In Proposition 3.6 one of these bounds has already been proven.

**Proposition 3.9.** *`Minimum Clique Cover` does not upper-bound `Clique-width`.*

*Proof.* This proof is heavily based on the proof of theorem 5 in the paper [Loz11] which shows that $F_{n,n}$ has `Clique-width` at least $\lfloor n/2 \rfloor$. Consider the graph class of all graphs $F_{n,n}$, where $F_{n,n}$ contains $n^2$ vertices arranged in $n$ rows with $n$ vertices each such that each two consecutive rows induce a bipartite graph, where every vertex is adjacent to every vertex in the other partition except for one, and choose the columns such that two consecutive vertices in the same column are non-adjacent. Now consider the graph class of all graphs $F'_{n,n}$, where $F'_{n,n}$ is a modification of $F_{n,n}$, such that every vertex $v_{i,j}$, where $i$ is the row and $j$ is the column of the vertex, is connected to every other vertex $v_{i',j'}$ if $i \bmod 2 = i' \bmod 2$ meaning that every vertex with an odd row number is connected to every other vertex with an odd row number and the same goes for vertices with even row numbers. The construction of $F'_{n,n}$ as well as how it differs from $F_{n,n}$ is illustrated in Figure 3.4. We now follow the proof described in the paper to prove $F'_{n,n}$ has `Clique-width` at least $\lfloor n/2 \rfloor$ as well. Consider a minimal $t$-expression $C$ which constructs $F'_{n,n}$. We look at the lowest node $a$ in $C$ such that after this operation the graph contains a full row of $F'_{n,n}$. We now color all vertices in the one subexpression below $a$ red, all nodes in the other subexpression blue and all other nodes yellow. Since $a$ is minimal, neither of its subexpressions contain a full row of $F'_{n,n}$. But, by the definition of $a$, it contains a full row of $F'_{n,n}$ meaning there is at least one non-yellow row which we denote by $r$. Without loss of generality we assume that $r \leq \lfloor n/2 \rfloor$, since otherwise we could choose to order the rows in reverse order, and $r$ contains at least $\lceil n/2 \rceil$ red vertices, since otherwise we could swap the colors red and blue. We will now show that at the node $a$ $F'_{n,n}$ contains at least $\lfloor n/2 \rfloor$ vertices with different labels by showing that there are $\lfloor n/2 \rfloor$ red vertices in $F'_{n,n}$ for each pair of which there exists a non-red vertex which has an edge to exactly one of them meaning that they must all have pairwise different labels. We do this by using the following procedure:

**Data:** a graph $F'_{n,n}$
**Result:** a set U of at least $\lfloor n/2 \rfloor$ red vertices with different labels
1. Set $i = r$, $U = \emptyset$ and $J = \{j | v_{r,j} \text{ is red}\}$.
2. Set $K = \{j \in J | v_{i+1,j} \text{ is non-red}\}$.
3. If $K \neq \emptyset$, add the vertices $\{v_{i,k} | k \in K\}$ to U. Remove members of $K$ from $J$.
4. If $J = \emptyset$ terminate the procedure.
5. Increase $i$ by 1. If $i = n$ choose an arbitrary $j \in J$, put
$U = \{v_{m,j} | r \leq m \leq n-1\}$ and terminate.
6. Go Back to Step 2.

**Algorithm 1:** Construction of a set of $\lfloor n/2 \rfloor$ vertices with different labels

If the procedure finishes in Step 5, we consider an arbitrary pair of vertices $v_{l,j}$ and $v_{m,j}$ in $U$, where $l < m$. If $m \bmod 2 = l \bmod 2$, then we can use the fact that $v_{m+1,j}$ is red and choose any non-red vertex from row $m+1$. This vertex is adjacent to $v_{m,j}$, since every vertex in row $m+1$ except for $v_{m+1,j}$ is adjacent to $v_{m,j}$ and it is not adjacent to $v_{l,j}$ as $|m+1-l| > 1$ and $(m+1) \bmod 2 \neq l \bmod 2$. If $m \bmod 2 \neq l \bmod 2$, we can distinguish them by choosing any non-red vertex from a row that is not adjacent to $l$ or $m$, since every such vertex will be in either an odd or an even row and thus be adjacent to exactly one of $v_{l,j}$ and $v_{m,j}$. There will also always exist such a row for

The graph $F'_{3,3}$ as an example for the construction in Proposition 3.9

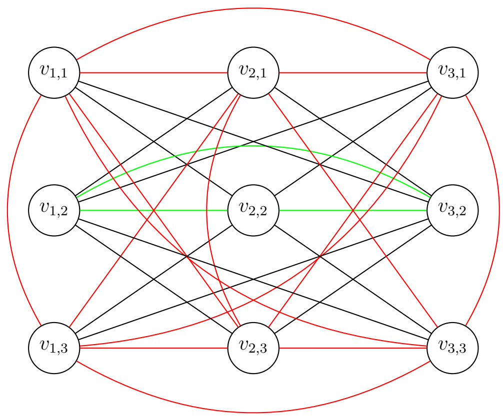



Figure 3.4: black edges are also in $F_{3,3}$, red edges connect all vertices in odd rows and green edges connect all vertices in even rows

sufficiently large $n \geq 7$. Since $v_{l,j}$ and $v_{m,j}$ were chosen arbitrarily from $U$, this proves that there are at least $\lfloor n/2 \rfloor$ vertices with different labels.
If the procedure finishes in Step 4, we consider an arbitrary pair of vertices $v_{l,j}$ and $v_{m,k}$ in $U$, where $l \leq m$. The procedure clearly guarantees that $j \neq k$ and that $v_{l+1,j}$ and $v_{m+1,k}$ are non-red. If $m \in \{l, l+2\}$, we can use $v_{l+1,j}$ to distinguish $v_{l,j}$ and $v_{m,k}$ since it is adjacent to $v_{m,k}$ and non-adjacent to $v_{l,j}$. If $m \notin \{l, l+2\}$ and $m \bmod 2 = l \bmod 2$, we can use the fact that $v_{m-1,k}$ is red, to choose any non-red vertex from row $m-1$ to distinguish $v_{l,j}$ and $v_{m,k}$, as $v_{m,k}$ is adjacent to every vertex in row $m-1$ except for $v_{m-1,k}$ and $v_{l,j}$ is non-adjacent to every vertex in row $m-1$ since $|m-1-l| > 1$ and $(m-1) \bmod 2 \neq l \bmod 2$. If $m \bmod 2 \neq l \bmod 2$ we can again distinguish them by choosing any non-red vertex from a row that is not adjacent to $l$ or $m$, since every such vertex will be in either an odd or an even row and thus be adjacent to exactly one of $v_{l,j}$ and $v_{m,k}$. Since $v_{l,j}$ and $v_{m,k}$ were chosen arbitrarily from $U$, this proves that there are at least $\lfloor n/2 \rfloor$ vertices with different labels.
Thus $F'_{n,n}$ has `Clique-width` at least $\lfloor n/2 \rfloor$. Since all vertices in even rows and all vertices in odd rows form a clique, $F'_{n,n}$ has `Minimum Clique Cover` two. Since `Minimum Clique Cover` is bounded in this graph class and `Clique-width` is unbounded, `Minimum Clique Cover` does not upper-bound `Clique-width`. □

**Lemma 3.10.** *Every graph in $\bar{P}_n$ has* ***`Clique-width`*** *at most 3 for $n \in \mathbb{N}$.*

*Proof.* We prove this by induction.
Base clause: The construction $1(v)$ is valid with cliquewidth one for $\bar{P}_1$.
The construction $1(v) \oplus 2(v)$ is valid with cliquewidth two for $\bar{P}_2$. Both of these also fulfill the requirements that we formulate in the induction hypothesis by labeling the last vertex of the path with the label 1 and the second to last vertex with the label 2.

