Technische Universität Berlin
Electrical Engineering and Computer Science
Institute of Software Engineering and Theoretical Computer Science
Algorithmics and Computational Complexity (AKT)

On Equilibria in Schelling Games: Robustness and Multimodality

Luca Kreisel

Thesis submitted in fulfillment of the requirements for the degree
“Bachelor of Science” (B. Sc.) in the field of Computer Science

June 2021

Supervisor and first reviewer: Prof. Dr. Rolf Niedermeier
Second reviewer: Prof. Dr. Markus Brill
Co-Supervisors: Niclas Böhmer, Dr. Vincent Froese

Abstract

Schelling games are a game-theoretic formulation of Schelling's model of segregation, which aims at explaining some of the dynamics of individual behavior leading to residential segregation. In the simplest formulation of Schelling games, agents of two types are to be positioned each on its own vertex of a given graph, called topology. Each agent aims at maximizing the fraction of agents of its type in its occupied neighborhood. Agents swap positions or jump to free vertices if doing so is profitable. We say that an assignment is an equilibrium if no two agents want to swap positions (swap-equilibrium) or no agent wants to jump to a free vertex (jump-equilibrium).
The contribution of this thesis is threefold. First, we show that deciding the existence of equilibria in our model is NP-hard, thereby extending results by Agarwal et al. [Aga+20] and Elkind et al. [Elk+19]. Second, we study the robustness of equilibria with regard to changes to the topology and define a measure for the robustness of an equilibrium as the minimum number of edges that need to be deleted to make an equilibrium unstable. We study the existence of equilibria with a certain robustness and provide tight lower and upper bounds on the robustness of equilibria for topologies from various graph classes. We find that the robustness of equilibria heavily depends on the underlying topology. Third, we study the existence of equilibria in so-called multimodal Schelling games, that are, Schelling games on multilayer graphs. A multilayer graph is a graph with multiple edge sets on a fixed set of vertices. An assignment is an equilibrium in a multimodal game if it is an equilibrium on every one of the layers. We find that swap-equilibria may fail to exist even on very simple multilayer graphs.

Zusammenfassung

Schelling Games sind ein spieltheoretisches Modell von Schellings Segregationsmodell, welches versucht, das Entstehen von Segregation durch die Modellierung von individuellem Verhalten zu erklären. In der einfachsten Form von Schelling Games werden Agenten von zwei Typen jeweils auf einem eigenen Knoten eines gegebenen Graphen, der Topologie, positioniert. Jeder Agent möchte den Anteil der Agenten seines Typs unter den benachbarten Agenten maximieren. Agenten tauschen ihre Positionen oder springen zu freien Knoten, wenn dies für sie profitabel ist. Eine Zuordnung der Agenten zu den Knoten ist ein Equilibrium, wenn keine zwei Agenten Positionen tauschen wollen (Swap-Equilibrium) oder kein Agent zu einem freien Knoten springen will (Jump-Equilibrium).
Der Beitrag dieser Arbeit ist dreiteilig. Erstens beweisen wir, dass es NP-schwer ist, die Existenz eines Jump- oder Swap-Equilibriums in unserem Modell zu entscheiden und erweitern damit Ergebnisse von Agarwal u. a. [Aga+20] und Elkind u. a. [Elk+19]. Zweitens untersuchen wir die Robustheit von Equilibria in Bezug auf Änderungen in der Topologie und definieren ein Maß für die Robustheit eines Equilibriums als die minimale Anzahl von Kanten, die gelöscht werden müssen, um ein Equilibrium instabil zu machen.

Wir untersuchen die Existenz von Equilibria mit einer bestimmten Robustheit und zeigen untere und obere Schranken für die Robustheit von Equilibria auf Topologien aus verschiedenen Graphklassen. Wir stellen fest, dass die Robustheit von Equilibria stark von der zugrundeliegenden Topologie abhängt. Drittens untersuchen wir die Existenz von Equilibria in sogenannten multimodalen Schelling Games, das heißt Schelling Games auf Multilayer-Graphen. Ein Multilayer-Graph ist ein Graph mit einer festen Knotenmenge und mehreren Kantenmengen. Eine Zuordnung ist ein Equilibrium in einem multimodalen Spiel, wenn sie ein Equilibrium auf jedem der Layer ist. Wir geben Beispiele für einfache Instanzen von Multilayer-Graphen, die kein Swap-Equilibrium besitzen.

Contents

Chapter 1

Introduction

Residential segregation, that is, the occurrence of large separated and homogeneous areas inhabited by residents from the same social group has been of major interest in sociology (see, e.g., [Cha03; MD88]). A real-world example is depicted in Figure 1.1. Segregation has been observed to be problematic, as it can, for example, lead to inequal access to health-care resources [WC01; WHW12]. While the reasons for the emergence of segregation patterns are likely complex and multi-causal in reality, Thomas Schelling proposed a deliberately simplistic random process [Sch69; Sch71] that models individual behavior and thereby aims at explaining some of the dynamics leading to residential segregation.

Schelling's Model of Segregation. In Schelling's model, we consider agents of two types that are to be positioned on the vertices of a given graph (called *topology*). Each vertex v can be occupied by at most one agent and is called *unoccupied* if no agent is positioned on v. Initially, each agent is placed uniformly at random on an individual vertex of the topology (in the original model, usually a grid graph). We say an agent is *happy* if at least a τ-fraction of its occupied neighborhood is occupied by agents of its type for some given tolerance parameter $\tau \in (0, 1]$. Happy agents do not change location, while, depending on the model, unhappy agents either randomly swap vertices with other unhappy agents or randomly jump to empty vertices. A surprising observation made by Schelling [Sch69; Sch71] is that even for relatively tolerant agents with $\tau \sim \frac{1}{3}$, segregation patterns are likely to occur. This shows that individual (local) preferences of moderately tolerant agents can lead to the emergence of global segregation patterns. Over the last 50 years, Schelling's model has been thoroughly studied both from an empirical (see, e.g., [CF08; CT18]) and a theoretical (see, e.g., [BEL14; BMR14; Bra+12; Imm+17]) perspective in various disciplines including computer science, economics, and sociology. Most works focused on explaining under which circumstances and how quickly segregation patterns occur. Moreover, extensions and variants of the original model have been considered, see for example the works by Benard and Willer [BW07] and Mantzaris [Man20] that incorporate financial aspects.

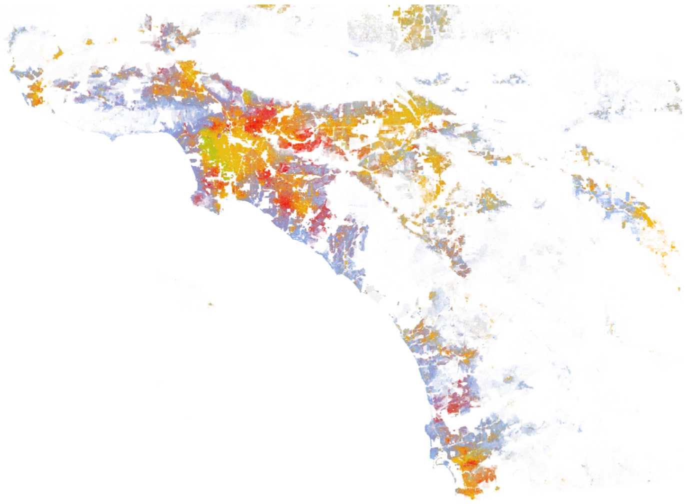
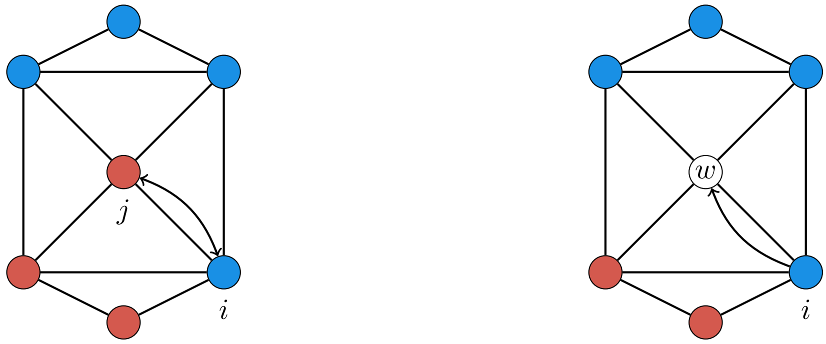


Figure 1.1: Residential Segregation in Los Angeles. Every dot represents a citizen and is colored by ethnicity. Taken from the Racial Dot Map [Cab13] (created using 2010 US Census data).

Game-Theoretic Formulations of Schelling's Model. As described above, in Schelling's model, the movement of unhappy agents is random. This, however, seems unrealistic, as real-world agents would realistically only change their location if doing so is profitable. Recently, motivated by the assumption that agents behave strategically, game-theoretic formulations of Schelling's model (so-called Schelling games) have attracted considerable attention [Aga+20; Bil+20; CLM18; Ech+19; Elk+19; KKV20; KKV21]. In these models, agents have a *utility* that depends on the composition of their neighborhood, and they swap positions or jump to unoccupied vertices if doing so increases their utility (we call such a jump or swap *profitable*). However, there is no unified formalization of the agents' utilities in the different game-theoretic models. For example, in the first game-theoretic works by Chauhan, Lenzner, and Molitor [CLM18] and Echzell et al. [Ech+19], the utility of an agent a depends on the minimum of the threshold parameter τ and the fraction of agents of a's type in the occupied neighborhood of a. That is, a only aims to maximize the fraction of agents of the same type in a's neighborhood up to τ. Additionally, in the model by Chauhan, Lenzner, and Molitor [CLM18], each agent has a favorite vertex and wants to minimize its distance to this vertex (in addition to maximizing the fraction of agents of the same type). Elkind et al. [Elk+19] introduced a simpler model where the utility of an agent only depends on the fraction of agents of the same type in the occupied neighborhood (i.e., $\tau = 1$ and the agents do not have favorite vertices). In this model, there are *strategic* agents that aim to maximize their utility and *stubborn* agents that are positioned on a fixed vertex which they never leave. In this work, we consider a simpler variant of this model in that we assume that all agents behave strategically (i.e., there are no stubborn agents).

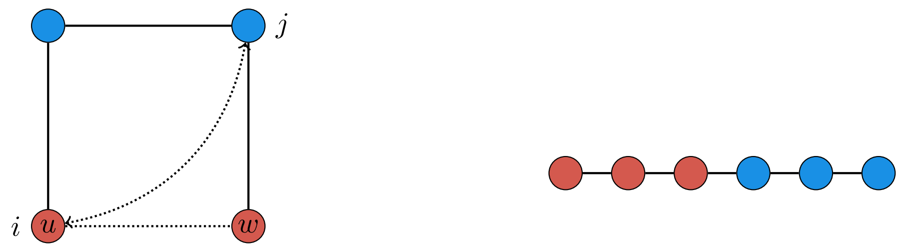


(a) The depicted assignment is not a swap-equilibrium, as agents i and j can both increase their utility by swapping. The resulting assignment is a swap-equilibrium.

(b) The depicted assignment is not a jump-equilibrium, as agent i can increase its utility by jumping to the unoccupied vertex w. The resulting assignment is a jump-equilibrium.

Figure 1.2: Examples for assignments in our game-theoretic model where a profitable jump or swap exists. The color of a vertex represents the type of the agent occupying this vertex, unoccupied vertices are uncolored. Recall that the utility of an agent is given by the fraction of agents of the same type in its occupied neighborhood.

Notions of Stability. As we want to model the individual behavior of residents, a natural way to define stability in these models is to consider, given a placement of the agents, if there exists an agent that has an incentive to jump to an unoccupied vertex or a pair of agents that want to swap positions. The central game-theoretic solution concept of (Nash) equilibria captures this notion and has been applied to these models [Aga+20; Bil+20; CLM18; Ech+19; Elk+19]. We say that an assignment of the agents to vertices is an *equilibrium* if there is no profitable swap for two agents (swap-equilibrium) or no profitable jump to an unoccupied vertex (jump-equilibrium). We give examples in Figures 1.2a and 1.2b. Investigating when these equilibria are guaranteed to exist has been one of the main focuses in previous work (see Section 1.1). Since the existence of an equilibrium clearly depends on the topology, most works investigate the influence of the structure of the topology thereon. While we also follow this approach in this work, we argue for a more fine-grained view on equilibria in Schelling games: As Schelling's model was originally designed for modeling segregation in an urban setting, edges might resemble roads or bus routes that can be unavailable at certain times (due to roadworks or bus timetables). Hence, viewing the topology as static seems unrealistic. In this setting, we might thus be interested in finding equilibria that remain stable when the underlying topology is subject to a certain number of changes. In this thesis, we capture this perspective as the *robustness* of an equilibrium (with regard to edge deletions), and study the existence of equilibria from this perspective. We find that the robustness of equilibria in Schelling games heavily depends on the structure of the underlying topology. An example for this is given in Figure 1.3, where we provide a topology with a swap-equilibrium that can be made unstable by deleting a single edge, and another topology with a swap-equilibrium that remains stable after deleting any arbitrary set of edges.



(a) After deleting the edge $\{u, w\}$, agents i and j want to swap.

(b) A swap-equilibrium that remains robust after deleting any set of edges.

Figure 1.3: A swap-equilibrium that can be made unstable by deleting a single edge and a swap-equilibrium that remains robust after deleting any set of edges.

Furthermore, while urban networks can be naturally modeled as simple graphs, one might be interested in distinguishing different kinds of connections in such networks. For example, in a city, there are usually both public (e.g., buses or trains) and private (e.g., cars) means of transportation. Moreover, in reality, residents often only have access to (or choose to use) some of the available transport systems. Thus, if we do not distinguish these different kinds of connections, then an agent could be adjacent to other agents even if she does not use the type of connection connecting them. Hence, we may want to find placements that are stable with regard to every one of the different systems. One way to model such networks with different kinds of connections are *multilayer graphs*. A multilayer graph is a graph with multiple sets of edges (representing the different kinds of connections) over a fixed vertex set. A *layer* of a multilayer graph is the simple graph given by the vertex set and one of the edge sets. We study Schelling games on multilayer graphs, which we call *multimodal* Schelling games. As motivated before, we say that an assignment is an equilibrium in a multimodal Schelling game if it is an equilibrium on every one of the layers. We study the existence of such multimodal equilibria and show that swap-equilibria in multimodal Schelling games may fail to exist even on very simple multilayer graphs.

1.1 Related Work

As mentioned above, many game-theoretic formulations of Schelling's model have been considered [Aga+20; Bil+20; CLM18; Ech+19; Elk+19; KKV20; KKV21]. In the following, we restrict our attention to the works by Agarwal et al. [Aga+20], Bilò et al. [Bil+20], and Elkind et al. [Elk+19], as the models considered therein are identical to the model in this work, with the exception of the existence of stubborn agents (all agents in our model are strategic).
All three works focus on the following aspects: existence and complexity of computing equilibria, and social welfare and price of anarchy and stability. We first review the results in the area most closely related to this work, which is the existence and computational complexity of equilibria. Elkind et al. [Elk+19] consider jump-equilibria and show that a jump-equilibrium is guaranteed to exist on stars and graphs with maximum

degree of at most 2. However, on the negative side, they show that even on a tree a jump-equilibrium may fail to exist. Furthermore, they study the computational complexity and prove that it is NP-hard to decide whether a given Schelling game admits a jump-equilibrium. Importantly, their model allows for stubborn agents that remain fixed on a given position and their results do not imply hardness in the absence of stubborn agents (which is the model considered in this work). For swap-equilibria, Agarwal et al. [Aga+20] show that in a Schelling game on a tree a swap-equilibrium may also fail to exist. Moreover, they show the NP-hardness of deciding whether a Schelling-game admits a swap-equilibrium, however again the reduction does not imply NP-hardness without stubborn agents. We prove that finding jump- and swap-equilibria remains NP-hard in the absence of stubborn agents in Chapter 3. Bilò et al. [Bil+20] extend the analysis of swap-equilibria existence and prove that a swap-equilibrium always exists on paths, almost regular graphs, and 4- and 8-grids. Furthermore, they investigate the impact of restricting the game to *local swaps*, where only neighboring agents are allowed to swap positions. In contrast to general swap-equilibria, when restricted to local swaps a *local swap-equilibrium* is guaranteed to exist on a tree.
As mentioned above, Agarwal et al. [Aga+20] and Bilò et al. [Bil+20] additionally study the social welfare and price of anarchy and stability of assignments. The social welfare of an assignment is the sum of the utilities of all agents. The price of anarchy is given by the ratio between the optimal social welfare of any assignment and the social welfare in the worst equilibrium assignment. While the results therein are not directly relevant to this thesis, Bilò et al. [Bil+20] use a similar approach in their analysis, as they also consider games on topologies from different graph classes.

1.2 Our Contribution

The main contributions of this work can be divided as follows: In the first, more technical, part of this thesis, we prove that deciding jump- and swap-equilibria existence remains NP-hard in the absence of stubborn agents. We later use these more general results to prove hardness results for robustness and multimodal Schelling games. In the second, more conceptual, part of this thesis, we introduce the notion of robustness and analyze the existence of equilibria from this perspective. Moreover, we study multimodal Schelling games on multilayer graphs. In the analysis of both of these aspects, we study the impact of the underlying topology.

Complexity of Finding Equilibria. We generalize the results by Elkind et al. [Elk+19] and Agarwal et al. [Aga+20] on the NP-hardness of deciding jump- and swap-equilibria existence to our simpler model (i.e., to the case without stubborn agents; see Chapter 3). This notably answers an open question posed by Elkind et al. [Elk+19], asking whether deciding jump-equilibrium existence remains NP-hard in this case.

Robustness. As mentioned before, with the robustness of an equilibrium, we introduce a new perspective for the analysis of equilibria in Schelling games in Chapter 4. We say that an equilibrium has robustness $r \in \mathbb{N}_0$ if it remains stable upon the deletion of any set of edges of size at most r but not under the deletion of $r + 1$ edges. We study the existence of equilibria with a given robustness. In addition to proving the (non-)existence of equilibria with a certain robustness, we also consider the *robustness-ratio*, which is given by the fraction between the least and most robust equilibrium on a topology. The robustness-ratio quantifies how the robustness of different equilibria in a fixed Schelling game may vary. From a practical perspective, a large robustness-ratio might justify putting more effort into finding a more robust equilibrium. We mostly restrict our analysis to swap-equilibria, but also shortly apply robustness to local swap-equilibria (where only adjacent agents are allowed to swap) and jump-equilibria.

In the analysis of the existence of swap-equilibria with a certain robustness, we follow the approach from most previous works and investigate the influence of the structure of the topology (e.g., [Bil+20; Elk+19]), as the robustness of equilibria clearly depends on the given topology. That is, we show upper and lower bounds on the robustness of swap-equilibria for topologies from various graph classes (see Section 4.1), summarized in Table 1.1. We analyze cliques and cycles, where we find that any swap-equilibrium can be made unstable by deleting only one edge. Turning to paths, we prove that on any (large enough) path, there exists a swap-equilibrium that can be made unstable by deleting a single edge and a swap-equilibrium that remains stable upon the deletion of any set of edges. This shows that the robustness-ratio of a Schelling game can be arbitrarily large. Moreover, for all graph classes mentioned above, we show that on any game on a graph from these classes, swap-equilibria with robustness matching both the upper and lower bounds are guaranteed to exist. We furthermore investigate grids (one of the graph classes most often considered in Schelling's original model), where we prove an upper bound for the robustness of a swap-equilibrium on a grid, and provide an infinite subclass of Schelling games on grids with robustness-ratio larger than one. In order to find a graph class where every swap-equilibrium has robustness larger than zero, we define *α-star-constellation graphs*. These graphs (formally defined in Chapter 2) consist of stars, where the central vertices of the stars can be arbitrarily connected such that it holds that every central vertex of a star is adjacent to at least α more degree-one vertices than other central vertices. We prove that every swap-equilibrium on an α-star-constellation graph has at least robustness α and find a subclass of α-star-constellation graphs where a swap-equilibrium is guaranteed to exist. Lastly, we apply robustness to local swap-equilibria (Section 4.1.3) and jump-equilibria (Section 4.2) and observe some differences between the different settings. We show that on a connected topology, contrary to swap-equilibria, the robustness of jump-equilibria is upper bounded by the maximum degree of the topology.

Our results for paths show that the robustness-ratio between the robustness of the most and least robust swap-equilibrium can be arbitrarily large. As more robust equilibria might be desirable in many real-world settings, we investigate computational aspects of robustness in Section 4.3. Here, we find both a positive and a negative result. On the positive side, we provide an efficient algorithm for deciding whether a given swap- or jump-equilibrium has at least robustness r for a given integer $r \in \mathbb{N}_0$. However, on the negative side, we show that deciding whether there exists a swap- or jump-equilibrium with at least a given robustness is NP-complete, using the general hardness results from Chapter 3.

Table 1.1: Overview of robustness bounds for swap-equilibria for various graph classes. For each considered class, there exists a Schelling game on a graph from this class with a swap-equilibrium whose robustness matches the depicted lower and upper bound. For bounds marked with †, on a graph from this class an equilibrium with this robustness is guaranteed to exist in every Schelling game. For the bound marked with ‡, this only holds if we have at least four agents of one of the types. Note that we add one to both the numerator and the denominator for the robustness-ratio, since an equilibrium can have a robustness of zero (for the formal definitions, see Chapter 4).

	Lower Bound	Upper Bound	Robustness-Ratio
Cliques (Prop. 4.7)	$0^\dagger$	$0^\dagger$	$= 1$
Cycles (Prop. 4.8)	$0^\dagger$	$0^\dagger$	$= 1$
Grids	0 (Prop. 4.13)	1 (Thm. 4.11)	$\in [1, 2]$
Paths	$0^\ddagger$ (Prop. 4.10)	$\lvert E(G)\rvert^\dagger$ (Prop. 4.9)	$= \lvert E(G)\rvert + 1$
α-star-constellation graphs	α (Thm. 4.14)	$\lvert E(G)\rvert$ (Prop. 4.18)	$\in [1, \frac{\lvert E(G)\rvert+1}{\alpha+1}]$

Multimodality. We introduce multimodal Schelling games in Chapter 5. We observe that our results from Chapter 3 also imply NP-hardness for deciding the existence of multimodal jump- and swap-equilibria (see Section 5.2). Furthermore, we analyze the existence of multimodal swap-equilibria on some simple graphs in Section 5.1. Here, we find that a multimodal swap-equilibrium may fail to exist even when all layers are isomorphic (i.e., structurally identical). This indicates that only considering the structure of the layers independently seems to be insufficient for analyzing equilibrium existence in multimodal Schelling games. Thus, we define an additional property of a multimodal graph, which captures the correspondence between the vertices of the layers. By incorporating this property, we are then able to define a subclass of multilayer graphs where a multimodal swap-equilibrium is guaranteed to exist.

Chapter 2

Preliminaries

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0$ the set of non-negative integers. For two integers $i < j \in \mathbb{N}_0$, we denote by $[i, j]$ the set $\{i, i + 1, \ldots, j - 1, j\}$ and by $[i]$ the set $[1, i]$.

Graph theory

Let $G = (V, E)$ denote an undirected graph, where V denotes the set of vertices and $E \subseteq \{\{v, w\} \mid v, w \in V, v \neq w\}$ denotes the set of edges. For a graph G, we also write $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G, respectively.

We denote by:

$N_G(v)$ The *neighborhood* of v, formally, $N_G(v) := \{u \in V \mid \{u, v\} \in E(G)\}$.

$\deg_G(v)$ The *degree* of v, formally, $\deg_G(v) := |N_G(v)|$.

$\Delta(G)$ The *maximum degree* of G, formally, $\Delta(G) := \max_{v \in V}\{\deg_G(v)\}$.

$G[V']$ The *induced subgraph* of G on $V' \subseteq V$. Formally, $G[V'] := (V', \{\{v, w\} \in E(G) \mid v, w \in V'\})$.

$G - S$ The graph obtained from G by deleting the edges $S \subseteq E(G)$, formally, $G - S := (V(G), E(G) \setminus S)$.

Schelling Games

A *Schelling game* I consists of a set $N = [n]$ of $n \geq 4$ agents partitioned into two disjoint *types* T_1 and T_2, and an undirected graph $G = (V, E)$ with $|V| \geq n$, called the *topology*. The strategy of agent $i \in N$ consists of picking some *position* $v_i \in V(G)$ with $v_i \neq v_j$ for $i, j \in N$ and $i \neq j$. The *assignment vector* $\mathbf{v} = (v_1, \ldots, v_n)$ defines the positions of all agents. A vertex $v \in V(G)$ is *unoccupied* if $v \neq v_i$ for all $i \in N$. In the following, we refer to an agent i and its position $v_i \in V(G)$ interchangeably. For example, we say agent i has an edge to an agent j, if $\{v_i, v_j\} \in E(G)$. For some agent $i \in T_l$, we call all

other agents of the same type $F_i = T_l \backslash \{i\}$ *friends* of i and define the set of *neighbors* as $N_i(\mathbf{v}) \coloneqq \{j \neq i \mid \{v_i, v_j\} \in E\}$. We define $a_i(\mathbf{v}) \coloneqq |N_i(\mathbf{v}) \cap F_i|$ as the number of friends in the neighborhood of agent i and $b_i(\mathbf{v}) \coloneqq |N_i(\mathbf{v}) \setminus F_i|$ as the number of neighbors of a different type.
Given some assignment $\mathbf{v}$, the utility of agent i on topology G is:

$$u_i^G(\mathbf{v}) \coloneqq \begin{cases} 0 & \text{if } N_i(\mathbf{v}) = \emptyset, \\ \frac{a_i(\mathbf{v})}{|N_i(\mathbf{v})|} & \text{otherwise.} \end{cases}$$

If the topology is clear from the context, then we omit the superscript G. Observe that if $|T_j| = 1$, then the only agent $i \in T_j$ has no friends and $u_i(\mathbf{v}) = 0$ on any vertex. We therefore assume $|T_j| \geq 2$ for all $j \in \{1, 2\}$.
Given some assignment $\mathbf{v}$, agent $i \in N$ and an unoccupied vertex v, we denote by $\mathbf{v}^{i \to v} = (v_1^{i \to v}, \ldots, v_n^{i \to v})$ the assignment obtained from $\mathbf{v}$ where i jumps to v: $v_i^{i \to v} = v$ and $v_j^{i \to v} = v_j$ for all $j \in N \setminus \{i\}$. Note that v_i is now unoccupied in $\mathbf{v}^{i \to v}$. We call a jump *local* if v_i is a neighbor of v. Agent i jumps to v if and only if the jump is *profitable*, that is, it holds that $u_i\left(\mathbf{v}^{i \to v}\right) > u_i(\mathbf{v})$. An assignment $\mathbf{v}$ is a (local) *Nash equilibrium* if no profitable (local) jump exists. An example is given in Figure 2.2. We refer to Nash equilibria as *jump-equilibria* to differentiate from another equilibrium concept defined below. Given a game I, let $\mathrm{JE}(I)$ and $\mathrm{LJE}(I)$ denote the sets of jump-equilibria and local jump-equilibria for I. Note that if $n = |V|$, every assignment is a jump-equilibrium since no unoccupied vertex exists and no jump is possible. We therefore assume $n < |V|$ in the analysis of jump-equilibria.
For two agents $i, j \in N$ and some assignment $\mathbf{v}$, we define $\mathbf{v}^{i \leftrightarrow j} = (v_1^{i \leftrightarrow v}, \ldots, v_n^{i \leftrightarrow v})$ as the assignment that is obtained by swapping the vertices of i and j. That is, $v_i^{i \leftrightarrow j} = v_j$, $v_j^{i \leftrightarrow j} = v_i$, and $v_k^{i \leftrightarrow j} = v_k$ for all $k \in N \setminus \{i, j\}$. A swap is *local* if i and j are neighbors. Agents i and j swap their positions if and only if the swap is *profitable*: $u_i\left(\mathbf{v}^{i \leftrightarrow j}\right) > u_i(\mathbf{v})$ and $u_j\left(\mathbf{v}^{i \leftrightarrow j}\right) > u_j(\mathbf{v})$. We call $\mathbf{v}$ a (local) *swap-equilibrium* if no (local) profitable swap exists. See Figure 2.1 for an example. We denote the sets of swap-equilibria and local swap-equilibria for a game I by $\mathrm{SE}(I)$ and $\mathrm{LSE}(I)$, respectively. Note that unoccupied vertices cannot be involved in a swap and do not contribute to the utility of agents. Hence, for the analysis of swap-equilibria, we assume that all vertices are occupied, formally $n = |V|$.
For a *l-modal Schelling game*, we consider a multilayer graph $\mathcal{G} = (V, \{E_1, \ldots E_l\})$, that is, we consider $l \geq 2$ topologies with edge sets $E_1, \ldots, E_l$ on a fixed set V of vertices. We define $G_j = (V, E_j)$ for some $j \in \{1, \ldots, l\}$. Since the set of vertices remains fixed for all topologies, an assignment $\mathbf{v}$ defines the positions on all graphs. For some G_j, the *induced game* is the Schelling game on topology G_j. An assignment $\mathbf{v}$ is called a multimodal (local) jump- or swap-equilibrium, if $\mathbf{v}$ is an equilibrium in all induced games on G_j for all $j \in \{1, \ldots l\}$.
Finally, we call equilibrium assignments *stable* and refer to non-equilibrium assignments as *unstable*.

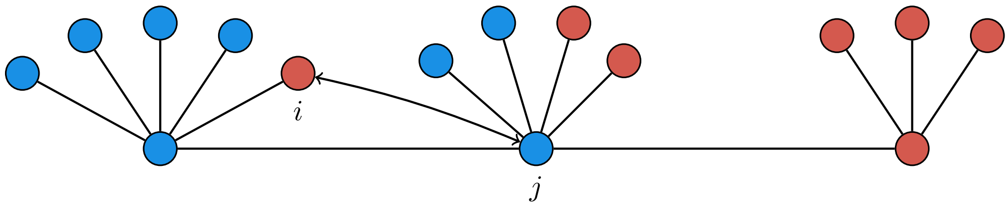


Figure 2.1: Schelling game with $|T_1| = 8$ and $|T_2| = 7$. Agents from T_1 are drawn in blue and agents from T_2 in red. Let $\mathbf{v}$ be the depicted assignment. It holds that $u_i(\mathbf{v}) = 0$ and $u_j(\mathbf{v}) = \frac{1}{2}$. Thus, swapping i and j is profitable, as $u_i(\mathbf{v}^{i\leftrightarrow j}) = \frac{1}{2} > 0 = u_i(\mathbf{v})$ and $u_i(\mathbf{v}^{i\leftrightarrow j}) = 1 > \frac{1}{2} = u_j(\mathbf{v})$. Hence, $\mathbf{v}$ is not a swap-equilibrium, as i and j have a profitable swap. However, the assignment resulting from this swap is a swap-equilibrium.

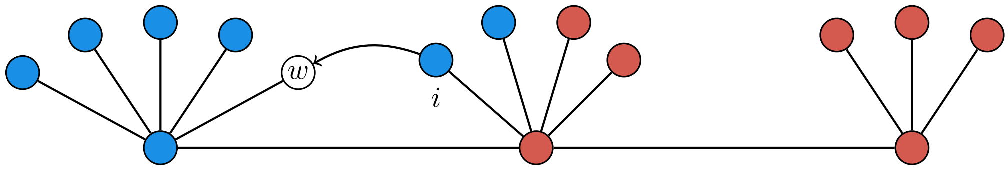


Figure 2.2: Schelling game with $|T_1| = |T_2| = 7$. Let $\mathbf{v}$ be the depicted assignment. In $\mathbf{v}$, the vertex w is the only unoccupied vertex. It holds that $u_i(\mathbf{v}) = 0$ and $u_i(\mathbf{v}^{i\to w}) = 1$. Thus, $\mathbf{v}$ is not a jump-equilibrium, as i has a profitable jump. The assignment resulting from this jump is a jump-equilibrium.

Graph Classes

Finally, we define the following graph classes which will play a prominent role in our analysis of the robustness of equilibria.

Path. A *path* of length n is a graph $G = (V, E)$ with $V = \{v_1, \ldots v_n\}$ and $E = \{\{v_i, v_{i+1}\} \mid i \in [n-1]\}$.

Cycle. A *cycle* of length n is a graph $G = (V, E)$ with $V = \{v_1, \ldots v_n\}$ and $E = \{\{v_i, v_{i+1}\} \mid i \in [n-1]\} \cup \{v_n, v_1\}$.

Clique. We call a graph $G = (V, E)$ with n vertices where every pair of vertices is connected by an edge a *clique* of size n. Formally, $V = \{v_1, \ldots v_n\}$ and $E = \{\{v, w\} \mid v, w \in V,\ v \neq w\}$.

Grid. For $x, y \geq 2$, we define the $(x \times y)$-*grid* $G = (V, E)$ as the graph formed by a lattice with x rows and y columns. That is, $V = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a \leq x, b \leq y\}$ and $E = \{\{(a, b), (c, d)\} \mid |a - c| + |b - d| = 1\}$.

Star. An x-*star* with $x \in \mathbb{N}$ is a graph $G = (V, E)$ with $V = \{v_0, \ldots, v_x\}$ and $E = \{\{v_0, v_i\} \mid i \in [x]\}$. The vertex v_0 is called the *central vertex* of the star.