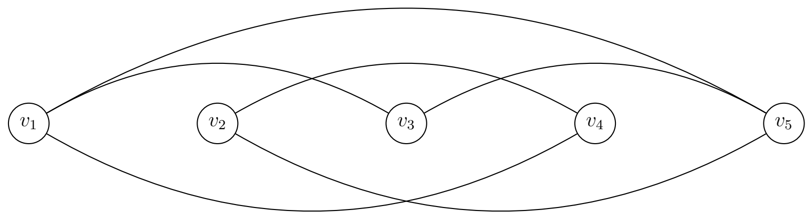



Figure 3.5: The graph $\bar{P}_5$ as an example for the constructions in Lemma 3.10 and Proposition 3.11

Induction hypothesis: For some $n \geq 2, n \in \mathbb{N}$, there exists a construction $\phi_n$ for $\bar{P}_n$ with clique-width at most three such that the all vertices on the path except for the last two have the label 3, the second to last vertex has the label 2 and the last vertex has the label 1.
Induction Step: Now we can use $\phi_n$ to derive a valid expression with cliquewidth three for $\bar{P}_{n+1}$ such that the first $n-1$ vertices on the path have the label 3, the $n^{\text{th}}$ vertex has the label 2 and the $n+1^{\text{th}}$ vertex has the label 1. In order to change the label of the $n-1^{\text{th}}$ vertex to 3 we use the relabeling operation $\rho$ and create the expression $\rho_{2,3}(\phi_n)$. Following that we change the label of the $n^{\text{th}}$ vertex to 2 by creating the expression $\rho_{1,2}(\rho_{2,3}(\phi_n))$. We then have to add the $n+1^{\text{th}}$ vertex with label 1 to the graph by extending the expression to $1(v) \oplus \rho_{1,2}(\rho_{2,3}(\phi))$. Finally, we have to connect the $n+1^{\text{th}}$ vertex to all vertices except for the $n^{\text{th}}$ vertex which we can do by adding an edge between all vertices with label 1 and 3 to the expression, since the $n+1^{\text{th}}$ vertex is the only vertex with the label 1 and the $n^{\text{th}}$ vertex is the only other vertex with a label other then 3. Thus, we have derived the expression $\eta_{1,3}(1(v) \oplus \rho_{1,2}(\rho_{2,3}(\phi)))$ which is a valid expression for $\bar{P}_{n+1}$ with clique-width three that fulfills the conditions set in the induction hypothesis. □

**Proposition 3.11.** *`Minimum Clique Cover`, `Distance to Perfect`, and `Clique-width` do not upper-bound `Chordality`.*

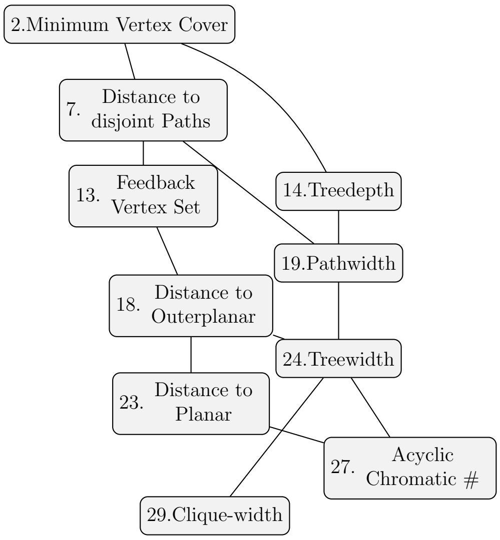
*Proof.* Consider the graph class of the complements $\bar{P}_n$ of all paths for $n \in \mathbb{N}$. The `Chordality` for $\bar{P}_n$ grows linearly in $n$ [CR89] while the `Minimum Clique Cover` of $\bar{P}_n$ is at most two, as all vertices with even and all vertices with odd indices form a clique. $P_n$ is perfect because every subgraph of $P_n$ with at least one edge has chromatic number and maximum clique two and every subgraph of $P_n$ without edges has maximum clique and chromatic number one. Since a graph is perfect, if and only if it's complement is perfect, it follows that $\bar{P}_n$ is perfect. Thus, $\bar{P}_n$ has `Distance to Perfect` zero. Lemma 3.10 shows that $\bar{P}_n$ has clique-width at most three. Since `Minimum Clique Cover`, `Distance to Perfect`, and `Clique-width` are bounded in this graph class and `Chordality` is unbounded, `Minimum Clique Cover`, `Distance to Perfect`, and `Clique-width` do not upper-bound `Chordality`. □

Lastly, `Distance to Co-Cluster` does not upper-bound any parameters that aren't below it while being below `Distance to Clique`, because it does not upper-bound `Boxicity` or `Distance to Chordal`. The fact that `Distance to Co-Cluster` does not upper-bound `Boxicity` has already been proven in Proposition 3.4.

**Proposition 3.12.** *`Distance to Co-Cluster` and `Distance to Bipartite` do not upper-bound `Distance to Chordal`, `Distance to Disconnected`, or `Domatic Number`.*

*Proof.* Consider the graph class of all graphs $K_{n,n}$ for $n \in \mathbb{N}$. Since $K_{2,2}$ and consequently every supergraph of it is not chordal, at least $n-1$ vertices have to be deleted from $K_{n,n}$ to create a chordal graph, thus `Distance to Chordal` for $K_{n,n}$ grows linearly in $n$. Similarly, every $K_{p,q}$ for $p > 0$ and $q > 0$ is connected and thus `Distance to Disconnected` grows linearly in $n$. Since any pair of one vertex from each of the partitions forms a dominating set and there is no dominating set of size one in $K_{n,n}$, $K_{n,n}$ has `Domatic Number` exactly $n$. The graph class $K_{n,n}$ is a subclass of the class of all co-cluster graphs because it's complement contains exactly all edges within each partition and is thus $2 \cdot K_n$ meaning it has `Distance to Co-Cluster` zero. Also $K_{n,n}$ contains by definition only bipartite graphs and thus has `Distance to Bipartite` zero. Since `Distance to Co-Cluster` and `Distance to Bipartite` are bounded in this graph class and `Distance to Chordal`, `Distance to Disconnected`, and `Domatic Number` are unbounded, `Distance to Co-Cluster` and `Distance to Bipartite` do not upper-bound `Distance to Chordal`, `Distance to Disconnected`, or `Domatic Number`. □

### 3.2.2 Parameters between Minimum Vertex Cover and Acyclic Chromatic Number




This section contains all parameters in the hierarchy between `Minimum Vertex Cover` and `Acyclic Chromatic Number` except for `h-index`. It also contains `Clique-width` because it is also upper-bounded by several parameters above `Acyclic Chromatic Number`. It consists of two main paths through `Treedepth` and `Distance to disjoint Paths` that are closely intertwined.
The parameters between `Distance to disjoint Paths` and `Distance to Planar` are independent of `Treedepth` because `Max Leaf Number` does not upper-bound `Girth` which is upper-bounded by `Treedepth` and because `Treedepth` does not upper-bound `Distance to Planar`.