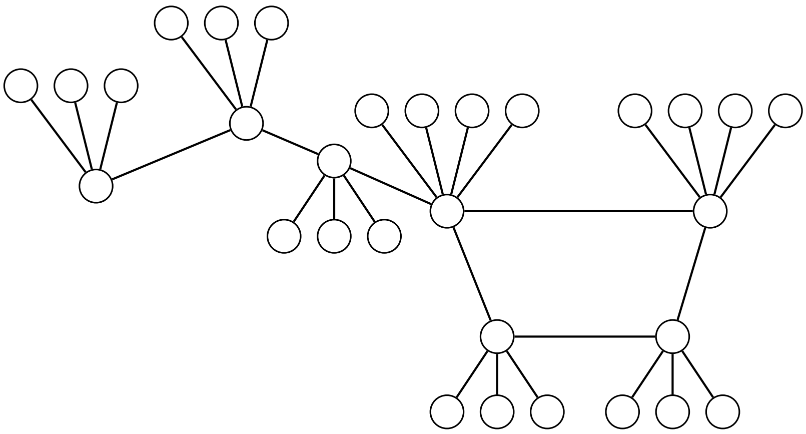


Figure 2.3: An example for a 1-star-constellation graph.

Star-constellation graph. We say a graph $G = (V, E)$ without isolated vertices is an *α-star-constellation graph*[1] for some $\alpha \in \mathbb{N}_0$, if every vertex with degree more than one is adjacent to at least α more degree-one vertices than vertices of degree at least two. Formally, it holds for all $v \in V$ with $\deg_G(v) > 1$ that $|\{w \in N_G(v) \mid \deg_G(w) = 1\}| \geq |\{w \in N_G(v) \mid \deg_G(w) > 1\}| + \alpha$. That is, G consists of stars, where the central vertices of the stars can be connected by edges such that every central vertex is adjacent to at least α more degree-one vertices than other central vertices. The graph in Figure 2.1 is an example for a 2-star-constellation graph, and a 1-star-constellation graph is depicted in Figure 2.3.

[1]A graph from this class consist of (connected) stars that form a *constellation* of stars, giving it its name.

Chapter 3

NP-Hardness of Equilibria Existence

In this chapter, we investigate the computational complexity of deciding the existence of jump- or swap-equilibria. Elkind et al. [Elk+19] and Agarwal et al. [Aga+20] proved that deciding jump- and swap-equilibria existence is NP-hard in a Schelling game with stubborn and strategic agents. A stubborn agent is positioned on a fixed vertex which it never leaves (such agents do not exist in our model). By "simulating" stubborn agents, we show that both problems remain NP-hard in the absence of stubborn agents.

3.1 Swap-Equilibria

First, we investigate the computational complexity of deciding whether a Schelling game admits a swap-equilibrium. Agarwal et al. [Aga+20] showed that it is NP-hard to decide whether a Schelling game with stubborn agents admits a swap-equilibrium. However, their result does not imply NP-hardness for the case where all agents are strategic, which is the model considered in this work. In order to prove the NP-hardness for this case, we reduce SWAP-EQ-STUB to SWAP-EQ. The decision problems are defined below. We first define the decision problem with stubborn agents which is NP-hard by Agarwal et al. [Aga+20].

Swap-Equilibrium Existence with stubborn agents (SWAP-EQ-STUB)

Input: A connected topology G, a set of agents $[|V(G)|] = N = R \dot{\cup} S$ partitioned into types T_1 and T_2, and a set of vertices $V_S = \{s_i \in V(G) \mid i \in S\}$.

Question: Does the Schelling game on G with types T_1 and T_2, strategic agents from R, and stubborn agents $i \in S$ on $s_i \in V_S$ admit a swap-equilibrium?

The decision problem for Schelling games without stubborn agents is almost identical, the only difference is that we do not partition the agents into R and S, and we do not have a set V_S of fixed positions for the stubborn agents, since all agents are strategic.

SWAP-EQUILIBRIUM EXISTENCE (SWAP-EQ)
Input: A topology G and a set of agents $[|V(G)|] = N$ partitioned into types T_1 and T_2.
Question: Does the Schelling game on G with types T_1 and T_2 admit a swap-equilibrium?

In our reduction, we reduce from a slightly restricted version of SWAP-EQ-STUB as defined in the following corollary. The NP-hardness of this version follows directly from the reduction by Agarwal et al. [Aga+20].

Corollary 3.1. SWAP-EQ-STUB *remains NP-hard if the following two properties hold.*

1. *For every vertex $v \notin V_S$ not occupied by a stubborn agent, there exist two adjacent vertices $s_i, s_i \in V_S$ occupied by stubborn agents $i \in T_1$ and $j \in T_2$.*
2. *There are at least 5 strategic agents and 3 stubborn agents of each type.*

Proof. The corollary follows directly from the reduction by Agarwal et al. [Aga+20], since, in their reduction, all constructed instances satisfy both properties. To verify that these properties hold, we give a short description of their construction below.
The reduction is from CLIQUE. An instance of CLIQUE consists of a connected graph $H = (X, Y)$ and an integer λ. Without loss of generality, they assume that $\lambda > 5$. Given an instance of CLIQUE, they construct a Schelling game with agents $N = R \dot{\cup} S$ partitioned into types T_1 and T_2 on a topology G. R contains λ strategic agents of type T_1 and $|X| + 5$ strategic agents of type T_2. Note that we thus have at least five strategic agents from each type. The stubborn agents will be defined together with the topology.
The graph G consists of three subgraphs G_1, G_2 and G_3, which are connected by single edges. G_1 is an extended copy of the given graph H with added degree-one vertices W_v for every $v \in X$. The vertices in W_v are occupied by stubborn agents from both types. The subgraph G_2 is a complete bipartite graph, where one of the partitions is fully occupied by stubborn agents from both types. G_3 is constructed by adding degree-one vertices to a given tree T. For every vertex $v \in V(T)$, at least ten vertices are added, half of which are occupied by stubborn agents from T_1 and the other half are occupied by stubborn agents from T_2. Thus, we have (more than) three stubborn agents of each type. Furthermore, it is easy to see that every vertex not occupied by a stubborn agent is adjacent to at least one stubborn agent from each type. □

Idea behind our Reduction. Before giving the full proof, we sketch the idea behind our reduction. Given an instance of SWAP-EQ-STUB on a topology G', we construct a Schelling game that simulates the given game without stubborn agents on a topology G. All stubborn agents are simulated by strategic agents in the constructed game. The topology G of the constructed instance is an extended copy of the given G' (see Figure 3.1). Moreover, we replace each stubborn agent by a strategic agent and add further strategic agents. In the construction, we ensure that if there exists a swap-equilibrium $\mathbf{v}'$ in the given game, then $\mathbf{v}'$ can be extended to a swap-equilibrium in the constructed game by replacing each stubborn agent with a strategic agent of the same type and filling empty vertices with further strategic agents. One particular challenge here is to ensure that the strategic agents that replace stubborn agents do not have a profitable swap. For this, recall that by Corollary 3.1, we assume that in G', every vertex not occupied by a stubborn agent in $\mathbf{v}'$ is adjacent to at least one stubborn agent of each type. Thus, in $\mathbf{v}'$ each strategic agent i is always adjacent to at least one friend and has utility:

$$u_i^{G'}(\mathbf{v}') \geq \frac{1}{\Delta(G')}.$$

Conversely, by swapping with agent i, an agent j of the other type can get utility at most

$$u_j^{G'}(\mathbf{v}'^{i\leftrightarrow j}) \leq \frac{\Delta(G')-1}{\Delta(G')}.$$

Our idea is now to "boost" the utility of a strategic agent j that replaces a stubborn agent in $\mathbf{v}$ by adding enough degree-one neighbors only adjacent to v_j in G, which we fill with agents of j's type when extending $\mathbf{v}'$ to $\mathbf{v}$ such that

$$u_j^G(\mathbf{v}) \geq \frac{\Delta(G')-1}{\Delta(G')} \geq u_j^G(\mathbf{v}^{i\leftrightarrow j}).$$

Additionally, we ensure that if the constructed game admits a swap-equilibrium $\mathbf{v}$ in the constructed game, then $\mathbf{v}$ restricted to $V(G')$, where some (strategic) agents are replaced by the designated stubborn agents of the same type, is a swap-equilibrium in the given game. The neighborhoods of all vertices in $V(G') \setminus V_{S'}$ are the same in G and G' and thus every swap that is profitable in the assignment in the given game would also be profitable in $\mathbf{v}$. So the remaining challenge here is to design G in such a way that the vertices occupied by stubborn agents of some type in the input game have to be occupied by agents of the same type in every swap-equilibrium in the constructed game. We achieve this by introducing an asymmetry between the types in the construction.

Theorem 3.2. SWAP-EQ *is NP-complete.*

Proof. We first observe that SWAP-EQ is in NP: To verify that an assignment $\mathbf{v}$ is a swap-equilibrium, we check if any pair of agents wants to swap by calculating their utilities before and after swapping.

For NP-hardness, we reduce from SWAP-EQ-STUB. An instance of SWAP-EQ-STUB consists of a connected topology G', a set of agents $[|V(G')|] = N' = R' \cup S'$ partitioned into types T_1', T_2', and a set of vertices $V_{S'} = \{v_i \in V(G') \mid i \in S'\}$. The agents in R' are strategic and the agents from S' are stubborn agents, with stubborn agent $i \in S'$ occupying $s_i \in V_{S'}$ in any assignment. By Corollary 3.1, we can assume without loss of generality that there are at least 5 strategic agents and at least 3 stubborn agents of each type in the input game. Furthermore, we assume that for every vertex $v \notin V_S$ not occupied by a stubborn agent, there exist two adjacent vertices $s_i, s_j \in V_S$ occupied by stubborn agents $i \in T_1$ and $j \in T_2$. Denote the sets of vertices occupied by stubborn agents from T_1' and T_2' as $V_{S_1'}$ and $V_{S_2'}$, respectively. We construct an instance of SWAP-EQ consisting of a topology $G = (V, E)$ and types T_1 and T_2 as follows.

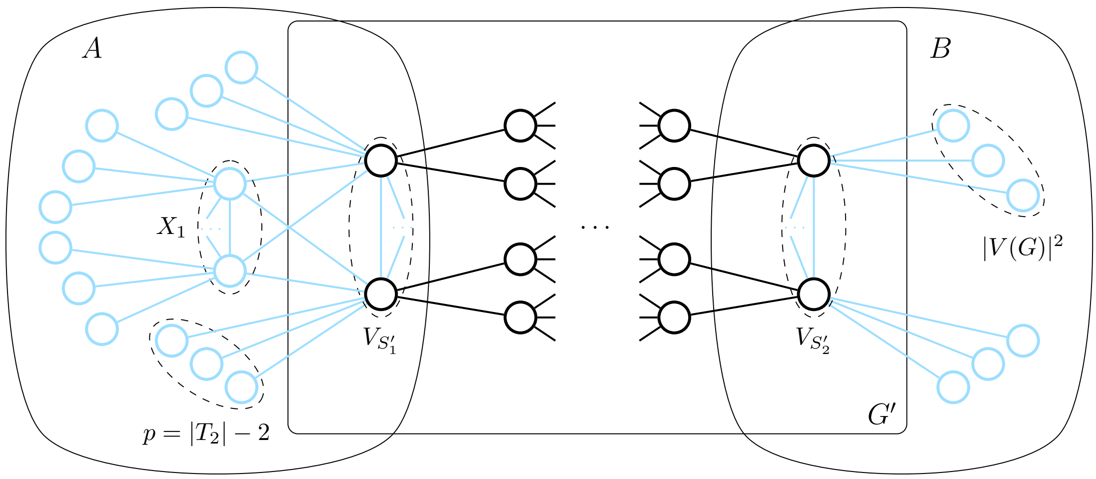
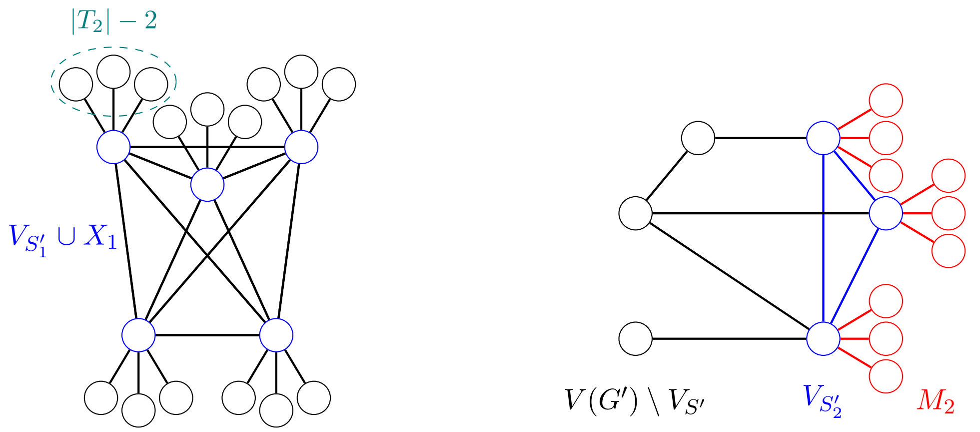


Figure 3.1: The constructed topology G. Modifications made to the given graph G' are colored light blue. Note that the vertices in $V_{S'_1} \cup X_1$ and $V_{S'_2}$ each form a clique.

- The graph G (sketched in Figure 3.1) is a modified copy of the given G' and contains all vertices and edges from G'. Additionally, we add three sets of vertices M_1, X_1, and M_2 as specified below. That is, $V = V(G') \cup M_1 \cup X_1 \cup M_2$. For every vertex $v \in V_{S'_2}$, we add $|V(G')|^2$ degree-one vertices only adjacent to v to M_2. We connect all vertices in $V_{S'_2}$ such that they form a clique by adding the edges $E_2 = \{\{v,w\} \mid v,w \in V_{S'_2}\}$ to G.
  Let $q := \Delta(G') + |V(G')|^2 + |V_{S'_2}|$. In order to achieve that $|V_{S'_1} \cup X_1| = q \cdot (|T'_2| + |M_2| + \Delta(G)) + 1 =: s$ (we use this property to introduce the mentioned asymmetry between the two types), let X_1 be a set of $s - |V_{S'_1}|$ vertices. We add the edges $E_1 = \{\{v,w\} \mid v,w \in V_{S'_1} \cup X_1\}$ such that the vertices in $V_{S'_1} \cup X_1$ form a clique. Let $p := |T'_2| + |M_2| - 2 > \Delta(G')^2$ (this choice of p is important to ensure that vertices in $V_{S'_1}$ are occupied by agents from T_1). For every vertex $v \in V_{S'_1} \cup X_1$, we add p degree-one vertices only adjacent to v to M_1. Notably, the neighborhood of all vertices in $V(G') \setminus V_{S'}$ is the same in G' and G.

- The set of agents $N = T_1 \dot\cup T_2$ is defined as follows. We have $|T_1| = |T'_1| + |M_1| + |X_1|$ agents in T_1 and $|T_2| = |T'_2| + |M_2|$ agents in T_2. By the construction of X_1 above, we have that $|V_{S'_1} \cup X_1| = q \cdot (|T'_2| + |M_2| + \Delta(G')) + 1 = q \cdot (|T_2| + \Delta(G')) + 1$. It also holds that $p = |T'_2| + |M_2| - 2 = |T_2| - 2$.

It is easy to see that the given construction can be computed in polynomial-time. Next, we address the correctness of the reduction. We define the sets $A, B \subseteq V$ as $A := M_1 \cup V_{S'_1} \cup X_1$ and $B := M_2 \cup V_{S'_2}$ (vertices from A should be occupied by agents from T_1, while vertices from B should be occupied by agents from T_2). Observe that the subgraph $G[A]$ (see Figure 3.2a) consists of $q \cdot (|T_2| + \Delta(G')) + 1$ stars which each have $(|T_2| - 1)$ vertices. The central vertices of these stars are connected such that they form a clique. Recall that $q = \Delta(G') + |V(G')|^2 + |V_{S'_2}| > |V_{S'_2}| \geq 3$, hence we have at least 3 stars in $G[A]$. Note that $V \setminus A = (V(G') \setminus V_{S'}) \cup B = (V(G') \setminus V_{S'}) \cup (M_2 \cup V_{S'_2})$. The subgraph $G[V \setminus A]$ (shown in Figure 3.2b) is connected, since every vertex in $V(G') \setminus V_{S'}$



(a) The subgraph $G[A]$. Observe that $G[A]$ consists of $q \cdot (|T_2| + \Delta(G')) + 1$ stars which each have $|T_2| - 1$ vertices and that are connected such that the central vertices form a clique.

(b) The subgraph $G[V \setminus A]$. Note that $G[V \setminus A]$ is connected, since every vertex in $V(G') \setminus V_{S'}$ is adjacent to at least one vertex in $V_{S'_2}$.

Figure 3.2: The induced subgraphs $G[A]$ and $G[V \setminus A]$. Recall that $A = V_{S'_1} \cup M_1 \cup X_1$.

is connected to a vertex in $V_{S'_2}$ (by our assumption) and the vertices in $V_{S'_2}$ form a clique (by the construction of G). Additionally, all vertices in M_2 are adjacent to exactly one vertex in $V_{S'_2}$. We start by proving Claims 1 and 2, which state that in every swap-equilibrium, the vertices in A and B have to be occupied by agents from T_1 and T_2, respectively. Using these claims, we will prove that every swap-equilibrium in the constructed game can be restricted to a swap-equilibrium for the given game.

Claim 1. *In any swap-equilibrium* $\mathbf{v}$*, all vertices in* A *are occupied by agents from* T_1*.*

Proof. Suppose there are $x > 0$ agents from T_2 in A. We distinguish the following three cases and prove that such an assignment can not be stable.

Case 1: Assume that $x < |T_2| - 1$. Since all stars in $G[A]$ have $|T_2| - 1 > x$ vertices, at least one of the stars has to contain agents from both types. Thus, there exists an agent $i \in T_t$ for some $t \in \{1, 2\}$ on a degree-one vertex in A with $u_i(\mathbf{v}) = 0$. Since $|T_2| - x > 0$ and $|T_1| > |A|$, there have to be agents from both T_1 and T_2 in $G[V \setminus A]$. As observed before, $G[V \setminus A]$ is connected. Thus, there exist agents $i' \in T_t$ and $j' \in T_{t'}$ with $t' \neq t$ in $G[V \setminus A]$ that are adjacent and hence have $u_{i'}(\mathbf{v}) < 1$ and $u_{j'}(\mathbf{v}) < 1$. Then, swapping i and j' is profitable, since we have $u_i(\mathbf{v}) = 0 < u_i(\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{j}'})$ and $u_{j'}(\mathbf{v}) < 1 = u_{j'}(\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{j}'})$.

Case 2: Assume that $x = |T_2|$. Again, note that all stars in $G[A]$ have $|T_2| - 1 < x$ vertices. Thus, there have to be agents from T_2 on at least two stars. Since it also holds that $x < 2 \cdot (|T_2| - 1)$, there are agents from both types on at least one of the stars. Let v_j be the central vertex of this star occupied by some agent j. There exists an agent $i \in T_t$ for some $t \in \{1, 2\}$ on a degree-one vertex adjacent to v_j with $u_i(\mathbf{v}) = 0$. The agent j is from type $T_{t'}$ with $t' \neq t$. As noted before, $G[A]$ contains at least 3 stars.

Since $x < 2 \cdot (|T_2| - 1)$, there also have to be agents from T_1 on at least two of the stars. Now consider the remaining central vertices in $G[A]$.
First, suppose that all central vertices are occupied by agents from $T_{t'}$. As noted above, it holds for both types that agents of this type occupy vertices from at least two stars. Hence, there exists an agent $i' \in T_t$ on a degree-one vertex adjacent to another central vertex $w \neq v_j$ with $u_{i'}(\mathbf{v}) = 0$. Let $j' \in T_{t'}$ be the agent on w. Then, swapping i and j' is profitable, since $u_{j'}(\mathbf{v}) < 1 = u_{j'}(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}'})$ and $u_i(\mathbf{v}) = 0 < u_i(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}'})$.
Now, consider that all remaining central vertices are occupied by agents from T_t. Then, there exists an agent $j' \in T_{t'}$ with $u_{j'}(\mathbf{v}) = 0$ on a degree-one vertex adjacent to another central vertex $w \neq v_j$ that is occupied by an agent from T_t. Swapping i and j' is profitable, since $u_{j'}(\mathbf{v}) = 0 < 1 = u_{j'}(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}'})$ and $u_i(\mathbf{v}) = 0 < 1 = u_i(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}'})$.
Therefore, the central vertices different from v_j have to be occupied by agents from both types. That is, there exists an agent $j' \in T_{t'}$ on a central vertex $v_{j'} \neq v_j$ and an agent $i' \in T_t$ on another central vertex $v_{i'} \neq v_j$. We have $u_{j'}(\mathbf{v}) < 1$, since $v_{j'}$ is adjacent to $v_{i'}$. Then however, swapping i and j' is profitable, since $u_{j'}(\mathbf{v}) < 1 = u_{j'}(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}'})$ and $u_i(\mathbf{v}) = 0 < u_i(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}'})$.

Case 3: Assume that $x = |T_2| - 1$. If the x agents from T_2 occupy vertices from two or more stars, then at least two stars contain agents from both types (illustrated in Figure 3.3). That is, there exists an agent $i \in T_{t_1}$ for some $t_1 \in \{1, 2\}$ with $u_i(\mathbf{v}) = 0$ on a degree-one vertex adjacent to an agent $j \in T_{t'_1}$ with $t'_1 \neq t_1$. The agent j on the central vertex has $u_j(\mathbf{v}) < 1$. Without loss of generality, assume that $t_1 = 1$ and thus $t'_1 = 2$. As argued above, another star has to contain agents from both types. Thus, there exists another agent $i' \in T_{t_2}$ for some $t_2 \in \{1, 2\}$ with $u_i(\mathbf{v}) = 0$ on a degree-one vertex adjacent to an agent $j' \in T_{t'_2}$ with $t'_2 \neq t_2$. Again, the agent j' on the central vertex has $u_{j'}(\mathbf{v}) < 1$. If $t_2 = 1$, then swapping i and j' is profitable. We have $u_i(\mathbf{v}) = 0 < u_i(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}'})$ and $u_{j'}(\mathbf{v}) < 1 = u_{j'}(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}'})$. Otherwise, if $t_2 = 2$, then swapping i and i' is profitable. It holds that $u_i(\mathbf{v}) = 0 < 1 = u_i(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{i}'})$ and $u_{i'}(\mathbf{v}) = 0 < 1 = u_{i'}(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{i}'})$. Summarizing, if the x agents from T_2 occupy vertices from two or more stars, then the assignment can not be a swap-equilibrium.
Hence, the agents from T_2 have to occupy all $|T_2| - 1$ vertices of one of the stars in A. Let agent $i \in T_2$ be the agent on the central vertex v_i of this star. Observe that $\deg_G(v_i) \geq (|T_2| - 2) + q \cdot (|T_2| + \Delta(G'))$, since v_i is adjacent to $|T_2| - 2$ degree-one neighbors and the $q \cdot (|T_2| + \Delta(G'))$ vertices in $(S_1 \cup X_1) \setminus \{v_i\}$. It follows that agent $i \in T_2$ has utility:

$$u_i(\mathbf{v}) \leq \frac{|T_2|}{(|T_2| - 1) + q \cdot (|T_2| + \Delta(G'))} < \frac{|T_2|}{|T_2| \cdot q} = \frac{1}{q}.$$

Since $x = |T_2| - 1$, there is one agent $i' \in T_2$ outside of A. As noted above, $G[V \setminus A]$ is connected. Thus, agent i' is adjacent to an agent $j \in T_1$ on $v_j \in V \setminus A$. Recall that $V \setminus A = (V(G') \setminus V_{S'}) \cup (M_2 \cup V_{S'_2})$. If $v_j \in V_{S'_2}$, then we have $\deg_G(v_j) \leq \Delta(G') + |V(G')|^2 + |V_{S'_2}|$. If $v_j \in M_2$, then we have $\deg_G(v_j) = 1$. If $v_j \in V(G') \setminus V_{S'}$, then we have $\deg_G(v_j) \leq \Delta(G')$. In any case, it holds that $\deg_G(v_j) \leq \Delta(G') + |V(G')|^2 + |V_{S'_2}| = q$. Since v_j is adjacent to $v_{i'}$, agent i has at least one adjacent friend after swapping to v_j.

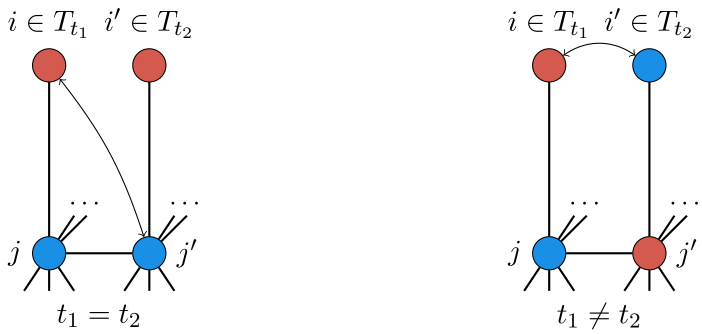


Figure 3.3: Case 3: There are exactly $|T_2| - 1$ agents from T_2 in A. If the agents from T_2 occupy two or more stars, then at least two of the stars have to contain agents from both types. That is, we have two agents $i \in T_{t_1}$ and $i' \in T_{t_2}$ on degree-one vertices with no adjacent friends.

The utility of agent i after swapping with j is:

$$u_i(\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{j}}) \geq \frac{1}{q} > u_i(\mathbf{v}).$$

Note that at least one of the at most q neighbors of j (specifically, agent i) is not a friend of j. Observe that the neighborhood of v_i consist of the $|T_2| - 2$ degree-one neighbors, the $q \cdot (|T_2| + \Delta(G'))$ vertices in $(S_1 \cup X_1) \setminus \{v_i\}$ and at most $\Delta(G')$ neighbors in $V(G') \setminus V_{S_1'}$. Thus, we have $\deg_G(v_i) \leq q \cdot (|T_2| + \Delta(G')) + |T_2| - 2 + \Delta(G') < (q+1) \cdot (|T_2| + \Delta(G'))$. Additionally, there are at least $q \cdot (|T_2| + \Delta(G'))$ agents of type T_1 adjacent to v_i (all agents in $(S_1 \cup X_1) \setminus \{v_i\}$). Therefore, it holds that agent $j \in T_1$ has utility:

$$u_j(\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{j}}) \geq \frac{q \cdot (|T_2| + \Delta(G'))}{(q+1) \cdot (|T_2| + \Delta(G'))} = \frac{q}{q+1} > \frac{q-1}{q} \geq u_j(\mathbf{v}).$$

Hence, swapping i and j is profitable and the assignment can not be a swap-equilibrium. Since we have exhausted all possible cases, the claim follows. ■

Next, we prove that all vertices in B have to be occupied by agents from T_2.

Claim 2. *In any swap-equilibrium* $\mathbf{v}$*, all vertices in* B *are occupied by agents from* T_2*.*

Proof. By Claim 1, A only contains agents from T_1 in any swap-equilibrium. The remaining $y := |T_1| - |A| = |T_1'| - |V_{S_1'}| < |V(G') \setminus V_{S'}|$ agents from T_1 and all agents from T_2 occupy vertices in $G[V \setminus A]$. Since it holds that $y < |V(G') \setminus V_{S'}|$, there exists an agent $j \in T_2$ with $v_j \in V(G') \setminus V_{S'}$. Recall that we assumed that in our input instance, every vertex not occupied by a stubborn agent is adjacent to at least one stubborn agent of each type. Thus, v_j is adjacent to $v_i \in V_{S_1'}$ occupied by agent i. It holds that $v_i \in A$, hence we have that agent i must be from T_1. The agents i and j have $u_i(\mathbf{v}) < 1$ and $u_j(\mathbf{v}) < 1$.
Suppose there are x agents of type T_1 in B, with $y \geq x > 0$. We prove that such an assignment $\mathbf{v}$ can not be a swap-equilibrium. As noted before, we have $x \leq y < |V(G')|$. Observe that the subgraph $G[B]$ consists of $|V_{S_2'}|$ stars where the central vertices form a

clique. Each star contains $|V(G')|^2+1 > x$ vertices. Therefore, at least one of the stars has to contain agents from both types. That is, there exists an agent $i' \in T_t$ for some $t \in \{1,2\}$ on a degree-one vertex in B with $u_{i'}(\mathbf{v}) = 0$. If $t = 1$, then swapping agent i' and agent j is profitable, since $u_{i'}(\mathbf{v}) = 0 < u_{i'}(\mathbf{v}^{\mathbf{i'}\leftrightarrow\mathbf{j}})$ and $u_j(\mathbf{v}) < 1 = u_j(\mathbf{v}^{\mathbf{i'}\leftrightarrow\mathbf{j}})$. Otherwise, if $t = 2$, then swapping agent i' and agent i is profitable, since $u_{i'}(\mathbf{v}) = 0 < u_{i'}(\mathbf{v}^{\mathbf{i'}\leftrightarrow\mathbf{i}})$ and $u_i(\mathbf{v}) < 1 = u_i(\mathbf{v}^{\mathbf{i'}\leftrightarrow\mathbf{i}})$. Therefore, there exists a profitable swap and the assignment can not be a swap-equilibrium. ■

Having established these two claims, we are now able to prove that the input game with stubborn agents admits a swap-equilibrium if and only if the constructed Schelling game admits a swap-equilibrium.

($\Rightarrow$): First, assume there exists a swap-equilibrium $\mathbf{v}'$ for the Schelling game with stubborn agents. Note that the vertices in $V_{S'_1}$ and $V_{S'_2}$ are occupied by stubborn agents from T'_1 and T'_2. We define an assignment $\mathbf{v}$ for the Schelling game without stubborn agents and prove that it is a swap-equilibrium. The vertices in $V \cap V(G')$ are occupied by agents of the same type as the agents in $\mathbf{v}'$. The vertices in X_1 are occupied by agents from T_1. For $t \in \{1,2\}$, the added degree-one vertices in M_t are occupied by agents from T_t. Hence, the vertices in $A = M_1 \cup V_{S'_1} \cup X_1$ are occupied by agents from T_1 and the vertices in $B = M_2 \cup V_{S'_2}$ are occupied by agents from T_2.

Next, we prove that $\mathbf{v}$ is a swap-equilibrium on G by showing that no profitable swap exists. We first observe that the utility of an agent i on a vertex $w \in M_1 \cup X_1$ is $u_i(\mathbf{v}) = 1$, since $N_G(w) \subseteq A$ and all agents in A are friends of i. The same holds analogously for any agent on a vertex in M_2. Therefore, the agents on vertices in $M_1 \cup M_2 \cup X_1$ can not be involved in a profitable swap. Note that the utility of any agent on a vertex $v \in V(G') \setminus V_{S'}$ is the same for $\mathbf{v}$ and $\mathbf{v}'$. Since $\mathbf{v}'$ is a swap-equilibrium, a profitable swap must therefore involve at least one agent i with $v_i \in V_{S'}$.

Let $Y = V_{S'_1} \cup X_1$ if $v_i \in V_{S'_1}$ and $Y = V_{S'_2}$ otherwise. Denote the number of added degree-one neighbors adjacent to v_i by x. By construction of G, it holds that $x > \Delta(G)^2$. The agent i is adjacent to the vertices in $Y \setminus \{v_i\}$ and the $x > \Delta(G')^2$ degree-one neighbors, which are all occupied by friends. It holds that $\deg_G(v_i) \leq x + |Y \setminus \{v_i\}| + \Delta(G')$. Thus, the utility of agent i on v_i is:

$$u_i^G(\mathbf{v}) \geq \frac{x + |Y \setminus \{v_i\}|}{x + |Y \setminus \{v_i\}| + \Delta(G')} > \frac{x}{x + \Delta(G')} > \frac{\Delta(G')^2}{\Delta(G')^2 + \Delta(G')} = \frac{\Delta(G')}{\Delta(G') + 1}.$$

We now distinguish between swapping i with an agent j with $v_j \in V_{S'}$ and with $v_j \in V_{S'}$ (note that this exhausts all cases as we have already argued above that all agents placed on newly added vertices can never be part of a profitable swap). First, consider swapping i with an agent j of the other type with $v_j \in V_{S'}$. On vertex v_j, agent i can at most have $\Delta(G')$ adjacent friends, as all vertices that are connected to v_j by edges added in the construction (that are, vertices in A or B) are occupied by friends of j. It holds that $\deg_G(v_j) \geq \Delta(G')^2$, since v_j is adjacent to at least $\Delta(G')^2$ degree-one neighbors. Therefore, the swap can not be profitable:

$$u_i^G(\mathbf{v}) \geq \frac{\Delta(G')}{\Delta(G') + 1} > \frac{1}{\Delta(G')} = \frac{\Delta(G')}{\Delta(G')^2} \geq u_i^G(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}}).$$

Hence, consider swapping i with an agent j of the other type on $v_j \in V(G') \setminus V_{S'}$. Recall that we assumed that every vertex not occupied by a stubborn agent is adjacent to at least one stubborn agent of each type in the input game. Since $v_j \in V(G') \setminus V_{S'}$, the agent j is adjacent to at least one friend. More precisely, we have:

$$u_j^{G'}(\mathbf{v}') \geq \frac{1}{\Delta(G')}.$$

As noted above, the neighborhood of v_j is identical in G and G'. We therefore have $u_j^G(\mathbf{v}) = u_j^{G'}(\mathbf{v}')$. By swapping with agent j, agent i can at most get the following utility:

$$u_i^G(\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{j}}) \leq \frac{\Delta(G') - 1}{\Delta(G')}.$$

It follows that swapping i and j can not be profitable:

$$u_i^G(\mathbf{v}) \geq \frac{\Delta(G')}{\Delta(G') + 1} > \frac{\Delta(G') - 1}{\Delta(G')} \geq u_i^G(\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{j}}).$$

Summarizing, no profitable swap is possible and $\mathbf{v}$ is a swap-equilibrium for the constructed Schelling game.