**Proposition 3.13.** *`Bandwidth` and `Treedepth` do not upper-bound `Distance to Planar`.*

*Proof.* Consider the graph class of $n \cdot K_5$ for $n \in \mathbb{N}$. As $K_5$ has distance to planar one, the `Distance to Planar` of $n \cdot K_5$ grows linearly in $n$ while it has bandwidth four and treedepth five because $K_5$ has bandwidth four and treedepth five. Since `Bandwidth` and `Treedepth` are bounded in this graph class and `Distance to Planar` is unbounded, `Bandwidth` and `Treedepth` do not upper-bound `Distance to Planar`. □

**Proposition 3.14.** *`Max Leaf Number` does not upper-bound `Girth`.*

*Proof.* Consider the graph class of all circles $C_n$ for $n \in \mathbb{N}$. The girth of $C_n$ grows linearly in $n$ while the size of the leaf number of every spanning tree of $C_n$ is two. Since `Max Leaf Number` is bounded in this graph class and `Girth` is unbounded, `Max Leaf Number` does not upper-bound `Girth`. □

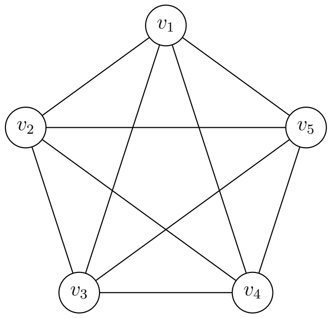



Figure 3.6: The graph $K_5$ as an example for the constructions in Proposition 3.13, 3.28, 3.33 and 3.35

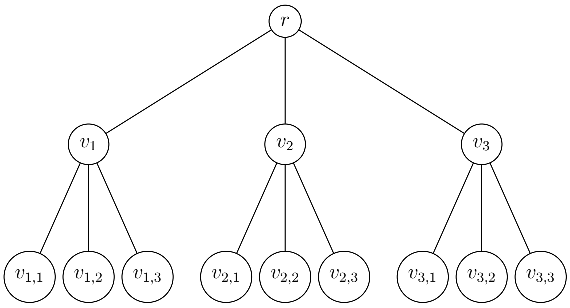



Figure 3.7: Example of a complete ternary tree: $T_{3,2}$

`Feedback Vertex Set`, `Distance to Outerplanar`, and `Distance to Planar` are independent of `Pathwidth` because `Feedback Edge Set` does not upper-bound `Pathwidth` and because `Bandwidth` does not upper-bound `Distance to Planar` as proven in Proposition 3.13.

**Lemma 3.15.** *The complete ternary tree $T_{3,n}$ of depth $n$ has* `Pathwidth` *at least $n$.*

*Proof.* We prove this by induction.
Base case: $T_{3,1}$ has pathwidth one.
Induction hypothesis: For some $n \in \mathbb{N}$, $T_{3,n}$ has pathwidth at least $n$.
Induction step: We prove this by contradiction. We assume there exists a path decomposition of $T_{3,n+1}$ with pathwidth less then $n + 1$. Without loss of generality assume that every bag contains at least two vertices. This path decomposition consists of any number $m$ of ordered bags. We define this order by assigning an index $i$ with $1 \leq i \leq m$ to each of the bags. Consider an arbitrary vertex $v_\alpha$ other than the root $r$ from the bag with index 1. By $A$ we refer to the child of $r$, in whose subtree $v_\alpha$ lies. Consider an arbitrary vertex $v_\omega$ other then $r$ from the bag with index $m$. By $C$ we refer to the child of $r$, in whose subtree $v_\omega$ lies. Since $r$ and the subtrees below $A$ and $C$ form a connected subgraph $G_{A,C}$ of $T_{3,n+1}$ and both the bag with index 1 and the bag with index $m$ contain a vertex from that subgraph, we know that every bag contains at least one vertex from that subgraph. There exists at least one child $B$ of $r$ such that $B \neq C$

and $B \neq A$. Since the subtree $G_B$ below $B$ is a subgraph of $T_{3,n+1}$, we know that any path decomposition of $T_{3,n+1}$ when restricted to the vertices of $G_B$ has to be a path decomposition of $G_B$. As $G_B$ is isomorphic to $T_{3,n}$ and $T_{3,n}$ has pathwidth at least $n$, we know that any path decomposition of $G_B$ has to contain a bag $M$ of size at least $n+1$. Furthermore, we know that every bag including $M$ has to contain at least one vertex from $G_{A,C}$. Since $G_{A,C}$ does not contain any vertices from $G_B$, we know that $M$ contains at least $n+2$ vertices meaning the path decomposition has pathwidth at least $n+1$. We have reached a contradiction. Thus, $T_{3,n}$ has `Pathwidth` at least $n+1$ which shows that `Pathwidth` is unbounded in $T_{3,n}$. □

**Proposition 3.16.** *`Feedback Edge Set` does not upper-bound `Pathwidth`.*

*Proof.* Consider the graph class of all complete ternary trees $T_{3,n}$ of depth $n$ for $n \in \mathbb{N}$. The graph class $T_{3,n}$ has unbounded pathwidth as proven in Lemma 3.15 and, since $T_{3,n}$ does not contain a cycle, it has `Feedback Edge Set` zero. Since `Feedback Edge Set` is bounded in this graph class and `Pathwidth` is unbounded, `Feedback Edge Set` does not upper-bound `Pathwidth`. □

the fact `Bandwidth` does not upper-bound `Distance to Planar` can also be used together with the fact that `Distance to Planar` does not upper-bound `Clique-width` to prove that `Distance to Planar` is independent of `Treewidth`.

**Proposition 3.17.** *`Genus` and `Distance to Planar` do not upper-bound `Clique-width`.*

*Proof.* Consider the graph class of all $G_{n,n}$ for $n \in \mathbb{N}$. [GR99] shows that $G_{n,n}$ has clique-width exactly $n+1$ but since $G_{n,n}$ is planar, it has genus and distance to planar zero. Since `Genus` and `Distance to Planar` are bounded in this graph class and `Clique-width` is unbounded, `Genus` and `Distance to Planar` do not upper-bound `Clique-width`. □

`Clique-width` and `Acyclic Chromatic Number` are independent because `Genus` does not upper-bound `Clique-width` as proven in Proposition 3.17 and because `Distance to Clique` does not upper-bound `Domatic Number` as proven in Proposition 3.7.
We have now proven all independencies within this section and will now proceed by showing that the parameters in this section do not upper-bound any parameters that are not below them in the hierarchy.
`Minimum Vertex Cover` does not upper-bound any parameters that aren't below it in the hierarchy because it does not upper-bound `Minimum Dominating Set`, `Genus`, `Maximum Degree`, or `Bisection Width`. The fact that `Minimum Vertex Cover` does not upper-bound `Minimum Dominating Set` has already been shown in Proposition 3.5.