($\Leftarrow$): Conversely, assume there exists a swap-equilibrium $\mathbf{v}$ for the constructed game. We define an assignment $\mathbf{v}'$ and prove that it is a swap-equilibrium for the given Schelling game with stubborn agents on G'. In $\mathbf{v}'$, a vertex $v \in V(G') \setminus V_{S'}$ is occupied by a strategic agent of the same type as the agent on v in $\mathbf{v}$. The vertices in $V_{S'}$ have to be occupied by the respective stubborn agents. Note that by Claims 1 and 2, in $\mathbf{v}$, the vertices in $V_{S'_1} \subseteq A$ and $V_{S'_2} \subseteq B$ have to be occupied by agents from T_1 and T_2, respectively. Thus, in $\mathbf{v}'$, all vertices are occupied by agents of the same type as in $\mathbf{v}$.

Now, we will prove that $\mathbf{v}'$ is a swap-equilibrium on the given G'. Since stubborn agents never swap position, a profitable swap has to involve two strategic agents $i \in T'_1$ and $j \in T'_2$ with $v_i, v_j \in V(G') \setminus V_{S'}$. However, by construction of G, the neighborhoods of v_i and v_j are identical in G and G'. Additionally, it holds that in $\mathbf{v}'$, all vertices are occupied by agents of the same type as in $\mathbf{v}$. Since $\mathbf{v}$ is a swap-equilibrium on G, swapping i and j can not be profitable. It follows that $\mathbf{v}'$ is a swap-equilibrium, which completes the proof. □

3.2 Jump-Equilibria

We now turn to proving the NP-hardness of deciding the existence of a jump-equilibrium. Elkind et al. [Elk+19] already proved that this problem is NP-hard in a Schelling game with stubborn agents, as defined below.

Jump-Equilibrium Existence with stubborn agents (JUMP-EQ-STUB)

Input: A connected topology G, a set of agents $[n] = N = R \dot\cup S$ for some $n < |V(G)|$ partitioned into types T_1 and T_2 and a set of vertices $V_S = \{s_i \in V(G) \mid i \in S\}$.

Question: Does the Schelling game on G with types T_1 and T_2, strategic agents from R and stubborn agents $i \in S$ on $s_i \in V_S$ admit a jump-equilibrium?

Note that we now require that $n < |V(G)|$ such that there are unoccupied vertices. We show that deciding jump-equilibria existence remains NP-hard in the absence of stubborn agents, as conjectured by Elkind et al. [Elk+19]. The decision problem is defined below.

JUMP-EQUILIBRIUM EXISTENCE (JUMP-EQ)
Input: A topology G and a set of agents $[n] = N = R\dot{\cup}S$ for some $n < |V(G)|$ partitioned into types T_1 and T_2.
Question: Does the Schelling game on G with types T_1 and T_2 admit a jump-equilibrium?

Again, we reduce from a restricted version of JUMP-EQ-STUB, as defined in the following lemma. The hardness of this restricted version does not directly follow from the reduction by Elkind et al. [Elk+19], but can be proven by slightly modifying their reduction.

Lemma 3.3. *Let $\lambda = |V(G)| - n$ be the number of unoccupied vertices in an instance of* JUMP-EQ-STUB*. We call an instance of* JUMP-EQ-STUB *regularized, if the following five properties hold.*

1. *Every vertex $v \notin V_S$ is adjacent to at least one stubborn agent of each type.*
2. *Every vertex $v \in V_S$ has $\deg_G(v) < \lambda$.*
3. *Every vertex $v \in V_S$ is adjacent to a vertex $v \notin V_S$.*
4. *It holds that $\lambda > 0$.*
5. *There are at least two stubborn agents of each type.*

JUMP-EQ-STUB *remains NP-hard when restricted to regularized instances.*

Proof. We prove this statement by giving a reduction that is heavily based on the reduction by Elkind et al. [Elk+19], which is modified such that the constructed instance is always regularized. Most importantly, we modify the original construction such that the first property holds. All other properties already hold or are trivial to achieve.
We reduce from CLIQUE. An instance of CLIQUE consist of an undirected graph $H = (X, Y)$ and an integer s. It is a yes-instance if and only if H contains a clique of size s. Without loss of generality, we assume that $s \geq 6$. We construct an instance of JUMP-EQ-STUB as follows:

- There are two types T_1 and T_2. There are s strategic agents from T_1. All other agents are stubborn and will be defined along with the topology.
- The topology $G = (V, E)$ consists of three disjoint components G_1, G_2, and G_3, which are constructed as described below.
  - To define the graph $G_1 = (V_1, E_1)$, let W_v be a set of s vertices for every $v \in X$. Out of the vertices in W_v, one vertex is occupied by a stubborn agent from T_1 and the remaining $s-1$ vertices are occupied by stubborn agents from T_2. We set $V_1 = X \cup \bigcup_{v\in X} W_v$ and $E_1 = Y \cup \bigcup_{v\in X}\{\{v, w\} \mid w \in W_v\}$.

That is, G_1 is an extended copy of the given H, where every $v \in X$ is adjacent to s degree-one vertices in W_v, which are occupied by stubborn agents from both types.

– The graph G_2 is a bipartite graph with parts L and R. Let L be a set of $s-2$ vertices. For every $v \in L$, the set R contains $4s$ vertices only connected to v. Out of these $4s$ vertices, $2s+1$ are occupied by stubborn agents from T_1 and the remaining $2s-1$ vertices are occupied by stubborn agents from T_2.

– in G_3, only three vertices x, y, and z are not occupied by stubborn agents. The vertices x and y are connected by an edge. The remaining vertices are occupied by stubborn agents and defined in the following. First, the vertex x is connected to one degree-one vertex occupied by a stubborn agent from T_1 and two degree-one vertices occupied by stubborn agents from T_2. The vertex y is connected to 41 degree-one vertices occupied by stubborn agents from T_1 and 80 degree-one vertices occupied by stubborn agents from T_2. Finally, z is connected to 5 degree-one vertices occupied by stubborn agents from T_1 and 7 degree-one vertices occupied by stubborn agents from T_2.

Lastly, we pick an arbitrary vertex occupied by a stubborn agent from each of the three components G_1, G_2, G_3 and connect the three vertices to form a clique.
It is easy to verify that the constructed instance is regularized. The correctness of the reduction follows analogously to the proof by Elkind et al. [Elk+19]. The only difference is the exact utility of agents on G_1; however, the same inequalities still hold. □

We now reduce this restricted version of JUMP-EQ-STUB to JUMP-EQ. This reduction is based on the same underlying idea as the reduction for SWAP-EQ, as we also simulate the game with stubborn agents using a similar construction. However, the proof for JUMP-EQ is more involved, since in every assignment some vertices remain unoccupied. For instance, we do not only need to prove that only agents from T_1 are placed on vertices from A (which is more challenging because we have to deal with possibly unoccupied vertices), but also that all vertices from A are occupied.

Theorem 3.4. JUMP-EQ *is NP-complete.*

Proof. First, observe that JUMP-EQ is in NP, since we can iterate over all pairs of agents and unoccupied vertices and check if there exists a profitable jump. For NP-hardness, we reduce from JUMP-EQ-STUB. An instance of JUMP-EQ-STUB consists of a connected topology G', a set of agents $N' = R' \cup S'$ partitioned into types T'_1, T'_2, and a set of vertices $V_{S'} = \{s_i \in V(G') \mid i \in S'\}$. The agents in R' are strategic and the agents from S' are stubborn agents, with stubborn agent $i \in S'$ occupying $s_i \in V_{S'}$ in any assignment. Without loss of generality, we assume that the given instance is regularized and fulfills the properties from Lemma 3.3. Denote the sets of vertices occupied by stubborn agents from T'_1 and T'_2 as $V_{S'_1}$ and $V_{S'_2}$, respectively. We construct an instance of JUMP-EQ consisting of a topology $G = (V, E)$ and types T_1 and T_2 as follows.

- The graph G (depicted in Figure 3.4) is a modified copy of the given G' and contains all vertices and edges from G'. Additionally, we add three sets of vertices M_1, X_1 and M_2. That is, $V = V(G') \cup M_1 \cup X_1 \cup M_2$. For every vertex $v \in V_{S'_2}$, we

add $|V(G')|^2$ degree-one vertices only adjacent to v to M_2. We connect all vertices in $V_{S'_2}$ such that they form a clique by adding the edges $E_2 = \{\{v, w\} \mid v, w \in V_{S'_2}\}$ to G.
Next, we define q, which, as argued later, is an upper bound for the degree of a vertex in $(V(G') \setminus V_{S'_1}) \cup M_2$:

$$q \coloneqq \Delta(G') + |V(G')|^2 + |V_{S'_2}|.$$

We define s and z, this choice of parameters is important in the proof of Claim 2.1.

$$s \coloneqq q \cdot (|T'_2| + |M_2| + \Delta(G)) + 1$$
$$z \coloneqq s + |V(G')| + |M_2|$$

Let X_1 be a set of $z - |V_{S'_1}|$ vertices, such that $|X_1 \cup V_{S'_1}| = z$. We add the edges $E_1 = \{\{v, w\} \mid v, w \in V_{S'_1} \cup X_1\}$ such that the vertices in $V_{S'_1} \cup X_1$ form a clique. Finally, we define the number p of added degree-one neighbors for vertices in $V_{S'_1} \cup X_1$. Again, the choice of p is used in Claim 2.1.

$$p \coloneqq |T'_2| + |M_2| - 2 > |V(G')|^2$$

For every vertex $v \in V_{S'_1} \cup X_1$, we add p degree-one vertices only adjacent to v to M_1.

- The set of agents $N = T_1 \dot\cup T_2$ is defined as follows. We have $|T_1| = |T'_1| + |M_1| + |X_1|$ agents in T_1 and $|T_2| = |T'_2| + |M_2|$ agents in T_2. By the construction of X_1 above, we have that:

$$s = q \cdot (|T'_2| + |M_2| + \Delta(G')) + 1 = q \cdot (|T_2| + \Delta(G')) + 1.$$

It also holds that $p = |T'_2| + |M_2| - 2 = |T_2| - 2$. Finally, note that there are equally many unoccupied vertices in the constructed and the given instance, since it holds that $|V(G)| - |V(G')| = |M_1| + |X_1| + |M_2| = |N| - |N'|$. That is, $\lambda = \lambda' < |V(G')|$.

It is easy to see that the given construction can be computed in polynomial-time. Next, we address the correctness of the reduction. We approach the proof in three steps. First, we make some basic observations about the constructed G. Then, we prove Claims 1 to 3, which state useful properties of jump-equilibria for the constructed game. Finally, using these claims, we prove that the constructed game admits a jump-equilibrium if and only if the given game admits a jump-equilibrium.

We define the sets $A, B \subseteq V$ as $A \coloneqq M_1 \cup V_{S'_1} \cup X_1$ and $B \coloneqq M_2 \cup V_{S'_2}$. Observe that the subgraph $G[A]$ can be partitioned into z stars which each have $|T_2| - 1$ vertices as follows. The vertices in $X_1 \cup V_{S'_1}$ are the central vertices and are connected such that they form a clique. Each central vertex is adjacent to $|T_2| - 2$ degree-one vertices in M_1. Recall that $z > |V(G')| \geq |V_{S'}| > 3$, hence we have at least 3 stars in $G[A]$. Note that $V \setminus A = (V(G') \setminus V_{S'}) \cup B = (V(G') \setminus V_{S'}) \cup (M_2 \cup V_{S'_2})$. The subgraph $G[V \setminus A]$ is connected, since every vertex in $V(G') \setminus V_{S'}$ is connected to a vertex in $V_{S'_2}$ (by our assumption that the given instance is regularized) and the vertices in $V_{S'_2}$ form a clique (by the construction of G). Additionally, all vertices in M_2 are adjacent to exactly one

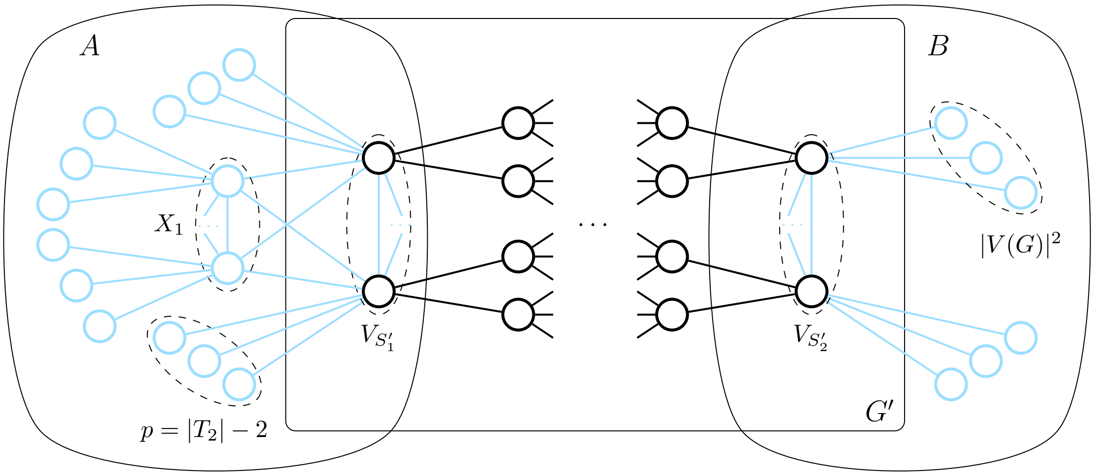


Figure 3.4: The constructed topology G. Modifications made to the given graph G' are colored light blue. Note that the vertices in $V_{S_1'} \cup X_1$ and $V_{S_2'}$ each form a clique.

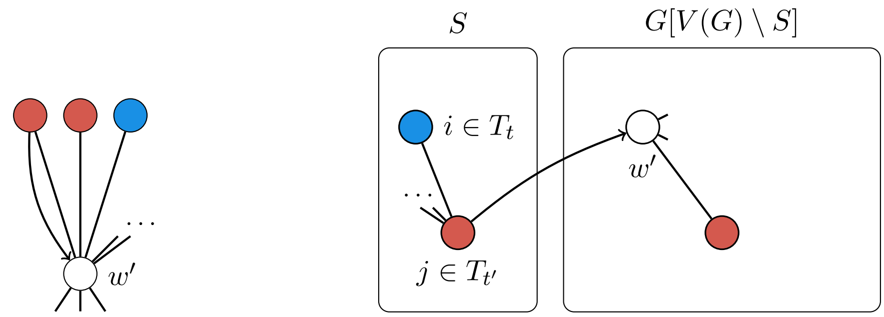
vertex in $V_{S_2'}$.
First, we show the following claim, which states that no agent on a degree-one vertex in A or B can have no adjacent friends and all central vertices have to be occupied. This property of jump-equilibria is then later used in the proof of Claims 2 and 3, where we prove that all vertices in A are occupied by agents from T_1 and all vertices from B are occupied by agents from T_2.

Claim 1. *In a jump-equilibrium* $\mathbf{v}$*, all agents on a degree-one vertex* $v \in M_1 \cup M_2$ *are adjacent to a friend and all central vertices* $w \in V_{S_1'} \cup X_1 \cup V_{S_2'}$ *are occupied.*

Proof. Suppose for the sake of a contradiction that there exists an agent $i \in T_t$ for some $t \in \{1, 2\}$ on a degree-one vertex v with no adjacent friend in $\mathbf{v}$. Let S be the set of vertices of the star which contains v. Recall that every star contains at least $|V(G')|^2 + 1 \geq |V(G')| + 3 \geq \lambda + 3$ vertices. Thus, at least three vertices in S have to be occupied. Since i has no adjacent friends, the central vertex $w \in S$ is either unoccupied or occupied by an agent of the other type. We distinguish these two cases and prove that there exists a profitable jump in both cases.

Case 1: First, assume that w is unoccupied. As mentioned above, at least three vertices in S have to be occupied by agents. Since w is unoccupied, all occupied vertices are degree-one vertices. By the pigeonhole principle, there exist two agents of the same type degree-one vertices in S. Both agents have no adjacent friends in $\mathbf{v}$ and can increase their utility by jumping to w. This concludes the first case and furthermore proves that no central vertex can be unoccupied.

Case 2: Second, we consider the case where w is occupied by an agent $j \in T_{t'}$ with $t' \neq t$. Note that both $|T_1| > |S|$ and $|T_2| > |S|$, thus there are agents from both types in $G[V(G) \setminus S]$. Next, we argue that $G[V(G) \setminus S]$ is connected. Let $Y = A$ if $S \subseteq A$ and $Y = B$ otherwise. It is easy to see that $G[V(G) \setminus Y]$ is connected (as argued for



(a) Case 1: Central vertex is unoccupied. (b) Case 2: Central vertex is occupied.

Figure 3.5: Illustration for Claim 1: In a jump-equilibrium, there can not exist an agent on a degree-one vertex with no friends in A or B.

$G[V(G) \setminus A]$ above and analogous for $G[V(G) \setminus B]$). The connected subgraph $G[Y \setminus S]$ consists of the remaining stars in $Y \setminus S$, where the central vertices form a clique. Note that there is at least one vertex $x \in V_{S'}$ in $Y \setminus S$, since by Lemma 3.3, we have assumed that there are at least two stubborn agents of each type. Furthermore, by Lemma 3.3, we have assumed that $x \in V_{S'}$ is adjacent to a vertex $y \in V(G') \setminus V_{S'}$ in $G[V(G) \setminus Y]$. Thus, as $G[V(G) \setminus Y]$ and $G[Y \setminus S]$ are each connected and connected by an edge, $G[V(G) \setminus S]$ is connected.
First, suppose that there is no unoccupied vertex in $V(G) \setminus S$, which implies that there need to be unoccupied vertices in S. Then, there are two adjacent agents $i' \in T_t$ and $j' \in T_{t'}$ in $G[V(G) \setminus S]$. It holds that $u_{j'}(\mathbf{v}) < 1$. Agent j' can increase its utility to 1 by jumping to any unoccupied vertex in S.
Therefore, there has to exist at least one unoccupied vertex in $G[V(G) \setminus S]$. If one of the unoccupied vertices in $G[V(G) \setminus S]$ is adjacent to an agent in T_t, then agent $i \in T_t$ can increase its utility by jumping to this vertex. Thus, the unoccupied vertices in $G[V(G) \setminus S]$ can only be adjacent to agents in $T_{t'}$ and unoccupied vertices. Since $G[V(G) \setminus S]$ is connected, there exists an unoccupied vertex w' that is adjacent to at least one agent in $T_{t'}$ on a vertex in $G[V(G) \setminus S]$. Note that agent j on the central vertex in S has $u_j(\mathbf{v}) < 1$, since $j \in T_{t'}$ is adjacent to $i \in T_t$. Hence, agent j can increase its utility by jumping to w', where j is only adjacent to friends. ■

Next, we prove that all vertices in A are occupied by agents from T_1.

Claim 2. *In every jump-equilibrium* $\mathbf{v}$*, A is fully occupied by agents from T_1.*

Proof. We prove this claim by splitting it into Claims 2.1 and 2.2. First, we prove that there are no agents from T_2 in A. Then, we prove that all vertices in A are occupied.

Claim 2.1. *In every jump-equilibrium* $\mathbf{v}$*, A only contains agents from T_1.*

Proof. Recall that by the construction, we have that $|T_1| \geq |A| = |X_1 \cup V_{S'_1}| \cdot (|T_2| - 1) = z \cdot (|T_2| - 1) = (s + |V(G')| + |M_2|) \cdot (|T_2| - 1)$. Thus, even if all vertices in $V(G) \setminus A = (V(G') \setminus V_{S'_1}) \cup M_2$ are occupied by agents from T_1, there are at least $s \cdot (|T_2| - 1)$ agents from T_1 in A. Since each star in A contains $|T_2| - 1$ vertices, there have to be agents

from T_1 on at least s stars. By Claim 1, it holds that at least s central vertices are occupied by agents from T_1.
Now suppose there are $x > 0$ agents from T_2 in A. We distinguish the following three cases based on the value of x (see Figure 3.6) and prove that such an assignment can not be a jump-equilibrium.

Case 1: First, assume there are $x < |T_2| - 1$ agents from T_2 in A. Recall Claim 1, which states that no agent on a degree-one vertex in A can have no adjacent friends in any jump-equilibrium. Thus, there exists an agent $i \in T_2$ on a central vertex in A. Let S be the set of vertices of the star which contains v_i. Since it holds that $x < |T_2| - 1 = |S|$, not all vertices in S can be occupied by agents from T_2. By Claim 1, these vertices have to be unoccupied. Summarizing, there exists an unoccupied degree-one vertex w adjacent to the central vertex occupied by i.
Next, we upper bound the utility of agent i. Note that $i \in T_2$ is adjacent to all other central vertices in A, of which at least $s = q \cdot (|T_2| + \Delta(G)) + 1$ are occupied by agents from T_1. It therefore holds that:

$$u_i(\mathbf{v}) \leq \frac{|T_2|}{q \cdot (|T_2| + \Delta(G)) + 1 + |T_2|} < \frac{1}{q}.$$

Since $x < |T_2|$ and $|T_1| \geq |A|$, there are agents from both types in $G[V(G) \setminus A]$. Furthermore, note that $G[V(G) \setminus A]$ is connected. Now consider a path between two arbitrary agents from T_1 and T_2 in $G[V(G) \setminus A]$. If there are two adjacent agents $i' \in T_1$ and $j' \in T_2$ on this path, then it holds that $u_{j'}(\mathbf{v}) < 1$ and jumping to w is profitable for j'. Thus, no such two agents can exist. However then, there exists an unoccupied vertex w' on the path that is adjacent to an agent in T_2. Note that $\deg_G(w') \leq \Delta(G') + |V(G')|^2 + |V_{S'_2}| = q$. Jumping to w' is profitable for agent i:

$$u_i(\mathbf{v}^{\mathbf{i} \to \mathbf{w}'}) \geq \frac{1}{q} > u_i(\mathbf{v}).$$

Case 2: Next, we consider the case where $x = |T_2|$. Since $2 \cdot (|T_2| - 1) > x > |T_2| - 1$, the agents from T_2 occupy vertices on at least two stars in A, but can not occupy all vertices of these stars. With Claim 1, there exists an unoccupied degree-one vertex w adjacent to a central vertex occupied by an agent from T_2. Now consider an agent $i \in T_2$ on another central vertex in A not adjacent to w. We have that $u_i(\mathbf{v}) < 1$, since i is adjacent to agents from T_1 on other central vertices. Then, agent i can increase her utility to 1 by jumping to w.

Case 3: Finally, we address the case where $x = |T_2| - 1$. Again, by Claim 1, at least one of the central vertices in A has to be occupied by an agent $i \in T_2$. Furthermore, if there are agents from T_2 on two or more stars, then there exists an unoccupied degree-one vertex adjacent to a central vertex occupied by an agent from T_2 and an agent from T_2 on another central vertex with utility less than 1. Analogous to Case 2, such an assignment can not be a jump-equilibrium. Therefore, the $|T_2| - 1$ agents from T_2 occupy all vertices of one star in A. Denote the set of vertices of this star by S.

All central vertices in A and B have to be occupied by Claim 1. Note that there is only one agent from T_2 outside of A. This agent can not occupy a degree-one vertex in B, since she would have no adjacent friends on such a vertex. If she occupies a central vertex, then there are agents from T_1 with no adjacent friends on degree-one vertices in this star (recall that at least three vertices of each star have to be occupied). Both possibilities contradict Claim 1. Hence, this agent occupies a vertex in $G[V(G') \setminus V_{S'}]$ and all central vertices in B are occupied by agents from T_2.
Next, we argue that both A and B are fully occupied. First, suppose there exists an unoccupied vertex w in A. By Claim 1, w has to be a degree-one vertex. Furthermore, since all $|T_2| - 1$ agents from T_2 in A fully occupy one star, w is adjacent to a central vertex occupied by an agent from T_1. Then, any agent from T_1 on a central vertex in A not adjacent to w can increase her utility to 1 by jumping to w. If there is an unoccupied degree-one vertex w in B, then this vertex is adjacent to a central vertex occupied by an agent from T_1 (recall that all central vertices in B are occupied by agents from T_1). Then again, any agent from T_1 on a central vertex in A has a profitable jump to w. Thus, both A and B are fully occupied and all λ unoccupied vertices are in $G[V(G') \setminus V_{S'}]$.
Now consider the central vertex v_i of the star in A occupied by the agents from T_2. Recall that we assumed that $\deg_{G'}(v) < \lambda$ for all $v \in V_{S'}$ in our input instance (by Lemma 3.3). In our construction, we only add edges within A to a vertex in A. Hence, as all λ unoccupied vertices need to be from $V(G') \setminus V_{S'}$, there exists an unoccupied vertex w in $G' - V_{S'}$ which is not adjacent to v_i. If w is adjacent to an agent in T_2, jumping to w is profitable for agent i. With an argument analogous to Case 1, we get that:

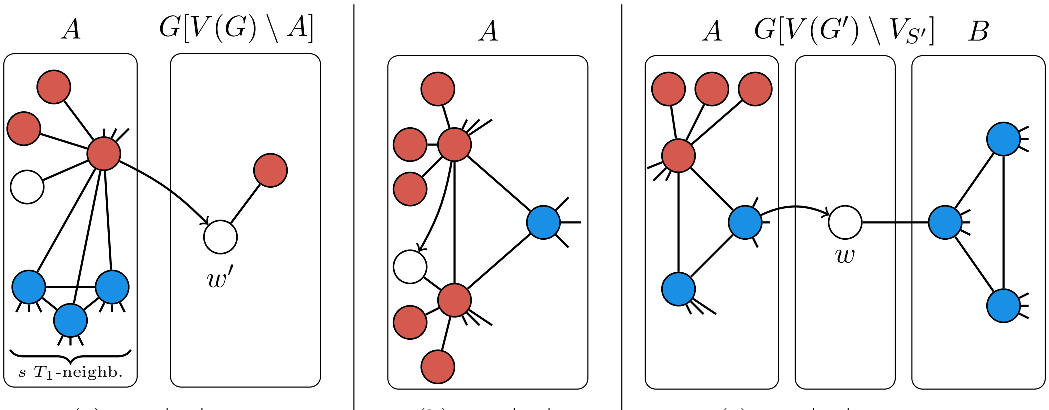
$$u_i(\mathbf{v}^{\mathbf{i} \to \mathbf{w}}) \geq \frac{1}{q} > u_i(\mathbf{v}).$$

Hence, w is not adjacent to any agent from T_2. Recall that (by Lemma 3.3) we assume that every vertex not occupied by a stubborn agent is adjacent to a stubborn agent of each type in our input instance. Thus, since $w \in V(G') \setminus V_{S'}$, the vertex w is adjacent to a central vertex in B, which is occupied by an agent from T_1. Then again, any agent from T_1 on a central vertex in A can increase her utility to 1 by jumping to w. This concludes Case 3. Since we have exhausted all possible cases, Claim 2.1 follows. ■

Next, we prove the second part of Claim 2 by proving that all vertices in A are occupied.

Claim 2.2. *In every jump-equilibrium* $\mathbf{v}$*, A is fully occupied.*

Proof. Suppose there exists an unoccupied vertex $v \in A$. Again, v has to be a degree-one vertex by Claim 1 and by Claim 1 and Claim 2.1 v needs to be adjacent to a vertex occupied by an agent from T_1. Note that $|T_2| > |V(G')|$, therefore there have to be agents from T_2 in B (as there are no agents from T_2 in A by Claim 2.1). By Claim 1, this implies that a central vertex in B is occupied by an agent from T_2. Since we also have that $|T_1| \geq |A|$, there are agents from T_1 outside of A (see Figure 3.7). If there is an agent from T_1 in B, there also exists an agent $i \in T_1$ on a central vertex in B. We have that $u_i(\mathbf{v}) < 1$, since i is adjacent to the central vertex occupied by an agent from T_2. However, agent i can then increase her utility to 1 by jumping to v. Hence, all remaining agents from T_1 not in A are in $V(G') \setminus V_{S'}$. Since all central vertices in B



(a) $x < |T_2| - 1$. (b) $x = |T_2|$. (c) $x = |T_2| - 1$.

Figure 3.6: Claim 2.1: If there are $x > 0$ agents from T_2 in A, then a profitable jump exists.

have to be occupied, there are agents from T_2 on all central vertices in B.
Now consider an arbitrary agent $i \in T_1$ in $V(G') \setminus V_{S'}$. As we assume that the given instance is regularized (by Lemma 3.3), i is adjacent to a central vertex in B, which is occupied by an agent from T_2. We thus have $u_i(\mathbf{v}) < 1$. Agent i can increase her utility to 1 by jumping to v, which completes the proof. ∎

With Claims 2.1 and 2.2, Claim 2 follows. ∎

Next, we prove the following analogous claim for B, where we show that all vertices in B are occupied by agents from T_2.

Claim 3. *In every jump-equilibrium* $\mathbf{v}$*,* B *is fully occupied by agents from* T_2*.*

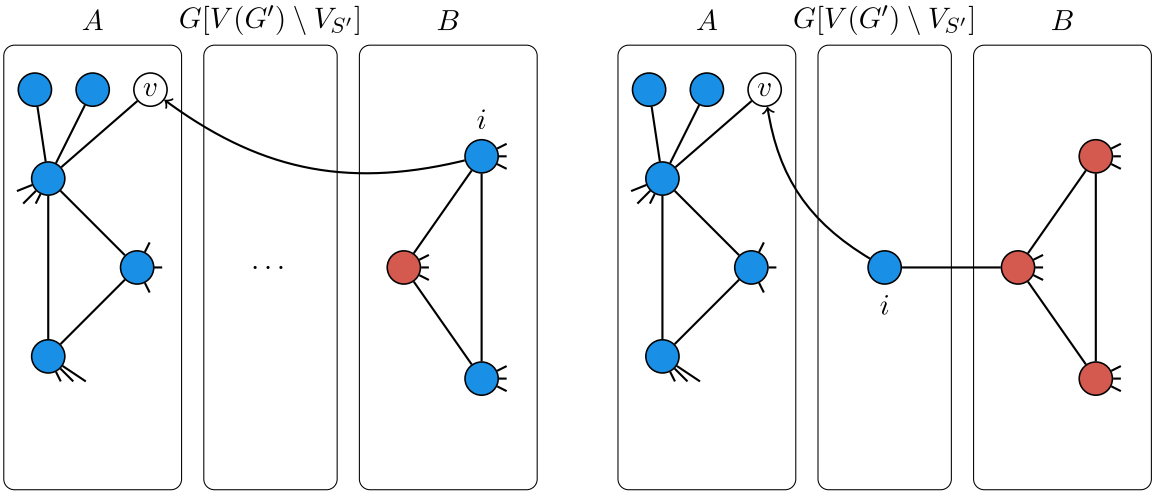
Proof. Again, we prove this claim by dividing it into Claims 3.1 and 3.2.

Claim 3.1. *In every jump-equilibrium* $\mathbf{v}$*,* B *only contains agents from* T_2*.*

Proof. Since A is fully occupied by agents from T_1 by Claim 2, there are $y := |T_1| - |A| < |V(G')|$ agents from T_1 outside of A. Suppose there are agents from T_1 in B. By Claim 1, there exists a central vertex $v \in B$ which is occupied by an agent from T_1. Furthermore, since $y < |V(G')|^2$ (recall that every star in B contains $|V(G')|^2 + 1$ vertices), there exists an unoccupied degree-one vertex w adjacent to v. Note that $|T_2| \geq |B|$. Since not all vertices in B are occupied by agents from T_2, there exists an agent $j \in T_2$ on a vertex in $V(G') \setminus V_{S'}$. This agent is adjacent to a central vertex in A. Let $i' \in T_1$ be the agent on this central vertex. We have $u_{i'}(\mathbf{v}) < 1$, since $i' \in T_1$ is adjacent to $j \in T_2$. Then, jumping to w is profitable for agent i', since $u_{i'}(\mathbf{v}) < 1 = u_{i'}(\mathbf{v}^{\mathbf{i'} \to \mathbf{w}})$. ∎

Claim 3.2. *In every jump-equilibrium* $\mathbf{v}$*,* B *is fully occupied.*

Proof. Suppose there exists an unoccupied vertex $v \in B$. By Claim 2, the vertex v has to be a degree-one vertex. Since all central vertices have to be occupied and B only contains agents from T_2 (by Claim 3.1), the vertex v is adjacent to a central vertex



(a) Agents from T_2 in B.

(b) No agents from T_2 in B.