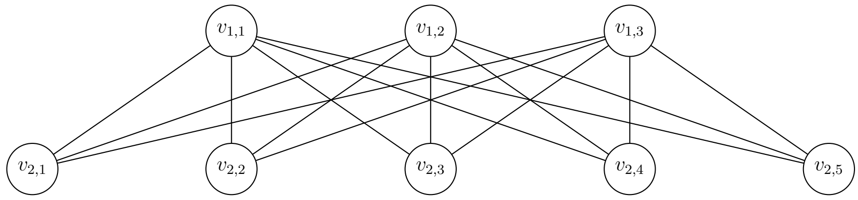



Figure 3.8: The graph $K_{3,5}$ as an example for the construction in Proposition 3.18

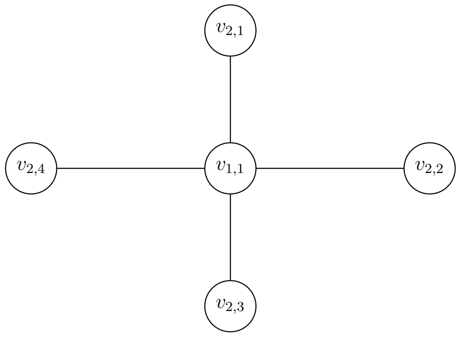



Figure 3.9: The graph $K_{1,4}$ as an example for the constructions in Proposition 3.19, 3.20 and 3.22

**Proposition 3.18.** *`Minimum Vertex Cover` does not upper-bound `Genus`.*

*Proof.* Consider the graph class of the complete bipartite graphs $K_{3,n}$ for $n \in \mathbb{N}$. The `Genus` of $K_{3,n}$ grows linearly in $n$[Bou78] but the size of the `Minimum Vertex Cover` is at most three in $K_{3,n}$ as the smaller partition has size at most three and in a bipartite graph either partition is a vertex cover. Since `Minimum Vertex Cover` is bounded in this graph class and `Genus` is unbounded, `Minimum Vertex Cover` does not upper-bound `Genus`. □

**Proposition 3.19.** *`Minimum Vertex Cover` does not upper-bound `Maximum Degree`.*

*Proof.* Consider the graph class of the complete bipartite graphs $K_{1,n}$ for $n \in \mathbb{N}$. The `Maximum Degree` for $K_{1,n}$ grows linearly in $n$ while the size of the `Minimum Vertex Cover` is at most one. Since `Minimum Vertex Cover` is bounded in this graph class and `Maximum Degree` is unbounded, `Minimum Vertex Cover` does not upper-bound `Maximum Degree`. □

**Proposition 3.20.** *`Minimum Vertex Cover` does not upper-bound `Bisection Width`.*

*Proof.* Consider the graph class of the complete bipartite graphs $K_{1,2n}$ for $n \in \mathbb{N}$. There exists a universal vertex in $K_{1,2n}$. Because this vertex has an edge to each vertex in the other bisection, the `Bisection Width` for $K_{1,2n}$ grows linearly in $n$ while the size of the `Minimum Vertex Cover` is at most one. Since `Minimum Vertex Cover` is bounded in

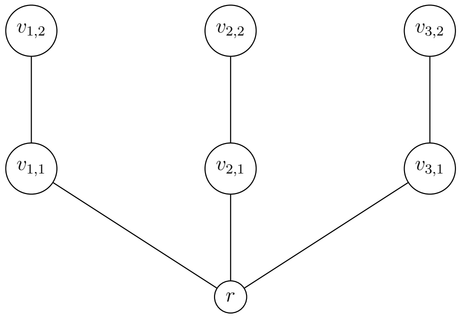



Figure 3.10: The graph $S_{2,3}$ for the construction in Proposition 3.21

this graph class and `Bisection Width` is unbounded, `Minimum Vertex Cover` does not upper-bound `Bisection Width`. □

`Distance to disjoint Paths` and all parameters below it do not upper-bound `Girth` because `Max Leaf Number` does not upper-bound `Girth` as proven in Proposition 3.14. `Feedback Vertex Set` does not upper-bound `Distance to Interval` because `Feedback Edge Set` does not upper-bound `Distance to Interval`.

**Proposition 3.21.** *`Feedback Edge Set` does not upper-bound `Distance to Interval`.*

*Proof.* Consider the graph class of $n \cdot S_{2,3}$ for $n \in \mathbb{N}$, where $S_{2,3}$ is the graph depicted in Figure 3.10. The graph $S_{2,3}$ is acyclic graph and has distance to interval one. As a result the `Distance to Interval` of $n \cdot S_{2,3}$ grows linearly in $n$ while $n \cdot S_{2,3}$ has `Feedback Edge Set` zero because it does not contain any cycles. Since `Feedback Edge Set` is bounded in this graph class and `Distance to Interval` is unbounded, `Feedback Edge Set` does not upper-bound `Distance to Interval`. □

Also `Feedback Vertex Set`, `Treedepth` and the parameters below them do not upper-bound `h-index` because `Feedback Edge Set` and `Treedepth` do not upper-bound `h-index`.

**Proposition 3.22.** *`Treedepth` and `Feedback Edge Set` do not upper-bound `h-Index`.*

*Proof.* Consider the graph class of $n \cdot K_{1,n}$ for $n \in \mathbb{N}$. The `h-Index` for $n \cdot K_{1,n}$ grows linearly in $n$ while $n \cdot K_{1,n}$ has feedback edge set zero because it does not contain any cycles and $K_{1,n}$ has `Treedepth` two. Since `Feedback Edge Set` and `Treedepth` are bounded in this graph class and `h-Index` is unbounded, `Treedepth` and `Feedback Edge Set` do not upper-bound `h-Index`. □

Finally, we show directly that `Distance to Outerplanar` and `Treedepth` do not upper-bound `Distance to Perfect`.

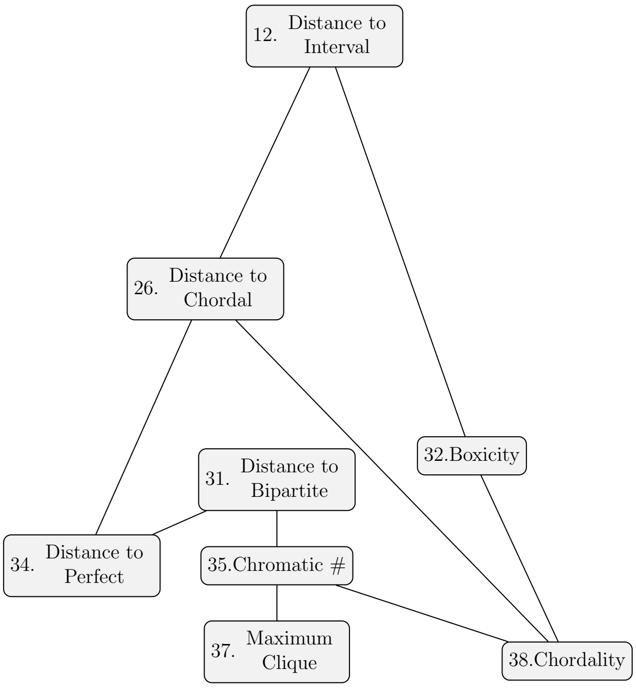
**Proposition 3.23.** *`Distance to Outerplanar` does not upper-bound `Distance to Perfect`.*

*Proof.* Consider the graph class $n \cdot P_4$ for $n \in \mathbb{N}$. The graph $P_4$ has distance to perfect one and distance to outerplanar zero and thus $n \cdot P_4$ has `Distance to Perfect` $n$ and `Distance to Outerplanar` zero. Since `Distance to Outerplanar` is bounded in this graph class and `Distance to Perfect` is unbounded, `Distance to Outerplanar` does not upper-bound `Distance to Perfect`. □

**Proposition 3.24.** *`Bandwidth`, `Genus`, and `Treedepth` do not upper-bound `Distance to Perfect`.*

*Proof.* Consider the graph class of $n \cdot C_5$ for $n \in \mathbb{N}$. As $C_5$ has distance to perfect one, the `Distance to Perfect` of $n \cdot C_5$ grows linearly in $n$ while it has `Bandwidth` two and `Treedepth` four because $C_5$ has bandwidth two and treedepth four. Because $C_5$ is planar each graph $n \cdot C_5$ is as well and thus they have genus zero. Since `Bandwidth`, `Genus`, and `Treedepth` are bounded in this graph class and `Distance to Perfect` is unbounded, `Bandwidth`, `Genus`, and `Treedepth` do not upper-bound `Distance to Perfect`. □