Figure 3.7: Claim 2.2: If not all vertices in A are occupied, there exists an agent from T_2 outside of A that has a profitable jump.

occupied by an agent from T_2. It holds that $|T_2| \geq |B|$. Since at least one vertex in B is unoccupied, there exists an agent $i \in T_2$ on a vertex in $V(G') \setminus V_{S'}$. We have that $u_i(\mathbf{v}) < 1$, since i is adjacent to a central vertex in A, which is occupied by an agent from T_1 by Claim 2. Agent i can increase her utility to 1 by jumping to v. ■

Claim 3 follows from Claims 3.1 and 3.2. ■

Finally, we prove that the input game with stubborn agents admits a jump-equilibrium if and only if the constructed game admits a jump-equilibrium.
($\Rightarrow$): First, assume there exists a jump-equilibrium $\mathbf{v}'$ for the Schelling game with stubborn agents. Note that the vertices in $V_{S'_1}$ and $V_{S'_2}$ are occupied by stubborn agents from T'_1 and T'_2. We define an assignment $\mathbf{v}$ for the Schelling game without stubborn agents and prove that it is a jump-equilibrium. The occupied vertices in $V \cap V(G')$ are occupied by agents of the same type as the agents in $\mathbf{v}'$. If a vertex is unoccupied in $\mathbf{v}'$, then it is also unoccupied in $\mathbf{v}$. The vertices in X_1 are occupied by agents from T_1. For $t \in \{1, 2\}$, the added degree-one vertices in M_t are occupied by agents from T_t. Hence, the vertices in $A = M_1 \cup V_{S'_1} \cup X_1$ are all occupied by agents from T_1 and the vertices in $B = M_2 \cup V_{S'_2}$ are all occupied by agents from T_2.
Next, we prove that $\mathbf{v}$ is a jump-equilibrium on G by showing that no profitable jump exists. We first observe that the utility of an agent i on a vertex $w \in M_1 \cup X_1$ is $u_i(\mathbf{v}) = 1$, since $N_G(w) \subseteq A$ and all agents in A are friends of i. The same holds analogously for any agent on a vertex in M_2. Therefore, the agents on vertices in $M_1 \cup M_2 \cup X_1$ do not want to jump to an unoccupied vertex. Furthermore, note that all unoccupied vertices are in $V(G') \setminus V_{S'}$ and that the neighborhood of all vertices in $V(G') \setminus V_{S'}$ is identical for $\mathbf{v}$ and $\mathbf{v}'$. Therefore, no agent on a vertex in $V(G') \setminus V_{S'}$ can have a profitable jump, since this jump would then also be profitable on G', contradicting that $\mathbf{v}'$ is a jump-equilibrium. Thus, a profitable jump can only exist for an agent i on a vertex $v_i \in V_{S'}$

to an unoccupied vertex $v \in V(G') \setminus V_{S'}$.

Let $Y = V_{S'_1} \cup X_1$ if $v_i \in V_{S'_1}$ and $Y = V_{S'_2}$ otherwise. Denote the number of degree-one neighbors adjacent to v_i added in the construction of G by x. By construction of G, it holds that $x > \Delta(G)^2$. The agent i is, among others, adjacent to the vertices in $Y \setminus \{v_i\}$ and the $x > \Delta(G')^2$ degree-one neighbors, which, by construction of $\mathbf{v}$, are all occupied by friends. It holds that $\deg_G(v_i) \leq x + |Y \setminus \{v_i\}| + \Delta(G')$. Thus, the utility of agent i on v_i is:

$$u_i^G(\mathbf{v}) \geq \frac{x + |Y \setminus \{v_i\}|}{x + |Y \setminus \{v_i\}| + \Delta(G')} > \frac{x}{x + \Delta(G')} > \frac{\Delta(G')^2}{\Delta(G')^2 + \Delta(G')} = \frac{\Delta(G')}{\Delta(G') + 1}.$$

Now consider an unoccupied vertex $v \in V(G') \setminus V_{S'}$. Recall that we assume that vertex not occupied by a stubborn agent is adjacent to at least one stubborn agent of each type in the input game. Since $v \in V(G') \setminus V_{S'}$, the vertex is adjacent to agents from both T_1 and T_2 in A and B, respectively. By construction of G, the neighborhood of v is identical in G and G'. We thus have that $\deg_G(v) = \deg_{G'}(v) \leq \Delta(G')$. Since at least one agent adjacent to v is not a friend of i, the jump can not be profitable:

$$u_i^G(\mathbf{v}^{\mathbf{i}\to\mathbf{v}}) \leq \frac{\Delta(G') - 1}{\Delta(G')} < \frac{\Delta(G')}{\Delta(G') + 1} \leq u_i^G(\mathbf{v}).$$

To sum up, no profitable jump is possible and $\mathbf{v}$ is a jump-equilibrium for the constructed Schelling game.

($\Leftarrow$): Conversely, assume there exists a jump-equilibrium $\mathbf{v}$ for the constructed game. We define an assignment $\mathbf{v}'$ on G' and prove that it is a jump-equilibrium for the given Schelling game with stubborn agents. In $\mathbf{v}'$, an occupied vertex $v \in V(G') \setminus V_{S'}$ is occupied by a strategic agent of the same type as the agent on v in $\mathbf{v}$. If a vertex is unoccupied in $\mathbf{v}$, then it is also unoccupied in $\mathbf{v}'$. The vertices in $V_{S'}$ have to be occupied by the respective stubborn agents. Note that by Claims 2 and 3, in $\mathbf{v}$, the vertices in $V_{S'_1} \subseteq A$ and $V_{S'_2} \subseteq B$ have to be occupied by agents from T_1 and T_2, respectively. Thus, in $\mathbf{v}'$, all occupied vertices are occupied by agents of the same type as in $\mathbf{v}$.

Next, we prove that $\mathbf{v}'$ is a jump-equilibrium on G'. Since the agents on vertices in $V_{S'}$ are stubborn, only agents on vertices in $V \setminus V(G')$ can be involved in a profitable swap. However, all unoccupied vertices are in $V(G') \setminus V_{S'}$ and the neighborhood of an agent on such a vertex is the same for $\mathbf{v}$ and $\mathbf{v}'$. A profitable swap for an agent on a vertex in $V(G') \setminus V_{S'}$ in $\mathbf{v}'$ would therefore also be a profitable jump in $\mathbf{v}$, thereby contradicting that $\mathbf{v}$ is a jump-equilibrium. Thus, no profitable jump is possible and $\mathbf{v}'$ is a jump-equilibrium on G', which completes the proof. □

Chapter 4

Robustness of Equilibria

In this chapter, we introduce the perspective of *robustness* for analyzing equilibria. To motivate this perspective, we revisit Schelling's original motivation of modeling segregation in cities. In such a setting, vertices would resemble locations and edges could resemble roads or bus routes. Assume that the residents of a city are positioned in an equilibrium such that no resident has an incentive to move. Now, suppose there are some roadworks in this city that make some roads (i.e., edges of the topology) unavailable. We might now be interested in the following questions: Does our initially stable placement of the residents remain stable? Which types of edges can potentially affect the stability of an equilibrium? Are there some cases for which we can guarantee that an equilibrium remains stable if we only remove a certain number of edges? In order to find answers to these questions, we begin by formalizing this view as the robustness of an equilibrium. We restrict our analysis to the deletion of edges and capture this notion in a worst-case measure: We say that an equilibrium is *r-robust* if it remains stable under the deletion of any set of at most r edges.

Definition 4.1. A (local) jump- or swap-equilibrium $\mathbf{v}$ is called r-robust for some $r \in \mathbb{N}_0$ if $\mathbf{v}$ is also a (local) jump- or swap-equilibrium for the topology $G - S$ for all $S \subseteq E(G)$ with $|S| \leq r$. The *robustness* of $\mathbf{v}$ is the largest $r \leq |E(G)|$ for which $\mathbf{v}$ is r-robust. We denote the robustness of $\mathbf{v}$ as $\mathrm{rob}(\mathbf{v})$.

The robustness of an equilibrium can also be interpreted as providing a budget for modifications (deletion of edges) such that the equilibrium is guaranteed to remain stable. Regarding Definition 4.1, note that we can make an assignment $\mathbf{v}$ that becomes unstable after deleting a certain set of edges stable again by deleting all edges $S = E(G)$, as we then only have isolated vertices and any assignment is an equilibrium. Thus, we require $\mathbf{v}$ to be stable for all S with $|S| \leq r$ and not only for all S with $|S| = r$.
As there might exist multiple equilibria for a given Schelling game, we are naturally interested in quantifying how much their robustness may vary. We formalize this in the notion of the *robustness-ratio*. In the following definition, we define this notion first for swap-equilibria and then extend it to the other types of equilibria.

Definition 4.2. The robustness-ratio for swap-equilibria for a Schelling game I on topology G is defined as $\frac{\max_{\mathbf{v} \in \mathrm{SE}(I)} \mathrm{rob}(\mathbf{v})+1}{\min_{\mathbf{v} \in \mathrm{SE}(I)} \mathrm{rob}(\mathbf{v})+1}$, where $\mathrm{SE}(I)$ is the set of all swap-equilibria in I.

Analogously, for local swap-equilibria and (local) jump-equilibria, the robustness-ratio is defined as the same ratio, where $\mathrm{SE}(I)$ is replaced by the respective set of equilibria.

Note that we add one to both the numerator and the denominator, as the robustness of an equilibrium can be zero. The robustness-ratio might also be of interest from a practical perspective. If the gap is low (in particular, if the gap is equal to one), then one might be satisfied with finding an arbitrary equilibrium. On the other hand, a high robustness-ratio might justify putting more effort into finding a more robust equilibrium. In the following section, we mostly focus on analyzing the robustness of swap-equilibria. However, we also shortly study local swap-equilibria (see Section 4.1.3) and jump-equilibria (see Section 4.2).

4.1 Swap-Equilibria

This section is divided into two parts. First, we make some general observations about the robustness of swap-equilibria. Then, we study the influence of the structure of the topology on the robustness of swap-equilibria and show bounds for the robustness of swap-equilibria on graphs from different graph classes.

4.1.1 General Observations

First, we study which types of edges can make a swap-equilibrium unstable by deleting them. Interestingly, we find that deleting edges between agents of different types can never make a swap-equilibrium unstable.

Lemma 4.3. *Let $\mathbf{v}$ be a swap-equilibrium for a Schelling game on a topology G. Let $S \subseteq E(G)$ be a set of edges such that all edges $\{v_i, v_j\} \in S$ only connect agents of different types, that is, $i \in T_l$ and $j \in T_{l'}$ with $l \neq l'$. Then, $\mathbf{v}$ is also a swap-equilibrium on $G - S$.*

Proof. Consider the game on topology $G-S$ and assume for the sake of contradiction that $\mathbf{v}$ is not a swap-equilibrium on $G - S$, that is, there exist agents $i \in T_1$ and $j \in T_2$ that want to swap. We therefore have $u_i^{G-S}(\mathbf{v}) < u_i^{G-S}(\mathbf{v^{i\leftrightarrow j}})$ and $u_j^{G-S}(\mathbf{v}) < u_j^{G-S}(\mathbf{v^{i\leftrightarrow j}})$. Since $\mathbf{v}$ is a swap-equilibrium on the original topology G, swapping i and j is not profitable for at least one of i or j on G. Without loss of generality, assume that the swap is not profitable for i, that is, it holds that $u_i^G(\mathbf{v}) \geq u_i^G(\mathbf{v^{i\leftrightarrow j}})$. As all edges in S are between agents of different types, no edges to friends of i are deleted in $G - S$ and it therefore holds that $u_i^G(\mathbf{v}) \leq u_i^{G-S}(\mathbf{v})$. Hence, we have:

$$u_i^G(\mathbf{v^{i\leftrightarrow j}}) \leq u_i^G(\mathbf{v}) \leq u_i^{G-S}(\mathbf{v}) < u_i^{G-S}(\mathbf{v^{i\leftrightarrow j}}) \qquad (\star)$$

Consider the vertex v_j that is occupied by agent $j \in T_2$ in assignment $\mathbf{v}$. Since all edges in S are between agents of different types, the only edges incident to v_j that have been deleted in $G - S$ are edges to agents in T_1. Therefore, the utility of agent $i \in T_1$ on v_j is lower on topology $G - S$, formally $u_i^{G-S}(\mathbf{v^{i\leftrightarrow j}}) \leq u_i^G(\mathbf{v^{i\leftrightarrow j}})$. This contradicts Equation ($\star$) and completes the proof. □

Next, we derive a useful corollary from the lemma above. By this corollary, given a set of edges S such that a swap-equilibrium $\mathbf{v}$ is unstable on $G - S$, it holds that $\mathbf{v}$ is also not a swap-equilibrium on $G - S'$ where $S' \subseteq S$ is the subset of edges from S that connect agents of the same type. Thus, this corollary allows us to only consider edges between agents of the same type when analyzing the robustness of swap-equilibria.

Corollary 4.4. *Let* $\mathbf{v}$ *be a swap-equilibrium for some Schelling game on topology* G*. Let* $S \subseteq E(G)$ *be a set of edges such that* $\mathbf{v}$ *is not a swap-equilibrium on* $G - S$*. Then,* $\mathbf{v}$ *is also not a swap-equilibrium on* $G - S'$*, where* $S' \subseteq S$ *is the subset of edges from* S *that connect agents of the same type, formally,* $S' = \{\{v_i, v_j\} \in S \mid i, j \in T_l \text{ for some } l \in \{1, 2\}\}$*.*

Proof. Assume for the sake of contradiction, that $\mathbf{v}$ is a swap-equilibrium assignment on $G - S'$. Notice that $X = S \setminus S'$ only contains edges between agents of different types. Hence, we can apply Lemma 4.3 and get that $\mathbf{v}$ is a swap-equilibrium on $G - S' - X = G - S$. This contradicts that, by definition of S, $\mathbf{v}$ is not a swap-equilibrium on $G - S$. Thus, $\mathbf{v}$ is also not a swap-equilibrium on $G - S'$. □

Regarding Definition 4.1, we discussed above that we can make a swap-equilibrium that is made unstable by deleting a set of edges stable again by deleting additional edges. In fact, if we delete all edges, any assignment is a swap-equilibrium on the topology that only consists of isolated vertices. In contrast to this, we show below that by only deleting additional edges between agents of the same type, a swap-equilibrium can not be made stable again.

Proposition 4.5. *Let* $\mathbf{v}$ *be a swap-equilibrium for some Schelling game on topology* G*. Let* $S \subseteq E(G)$ *be a set of edges such that* $\mathbf{v}$ *is not a swap-equilibrium on* $G - S$*. Then, for any set* $A \subseteq \{\{v_i, v_j\} \in E(G) \mid i, j \in T_l \text{ for some } l \in \{1, 2\}\}$ *of edges between agents of the same type,* $\mathbf{v}$ *is also not a swap-equilibrium on* $G - (S \cup A)$*.*

Proof. Consider a set of edges A that only contains edges between agents of the same type. Let $i \in T_1$ and $j \in T_2$ be a pair of agents that has a profitable swap on $G - S$. We will now argue that the swap is also profitable on $G - (S \cup A)$. Consider the vertex v_i that is occupied by agent i. Since A only contains edges between agents of the same type, we only delete edges to friends of i in the neighborhood of v_i. Hence, the utility of agent i before swapping with j on topology $G - (S \cup A)$ is at most as high as the utility on $G - S$. In the neighborhood of v_j, we only delete edges to agents in T_2. Therefore, the utility of i after swapping to v_j on $G - (S \cup A)$ has to be at least as high as on $G - S$. By symmetry, the same holds for agent j. Hence, the swap is profitable and $\mathbf{v}$ is also not a swap-equilibrium on $G - (S \cup A)$. □

We conclude our general observations with the lemma below, which describes a simple condition that directly implies that the swap-equilibrium in question has robustness zero. Concretely, we capture the situation depicted in Figure 4.1, where an agent $i \in T_1$ with only one adjacent friend and another agent $j \in T_2$ with utility less than 1 exist.

Lemma 4.6. *A swap-equilibrium* $\mathbf{v}$ *has robustness zero if there exists some agent* $i \in T_1$ *on vertex* v_i *with* $a_i = |N_i(\mathbf{v}) \cap F_i| = 1$ *and* $|N_i(\mathbf{v})| > 1$ *and another agent* $j \in T_2$ *on vertex* w *with* $u_j(\mathbf{v}) < 1$*,* $|N_j(\mathbf{v})| \geq 1$ *and* $w \notin N_G(v_i)$*.*

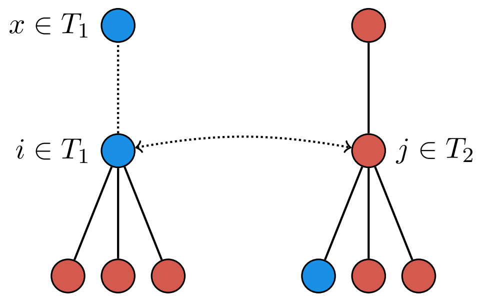


Figure 4.1: Lemma 4.6: There exists an agent $i \in T_1$ with one friend and an agent $j \in T_2$ with $u_j(\mathbf{v}) < 1$. After deleting $\{v_i, v_x\}$, agents i and j want to swap.