### 3.2.3 Parameters below Distance to Interval




This section contains all parameters in the hierarchy below `Distance to Interval`. It also contains the parameters between `Distance to Bipartite` and `Maximum Clique` because they are closely connected to the parameters below `Distance to Interval`. It splits up in three different paths ending in three of the six parameters in the graph parameter hierarchy who do not upper-bound any other parameters.
No parameter below `Distance to Interval` upper-bounds any parameter between `Distance to Bipartite` and `Maximum Clique` because `Distance to Clique` does not upper-bound `Maximum Clique` as shown in Proposition 3.7. It follows that the parameters between `Distance to Bipartite` and `Maximum Clique` are independent of `Distance to Interval` and `Distance to Chordal` because `Distance to Bipartite` does not upper-bound `Distance to Chordal` as shown in Proposition 3.12. Since [CFM11] has proven that `Distance to Bipartite` does not upper-bound `Boxicity`, it also follows that the parameters between `Distance to Bipartite` and `Maximum Clique` are independent of `Boxicity`.
`Distance to Chordal` and `Distance to Perfect` are independent of `Boxicity` because `Distance to Chordal` does not upper-bound `Boxicity` and because `Distance to Outerplanar` does not upper-bound `Distance to Perfect` as shown in Proposition 3.23.

**Proposition 3.25.** *`Distance to Chordal` does not upper-bound `Boxicity`.*

*Proof.* Consider the graph class of all split graphs. [CR83] shows that the class of split graphs has unbounded `Boxicity`. [FH77] shows that all split graphs are chordal and thus have Distance to Chordal zero. Since Distance to Chordal is bounded in this graph class and Boxicity is unbounded, Distance to Chordal does not upper-bound Boxicity. □

The fact that Distance to Outerplanar does not upper-bound Distance to Perfect also helps us to prove even more independencies. It proves in conjunction with the fact that Distance to Perfect does not upper-bound Chordality, as shown in Proposition 3.11, that Distance to Perfect is also independent of Chordality.
Additionally, it proves that Chromatic Number and Maximum Clique are independent of Distance to Perfect, since we already know that Chromatic Number and Maximum Clique are not upper-bounded by any parameter below Distance to Clique.
The last unproven independence within this section is the independence between Chordality and Maximum Clique. It is known that Chordality does not upper-bound Maximum Clique as it is below Distance to Clique, but it is unknown whether Maximum Clique upper-bounds Chordality. Thus, we have proven all independencies within this subgraph.
We will now show that the parameters in this subgraph do not upper-bound any vertices that are not below them. The only parameter which is below all parameters that are above Distance to Interval without being below Distance to Interval is Clique-width. [GR99] has shown that Distance to Interval does not upper-bound Clique-width by constructing a class of interval graphs with unbounded Clique-width. Distance to Bipartite does not upper-bound any parameters that are not below it because it does not upper-bound Domatic Number as shown in Proposition 3.12, Distance to Disconnected as shown in Proposition 3.12, or Clique-width.

**Proposition 3.26.** *Maximum Degree and Distance to Bipartite do not upper-bound Clique-width or Bisection Width.*

*Proof.* Consider the graph class $G_{n,n}$ of all $n \times n$-Grids for $n \in \mathbb{N}$. [GR99] shows that $G_{n,n}$ has Clique-width exactly $n+1$ and [Efe08] shows that $G_{n,n}$ has unbounded Bisection Width. Moreover, each $G_{n,n}$ has maximum degree at most four and distance to bipartite zero because $\{v_{n,m} | (n+m) \bmod 2 = 0\}$ and $\{v_{n,m} | (n+m) \bmod 2 = 1\}$ form a valid bipartition. Since Maximum Degree and Distance to Bipartite are bounded in this graph class and Clique-width and Bisection Width are unbounded, Maximum Degree and Distance to Bipartite do not upper-bound Clique-width or Bisection Width. □

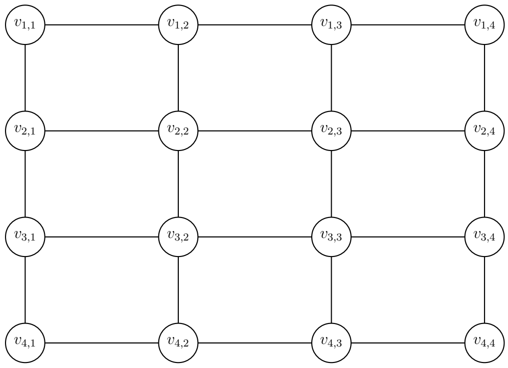
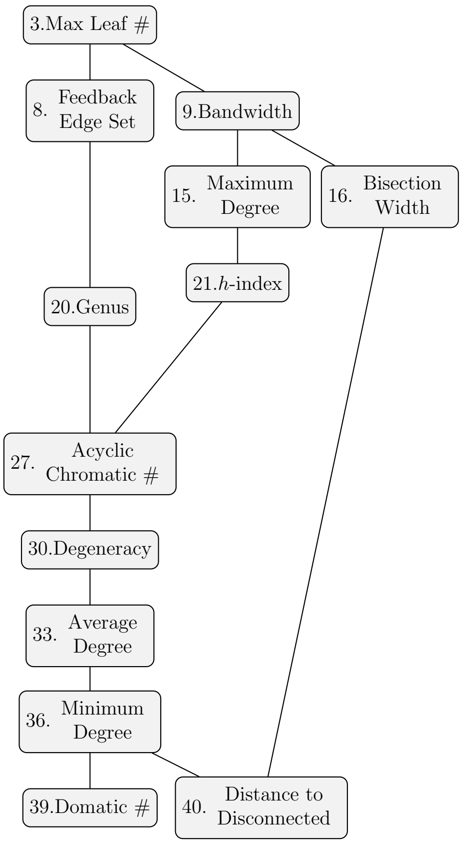



Figure 3.11: The graph $G_{4,4}$ as an example for the construction in Proposition 3.26, 3.32 and 3.17

### 3.2.4 Parameters between Max Leaf Number and Distance to Disconnected




This section contains the remaining parameters of the graph parameter hierarchy. It does not contain all parameters between `Max Leaf Number` and `Distance to Disconnected`, since many of them are also between `Minimum Vertex Cover` and `Acyclic Chromatic Number`, and it also contains `Domatic Number` since `Domatic Number` has many shared

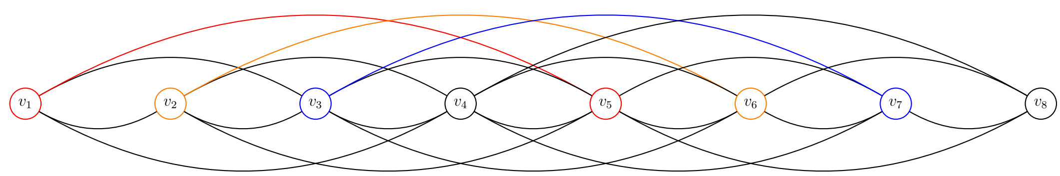



Figure 3.12: The graph $L_2$ as an example for the construction in Proposition 3.27, same colored vertices are contracted to create a supergraph of $K_{3,2}$

upper bounds with `Distance to Disconnected`.
`Feedback Edge Set` and `Genus` are independent of `Bandwidth`, `Maximum Degree`, and `h-index` because `Feedback Edge Set` does not upper-bound `h-index` as shown in Proposition 3.22 and `Bandwidth` does not upper-bound `Genus`.

**Proposition 3.27.** *`Bandwidth` does not upper-bound `Genus`.*

*Proof.* Consider the graph class of all graphs $L_n = (V, E)$ with $V = \{v_i | 1 \leq i \leq 4n\}$ and $E = \{\{v_i, v_j\} | 0 < |i - j| \leq 4\}$ for $n \in \mathbb{N}$. The Figure 3.12 shows an example of this construction. By contracting the edges $\{\{v_i, v_{i+4}\} | i \bmod 4 \neq 0\}$ we create three universal vertices in a graph with $n + 3$ vertices, which shows that $K_{3,n}$ is a minor of $L_n$. Since $K_{3,n}$ is a minor of $L_n$, its Genus grows at least linearly in $n$ [Bou78] while it has bandwidth four. Since `Bandwidth` is bounded in this graph class and `Genus` is unbounded, `Bandwidth` does not upper-bound `Genus`. □

`Bisection Width` is independent of all parameters between `Feedback Edge Set` and `Domatic Number` as well as all parameters between `Maximum Degree` and `Domatic Number`. This follows from the fact that `Bisection Width` does not upper-bound `Domatic Number` and the fact that neither `Feedback Edge Set` nor `Maximum Degree` upper-bounds `Bisection Width`. The fact that `Maximum Degree` does not upper-bound `Bisection Width` is shown in Proposition 3.26.

**Proposition 3.28.** *`Bisection Width` does not upper-bound `Domatic Number`.*

*Proof.* Consider the graph class $2 \cdot K_n$ for $n \in \mathbb{N}$. The `Domatic Number` of $2 \cdot K_n$ is $n$, since any pair of one vertex each from the two cliques form a dominating set and there is no dominating set of size one. Each graph $2 \cdot K_n$ contains two unconnected subgraphs of equal size meaning it has `Bisection Width` zero. Since `Bisection Width` is bounded in this graph class and `Domatic Number` is unbounded, `Bisection Width` does not upper-bound `Domatic Number`. □

**Proposition 3.29.** *`Feedback Edge Set` does not upper-bound `Bisection Width`.*

*Proof.* Consider the graph class of $K_{1,n}$ for $n \in \mathbb{N}$. As $K_{1,n}$ contains a universal vertex, any bisection would have to remove at least $n/2$ of its edges. Consequently, the `Bisection Width` for $K_{1,n}$ grows linearly in $n$ while $K_{1,n}$ has `Feedback Edge Set` zero because it does not contain any cycles. Since `Feedback Edge Set` is bounded in this graph class and `Bisection Width` is unbounded, `Feedback Edge Set` does not upper-bound `Bisection Width`. □

The fact that `Bisection Width` does not upper-bound `Domatic Number` also combines with the fact that `Domatic Number` does not upper-bound `Distance to Disconnected` to show the independence of `Distance to Disconnected` and `Domatic Number`.

**Proposition 3.30.** *`Domatic Number` does not upper-bound `Distance to Disconnected`.*

*Proof.* Consider the graph class of all $F_n = (V, E)$ for $n \geq 3$ and $n \in \mathbb{N}$, where $V = V_n \cap V'_n$ and $V_n$ contains $3n + 1$ vertices $v_i$ for $1 \leq i \leq 3n + 1$ while $V'_n$ contains a vertex $v_S$ for each subset $S \subset V_n$ with $|S| = n$. Each $v_S$ is connected to all vertices $v \in S$. As an example of this construction see Figure 3.13. [Zel83] shows that the `Domatic Number` of this graph class is at most two. Furthermore, each pair of vertices $(v_i, v_j)$ has $\binom{3n-1}{n-2} > n$ common neighbors, since that is the number of subsets of $V_n$ of size $n$ that contain both $v_i$ and $v_j$. Additionally, every vertex $v$ representing a subset has $d_{F_n}(v) = n$ with each adjacent vertex $w \in N_{F_n}(v)$ being in $V_n$. Thus, $F_n$ has distance to disconnected n, since $F_n - S$ with $S \subset V$ and $|S| < n$ is a connected graph for every $S$ because each vertex $v$ representing a subset has $d_{F_n}(v) = n$ and thus $d_{F_n - S}(v) \geq 1$ with each $w \in N_{F_n - S}(v)$ being in $V_n$ and each $v_i \in V$ is connected to each $v_j \in V_n$, since they have more than $n$ common neighbors. Since `Domatic Number` is bounded in this graph class and `Distance to Disconnected` is unbounded, `Domatic Number` does not upper-bound `Distance to Disconnected`. □

We have now proven all independencies within this subgraph and will prove next that no parameter in this subgraph upper-bounds a parameter that isn't below it in the hierarchy.
`Max Leaf Number` does not upper-bound any parameters that aren't below it because `Max Leaf Number` does not upper-bound `Girth` as shown in Proposition 3.14. Proposition 3.21 shows that `Feedback Edge Set` does not upper-bound `Distance to Interval`. Furthermore, `Bandwidth` does not upper-bound `Distance to Planar` or `Distance to Perfect`. This is shown in Proposition 3.13 and Proposition 3.24 respectively.
`Maximum Degree` does not upper-bound `Clique-width`, or any parameter above it as shown in Proposition 3.26. `Bisection Width` does not upper-bound any parameter besides `Distance to Disconnected` because it does not upper-bound `Chordality`, `Maximum Clique`, or `Clique-width`. We prove the absence of all of these bounds by considering two disconnected subgraphs of the same size with high `Chordality`, `Maximum Clique`,

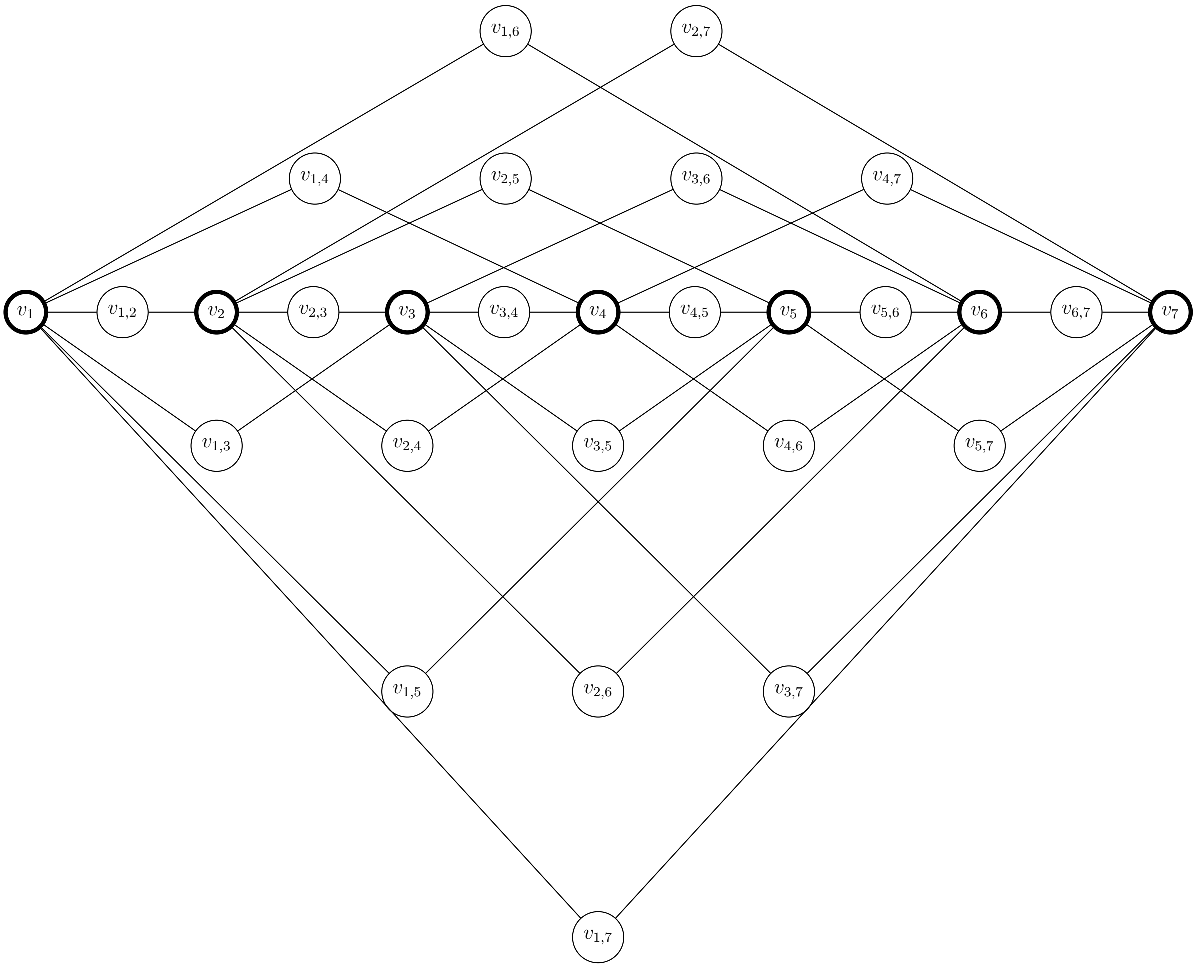



Figure 3.13: The graph $F_2$ as an example for the construction in Proposition 3.30

or `Clique-width` respectively.

**Proposition 3.31.** *`Bisection Width` does not upper-bound `Chordality`.*

*Proof.* Consider the graph class $2 \cdot N_n$ with $N_n$ being a clique of size n with a perfect matching removed for $n \in \mathbb{N}$. The Lemma 4.29 in [Sa19] shows that $N_n$ has unbounded `Chordality` meaning $2 \cdot N_n$ also has unbounded `Chordality`. $2 \cdot N_n$ contains two unconnected subgraphs of equal size meaning it has `Bisection Width` zero. Since `Bisection Width` is bounded in this graph class and `Chordality` is unbounded, `Bisection Width` does not upper-bound `Chordality`. □

**Proposition 3.32.** *`Bisection Width` does not upper-bound `Clique-width`.*

*Proof.* Consider the graph class $2 \cdot G_{n,n}$ for $n \in \mathbb{N}$. (on the clique-width of some perfect graph classes) shows that $G_{n,n}$ has `Clique-width` exactly $n+1$ meaning $2 \cdot G_{n,n}$ also has `Clique-width` exactly $n+1$. Furthermore, the graph class $2 \cdot G_{n,n}$ contains two unconnected subgraphs of equal size meaning it has `Bisection Width` zero. Since `Bisection Width` is bounded in this graph class and `Clique-width` is unbounded, `Bisection Width` does not upper-bound `Clique-width`. □

**Proposition 3.33.** *`Bisection Width` does not upper-bound `Maximum Clique`.*

*Proof.* Consider the graph class $2 \cdot K_n$ for $n \in \mathbb{N}$. The graph class $2 \cdot K_n$ has `Maximum Clique` $n$ and since $2 \cdot K_n$ contains two unconnected subgraphs of equal size, it has `Bisection Width` zero. Since `Bisection Width` is bounded in this graph class and `Maximum Clique` is unbounded, `Bisection Width` does not upper-bound `Maximum Clique`. □

`Genus` does not upper-bound anything that is not also below `Acyclic Chromatic Number` because it does not upper-bound `Clique-width` as shown in Proposition 3.17, `Distance to Perfect` as shown in Proposition 3.24, or `Distance to Planar`.

**Proposition 3.34.** *`Genus` does not upper-bound `Distance to Planar`.*

*Proof.* We assume towards a contradiction that there exists an upper bound such that a graph $G$ of genus one can have distance to planar at most $m$. Let $G_{max} = (V, E)$ be any graph with genus one and distance to planar $m$. Without loss of generality we assume that $G_{max}$ does not contain vertices of degree one or less, since these vertices do not affect the distance to planar or the genus of a graph. Note that $m > 0$, since $G_{max}$ is not planar. Thus, there exists a minimal set $S \subset V$ with $|S| = m$ such that $G - S$ is planar. Since $G_{max}$ has genus one, there exists an embedding $E_{max}$ of $G_{max}$ on a surface with genus one. We construct a graph following these steps:

**Data:** graph $G$ without vertices of degree less then three with an embedding $\Omega$

**Result:** graph G' with an embedding $\Omega'$ of equal genus but higher distance to planar

$V' = V$;
$E' = E$;
$\Omega' = \Omega$;
\\ *expand all* $v \in V$:
**forall** $v \in V$ **do**
  \\ *Let* $R_v$ *be the order of the neighbors of* $v$ *in* $\Omega'$*, such that for each consecutive pair of vertices* $a$ *and* $b$ *in* $R_v$ *it holds that* $\{v, a\}$ *and* $\{v, b\}$ *aren't separated by another edge that is incident to* $v$ *in* $\Omega$.
  **forall** $\{v, u\} \in E'$ **do**
    $V' = V' \cup v_{v,u}$;
    $E' = E' \setminus \{\{v, u\}\} \cup \{\{v, v_{v,u}\}, \{u, v_{v,u}\}\}$;
    add $v_{v,u}$, $\{v, v_{v,u}\}$ and $\{u, v_{v,u}\}$ to $\Omega'$ by replacing $\{v, u\}$ with them in $\Omega'$;
  **end**
  **for** $a$ *is the last vertex in* $R$ *and* $b$ *is the first vertex in* $R_v$ **do**
    $E' = E' \cup \{\{a, v_{v,b}\}\}$;
  **end**
  **forall** $a, b \in V'$ *with* $a$ *is directly ahead of* $b$ *in* $R_v$ **do**
    $E' = E' \cup \{\{a, v_{v,b}\}, \{v_{v,a}, v_{v,b}\}\}$;
    \\ *these two edges can also be embedded in* $\Omega'$*, since* $a$ *and* $b$ *are by definition of* $R_v$ *adjacent to the same face in* $\Omega'$ *and thus the same is true for* $a$, $v_{v,a}$, $v$ *and* $v_{v,b}$ *after subdividing the edges such that* $\{a, v_{v,b}\}$ *and* $\{v_{v,a}, v_{v,b}\}$ *are non-intersecting chords of that face of* $\Omega'$.
  **end**
**end**
return $G' = (V', E')$;

**Algorithm 2:** Creating a graph $G'$

An example of how this algorithm works, can be seen in Figure 3.14. During the algorithm $\Omega$ gets modified to $\Omega'$, such that $\Omega'$ is an embedding of $G'$ on the same surface as the embedding $\Omega$ meaning $G'$ can still be embedded on any surface that $G$ could be embedded on. The comments in the algorithm explain why all new edges are between vertices that are adjacent to the same face in $\Omega'$ and thus why they can be embedded without crossings. It follows that $G'$ has the same genus as $G$. To prove that $G'$ has higher distance to planar then $G$, we show that $G' - h$ has $G$ as a minor independent of the choice of $h \in V'$. We can construct $G$ as a minor of $G' - h$ by following these steps: We denote the Set of the vertices who were created while expanding $x$ and the vertex $x$ as $W_x$. It holds for each vertex $v$ that the induced subgraph $G'_v$ of the vertices $W_v$ in $G'$ is 2-vertex-connected, since $v$ is adjacent to every other vertex in $W_v$ and all other vertices in $W_v$ are connected in a path with the edges of the form $\{v_{v,a}, v_{v,b}\}$. Since $G'_v$ is 2-vertex-connected, we can contract it to a single vertex $v'$ even if $h \in W_v$. For every

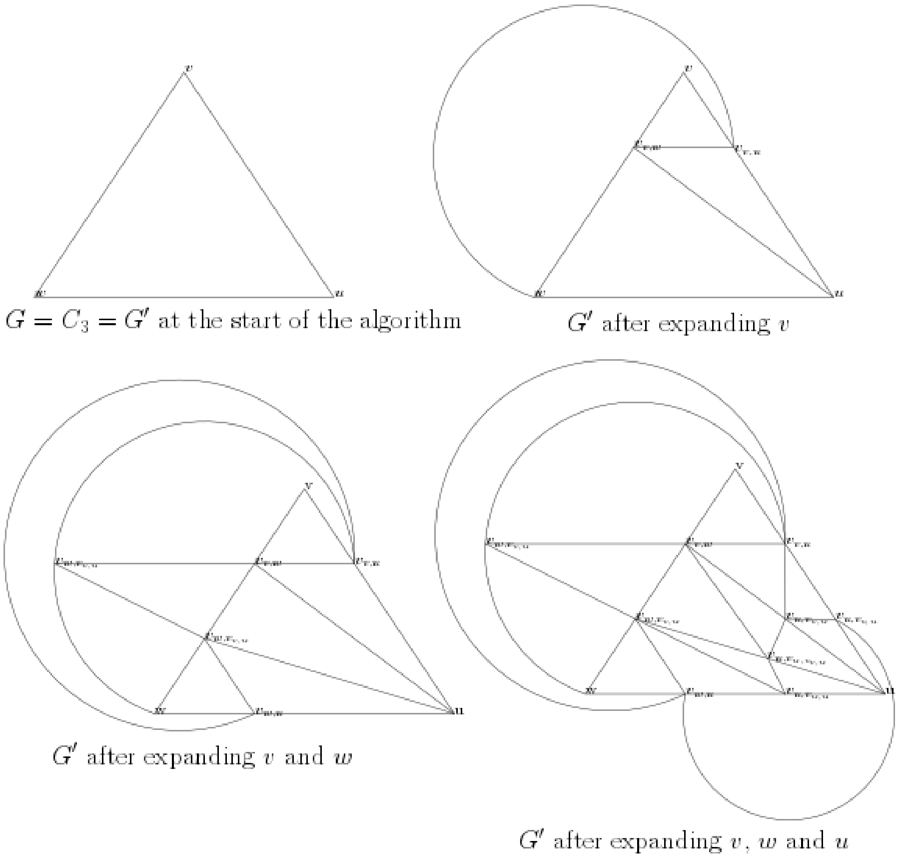



Figure 3.14: An example for the algorithm from Proposition 3.34 running on a $C_3$

vertex $w$, that was adjacent to $v$ before it was expanded, after the expansion there are two vertices in $W_v$ such that they have an edge to $w$. Thus, even if $h \in W_v$, $w$ will be adjacent to $v'$ after it was created by contracting the edges between vertices in $W_v$. It follows that the graph that gets generated from $G' - h$ by contracting specific edges and deleting specific vertices is isomorphic to $G$. Since $G'_{max} - h$ has $G_{max}$ as a minor for every $h \in V$ and $G_{max}$ has distance to planar $m$, it follows that $G'_{max} - h$ has distance to planar at least $m$. Because $h$ was chosen arbitrarily, this means that $G_{max}$ has distance to planar at least $m + 1$. We have reached a contradiction. Since there does not exist an upper bound on `Distance to Planar` for graphs with `Genus` one, `Genus` does not upper-bound `Distance to Planar`. □

To show that `Degeneracy` does not upper-bound any parameters that aren't below it, we would have to prove that `Degeneracy` does not upper-bound `Clique-width` but no such proof is known.
Lastly, `Average Degree` does not upper-bound any additional parameters since it does not upper-bound `Maximum Clique` or `Chordality`.

**Proposition 3.35.** *`Average Degree` does not upper-bound `Maximum Clique`.*

*Proof.* Consider the graph class of all $K_n + I_{(n^2-3n)/2}$ for $n \geq 3$ and $n \in \mathbb{N}$. Since the graph $K_n + I_{(n^2-3n)/2}$ contains $K_n$, it has `Maximum Clique` $n$. Furthermore, $K_n$ contains $(n^2 - n)/2$ edges and $I_{(n^2-3n)/2}$ contains zero edges. Thus, $K_n + I_{(n^2-3n)/2}$ has $(n^2 - n)/2$ edges and as $K_n + I_{(n^2-3n)/2}$ contains exactly $(n^2 - n)/2$ vertices, it has `Average Degree` one. Since `Average Degree` is bounded in this graph class and `Maximum Clique` is unbounded, `Average Degree` does not upper-bound `Maximum Clique`. □

**Proposition 3.36.** *`Average Degree` does not upper-bound `Chordality`.*

*Proof.* Consider the graph class of all $\bar{P}_n + I_{(n^2-5n+2)/2}$ for $n \geq 5$ and $n \in \mathbb{N}$. Since the graph $\bar{P}_n + I_{(n^2-5n+2)/2}$ contains $\bar{P}_n$, its `Chordality` grows linearly in $n$[CR89]. Furthermore the two endpoints of $\bar{P}_n$ have degree $n-2$ and all other vertices in $\bar{P}_n$ have degree $n-3$ meaning that $\bar{P}_n$ contains a total of $(n^2-3n+2)/2$ edges and $I_{(n^2-5n+2)/2}$ contains no edges. Thus, $\bar{P}_n + I_{(n^2-5n+2)/2}$ has $(n^2 - 3n + 2)/2$ edges and as $\bar{P}_n + I_{(n^2-5n+2)/2}$ contains exactly $(n^2 - 3n + 2)/2$ vertices, it has `Average Degree` one. Since `Average Degree` is bounded in this graph class and `Chordality` is unbounded, `Average Degree` does not upper-bound `Chordality`. □

# 4 Conclusion

This thesis expands the graph parameter hierarchy by proving some bounds and showing for nearly all combinations of parameters for which no bounds have been discovered, that they are indeed independent. The only connections for which this thesis is unable to provide a result are whether `Degeneracy` upper-bounds `Boxicity` and whether `Maximum Clique` upper-bounds `Chordality`.

There are several ways to expand the work on the graph parameter hierarchy. The most direct way to improve these results would be by finalizing the work on this hierarchy and proving whether `Degeneracy` upper-bounds `Boxicity` as well as whether `Maximum Clique` upper-bounds `Chordality`. With these two bounds the part of the graph parameter hierarchy on which this thesis is focused would be finished. Another way to expand on the results of this thesis would be by proving the optimality of all bounds and directly illustrating them in Figure 1.1 as there are many kinds of bounds from linear to exponential in this hierarchy that are not distinguished in the illustration of it. Furthermore, these results could be enhanced by expanding the scope of this hierarchy and including more graph parameters. As an example, it would be very helpful in the future to include parameters on which research is being done to keep this hierarchy up to date with state-of-the art graph theory. Finally, this hierarchy can be adapted and expanded for specific graph properties to explore bounds with auxiliary constraints. This could theoretically be done for any number of constraints but it would be especially helpful for properties that are common in practice and on which there already exists a lot of research to give an overview of the way common graph parameters are impacted by those constraints. Good examples of such properties could be connectedness and planarity of graphs.

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Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und eigenhändig sowie ohne unerlaubte fremde Hilfe und ausschließlich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe.

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