Proof. Let $\{x\} = N_i(\mathbf{v}) \cap F_i$. After deleting $\{v_i, v_x\}$, agents i and j want to swap:

$$u_i(\mathbf{v}) = 0 < u_i(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}}), \text{ since } w \text{ has neighbors in } T_1.$$
$$u_j(\mathbf{v}) < 1 = u_j(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}}), \text{ since all neighbors of } v_i \text{ are in } T_2.$$

□

4.1.2 Topological Influence on Robustness

In this section, we analyze the influence of the structure of the topology on the robustness of swap-equilibria. We prove upper and lower bounds on the robustness of swap-equilibria on topologies from various graph classes and find that the robustness of swap-equilibria heavily depends on the structure of the underlying topology. We first analyze cliques, cycles, paths and grids and find that on all (large enough) topologies from these classes there exist equilibria that can be made unstable by deleting a single edge. For paths, we observe that the robustness-ratio between the most and least robust swap-equilibrium can be arbitrarily large. Finally, with star-constellation graphs, we provide an infinite class of graphs where every swap-equilibrium has robustness larger than 0.

Cliques

For swap-equilibria on cliques, we show that deleting any edge between two agents of the same type makes every swap-equilibrium unstable.

Proposition 4.7. *For Schelling games where the topology is a clique, every swap-equilibrium has robustness of* 0*.*

Proof. Fix some Schelling game with $|T_1| = a$, $|T_2| = b$, and a topology G which is a clique of size $a + b \geq 4$. Let $\mathbf{v}$ be a swap-equilibrium. Consider some agent $i \in T_1$ on vertex v. Since G is fully connected, it holds that $u_i(\mathbf{v}) = \frac{a-1}{a+b-1}$ and there exists an edge $e = \{v_i, v_j\}$ to some other agent $j \in T_1$. All agents $l \in T_2$ have $u_l(\mathbf{v}) = \frac{b-1}{a+b-1}$.

After deleting e, we have $u_i(\mathbf{v}) = \frac{a-2}{a+b-2}$. Now, i and some agent $l \in T_2$ want to swap:

$$u_i(\mathbf{v}) = \frac{a-2}{a+b-2} < \frac{a-1}{a+b-1} = u_i(\mathbf{v^{i \leftrightarrow l}})$$
$$u_l(\mathbf{v}) = \frac{b-1}{a+b-1} < \frac{b-1}{a+b-2} = u_l(\mathbf{v^{i \leftrightarrow l}})$$

Summarizing, $\mathbf{v}$ is not a swap-equilibrium for the topology $G - \{e\}$, which completes the proof. □

Cycles

Turning to cycles, we find that in every swap-equilibrium there exists an agent from one type that has only one adjacent friend and an agent from the opposite type that has utility less than 1, as depicted in Figure 4.2. Thus, we can apply Lemma 4.6 and conclude that every swap-equilibrium has robustness zero.

Proposition 4.8. *For Schelling games where the topology G is a cycle, every swap-equilibrium has robustness zero.*

Proof. Let $\mathbf{v}$ be a swap-equilibrium. First, we show that there always exists a sequence of $l \geq 2$ vertices $w_1, \ldots, w_l$ occupied by agents of the same type that induce a path in G, where the agents at both ends of the path on w_1 and w_l are each adjacent to a different agent of the other type (see Figure 4.2). Assume that no two agents of the same type are neighbors, hence we have $u_i(\mathbf{v}) = 0$ for all $i \in N$. Then, any swap of two agents $i \in T_1$ and $j \in T_2$ is profitable: Since both neighbors of i and j respectively are of the other type, after swapping, both agents have at least one friend in their neighborhood. It therefore holds that $u_i(\mathbf{v}) = 0 < u_i(\mathbf{v^{i \leftrightarrow j}})$ and $u_j(\mathbf{v}) = 0 < u_j(\mathbf{v^{i \leftrightarrow j}})$. Hence, in any equilibrium assignment $\mathbf{v}$, there exists a sequence as defined above with $l \geq 2$. Now consider a maximal path $w_1, \ldots, w_l$ of vertices occupied by agents of the same type. Without loss of generality, assume that all agents are from T_1. Since we have at least two agents from T_2, the agents on w_l and w_1 need to be adjacent to agents $j \in T_2$ and $j' \in T_2$ with $j \neq j'$.
It follows that there always exists agent $i \in T_1$ with only one adjacent friend and agent $j \in T_2$ that is outside of the neighborhood of i and has utility of less than 1. We can therefore apply Lemma 4.6 and conclude that $\mathbf{v}$ has robustness zero. □

Paths

Next, we analyze swap-equilibria on paths. Interestingly, in contrast to cycles, we find that there always exists a swap-equilibrium with robustness $|E(G)|$ (i.e., a swap-equilibrium that remains stable after deleting any set of edges). The reason for this is that on a path, we can always position the agents such that there exists only one edge between agents of different types, yielding a swap-equilibrium with robustness $|E(G)|$ (depicted in Figure 4.3a). This is not possible on a cycle.

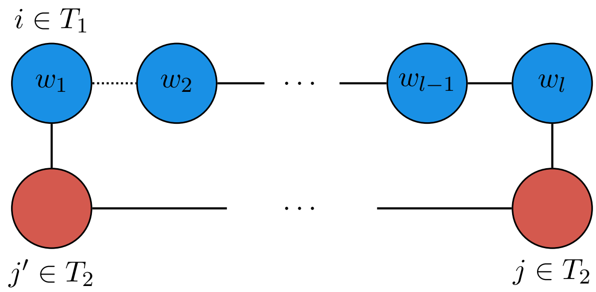


Figure 4.2: Sequence of vertices $w_1, \ldots, w_l$ on the cycle considered in Proposition 4.8. By Lemma 4.6, the agents $i \in T_1$ and $j \in T_2$ want to swap after deleting $\{w_1, w_2\} \in E(G)$.

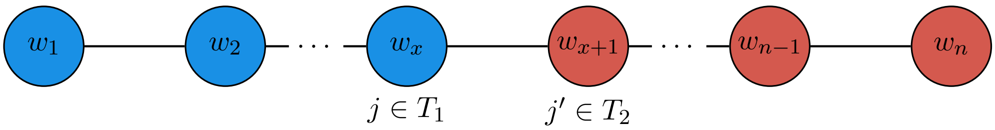
Proposition 4.9. *For Schelling games where the topology is a path of length $n = |T_1| + |T_2|$, there exists a swap-equilibrium with robustness $|E(G)|$.*

Proof. Fix some 2-swap game with $|T_1| = x$, $|T_2| = y$ and $x, y \geq 2$, and a topology G which is a path on vertices $(w_1, w_2, \ldots, w_n)$ with $n = x + y \geq 4$. We define an assignment $\mathbf{v}$ (see Figure 4.3a) as follows: The x agents from T_1 occupy the first x vertices $w_1, \ldots, w_x$ and the agents of type T_2 are placed on the remaining y vertices $w_{x+1}, \ldots, w_n$. The assignment $\mathbf{v}$ is a swap-equilibrium, since only the agents on w_x and w_{x+1} have utility less than 1 and swapping these two agents results in a utility of zero for both agents. We now show that $\mathbf{v}$ is $|E(G)|$-robust:
Let $S \subseteq E(G)$ be any set of edges and consider the topology $G - S$. Note that an agent i on vertex w_l, $l \neq x$ and $l \neq x+1$, only has neighbors of the same type in G. Hence, for the topology $G - S$, we have $u_i(\mathbf{v}) = 1$ if i has remaining neighbors and $u_i(\mathbf{v}) = 0$ if i occupies an isolated vertex. Therefore, only the agents j on vertex w_x and j' on w_{x+1} can be involved in a swap. Swapping j and j' results in $N_j(\mathbf{v}^{\mathbf{j} \leftrightarrow \mathbf{j}'}) \cap F_j = N_{j'}(\mathbf{v}^{\mathbf{j} \leftrightarrow \mathbf{j}'}) \cap F_{j'} = \emptyset$, since in G the vertex $w_x = v_{j'}^{j \leftrightarrow j'}$ is only connected to an agent from T_1 on w_{x-1} and to $w_{x+1} = v_j^{j \leftrightarrow j'}$ with $j \in T_1$, the same holds for w_{x+1}. Thus, the utility of both agents cannot be strictly greater and no swap is possible. □

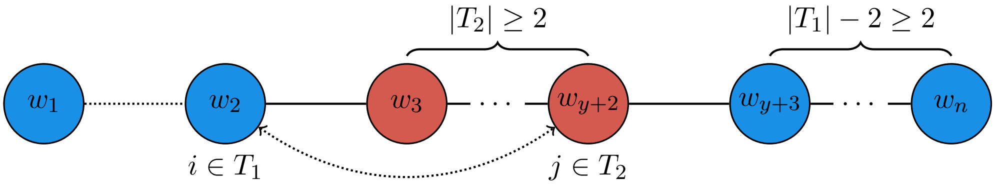
However, by placing the agents such that there are two agents from each type that are adjacent to agents from the opposite type (as shown in Figure 4.3b), we can construct a swap-equilibrium with robustness zero in a Schelling game on a path with sufficiently many agents.

Proposition 4.10. *For Schelling games where the topology is a path and it holds that $|T_1| \geq 4$ or $|T_2| \geq 4$, there exists a swap-equilibrium with robustness zero.*

Proof. Let $x = |T_1|$, $y = |T_2|$, and $n = x + y$. Assume without loss of generality that $x \geq 4$. The topology G is a path on vertices $w_1, \ldots, w_n$. We define the following assignment $\mathbf{v}$ (shown in Figure 4.3b). The vertices w_1 and w_2 are occupied by two agents from T_1. The $y \geq 2$ agents from T_2 are positioned on the vertices w_3 to w_{y+2} and the remaining $x - 2 \geq 2$ agents from T_1 are placed on the vertices w_{y+3} to w_n.
Since, by definition of $\mathbf{v}$, every agent has at least one friend and it holds that $\Delta(G) = 2$, we have $u_i(\mathbf{v}) \geq \frac{1}{2}$ for all agents $i \in N$. We also observe that any agent has at most



(a) The swap-equilibrium with robustness $|E(G)|$ constructed in Proposition 4.9. Note that only agents j and j' have neighbors of a different type.



(b) The swap-equilibrium with robustness zero from Proposition 4.10. After deleting $\{w_1, w_2\} \in E(G)$, swapping i and j is profitable.

Figure 4.3: Two swap-equilibria on paths with robustness zero and $|E(G)|$. This shows that the gap between the robustness of different swap-equilibria on a fixed topology can be arbitrarily large.

one neighbor of the other type. Therefore, by swapping agents $i \in T_1$ and $j \in T_2$ we get at most $u_i(\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{j}}) = \frac{1}{2} \leq u_i(\mathbf{v})$. Hence, no profitable swap is possible and $\mathbf{v}$ is a swap-equilibrium.
To show that $\mathbf{v}$ has robustness zero, observe that there exists an agent $i \in T_1$ on vertex w_2 with only one adjacent friend and an agent $j \in T_2$ on vertex w_{y+2} (outside of the neighborhood of v_i) with utility less than 1. Thus, with Lemma 4.6, it follows that $\mathbf{v}$ has robustness zero. □

Moreover, this shows that every Schelling game on a path with sufficiently many agents from both types has robustness-ratio $|E(G)| + 1$. Hence, the robustness-ratio can become arbitrarily large. As, from a practical perspective, more robust equilibria might be desirable in many settings, this raises the question whether one can efficiently check the robustness of an equilibrium or decide the existence of equilibria with a certain robustness. We study these computational aspects in Section 4.3.

Grids

Next, we turn to grids, which are one of the graph classes which have been most often considered in Schelling's original model. We first prove that every swap-equilibrium on a grid has robustness of at most 1. To prove this, we define the concept of *frames* of a grid (this notion also appears in the analysis of the price of anarchy of swap-equilibria on grids by Bilò et al. [Bil+20]). Let G be an $(x \times y)$-grid with $V(G) = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a \leq x, b \leq y\}$. We refer to the set $B(G)$ of vertices in the top row, bottom row, left column and right column as *border vertices* (formally, $B(G) = \{(a, b) \in V(G) \mid a \in \{1, x\} \text{ or } b \in \{1, y\}\}$). The first frame F_1 of G is the set of border vertices. The second frame F_2 of G is the set of border vertices of the grid that results from deleting the first frame from G. Further, for all $i > 1$, the frame F_i of G is the set of border vertices of the grid that results from

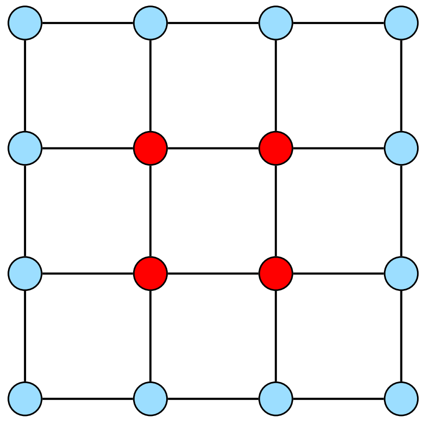


Figure 4.4: The two frames of a (4×4)-grid. The first frame F_1 is colored in a light blue and the second frame F_2 is colored in red.

deleting the frames $F_1, \dots, F_{i-1}$ from G. An example for the frames of a (4×4)-grid is given in Figure 4.4.

Theorem 4.11. *Let* $\mathbf{v}$ *be a swap-equilibrium on a grid* G. *Then,* $\mathbf{v}$ *has robustness at most one.*

Proof. For the sake of contradiction, assume that there exists a swap-equilibrium $\mathbf{v}$ which is 2-robust. By induction over the frames of G, we show that all vertices have to be occupied by agents from the same type in $\mathbf{v}$, which contradicts that we have at least two agents from both types.

Base case: First, consider the frame F_1 and assume that there are agents from both types. Then, there exist two adjacent agents $i \in T_1$ and $j \in T_2$. Since G is a grid, there exist two other adjacent agents $i' \in T_{t_1}$ and $j' \in T_{t_2}$ in G with $t_1, t_2 \in \{1, 2\}$ such that i' is adjacent to i and j' is adjacent to j. First, suppose that $t_1 \neq t_2$. Note that the (at most two) adjacent friends of i can only be on $v_{i'}$ and, if it exists, the other adjacent vertex $u \neq v_j$ of v_i on F_1. If we delete the edges to $v_{i'}$ and u (if it exists), then we have $u_i(\mathbf{v}) = 0$. If $t_1 = 1$, then swapping i and j' is profitable, since $u_i(\mathbf{v}) = 0 < u_i(\mathbf{v}^{i \leftrightarrow j'})$ and $u_{j'}(\mathbf{v}) < 1 = u_{j'}(\mathbf{v}^{i \leftrightarrow j'})$. Otherwise, if $t_1 = 2$, swapping i and i' is profitable. This contradicts that $\mathbf{v}$ has robustness 2.
Therefore, it has to hold that $t_1 = t_2$. By symmetry, assume that $t_1 = t_2 = 1$. Then, agent j is adjacent to two agents of the opposite type (i and j'). Consider another agent $x \in T_2$ with $x \neq j$ and an arbitrary agent from T_1. Since G is a grid, there exists a path from x to this agent that does not go through v_j. On this path, there exists an agent $x' \in T_2$ with $x' \neq j$ that is adjacent to an agent $y \in T_1$. We distinguish the following three cases and prove that in every case, there exists a profitable swap after deleting two edges.

Case 1: First, suppose that $y = i$. Note that in this case, both i and j are adjacent to at most one friend, since they both have at most degree three and are each adjacent to two agents of the opposite type. Thus, after deleting the at most two edges to their respective adjacent friends, we have that $u_i(\mathbf{v}) = 0$ and $u_j(\mathbf{v}) = 0$. Swapping i and j is profitable, since v_i is adjacent to agent $x' \in T_2$ and v_j is adjacent to $j' \in T_1$.

Case 2: Second, suppose that $y = j'$. Note that agent j has at most one adjacent friend. Thus, after deleting at most one edge, we have that $u_j(\mathbf{v}) = 0$. Additionally, after deleting the edge between $j' = y$ and i', we have that $u_{j'}(\mathbf{v}) \leq \frac{1}{3}$, since j' has at most three remaining neighbors and two of those agents are of the opposite type (x' and j). Then, swapping j and j' is profitable. It holds that $u_j(\mathbf{v}) = 0 < u_j(\mathbf{v}^{j \leftrightarrow j'})$ and $u_{j'}(\mathbf{v}) \leq \frac{1}{3} < \frac{1}{2} \leq u_{j'}(\mathbf{v}^{j \leftrightarrow j'})$.

Case 3: Finally, assume that $y \neq i$ and $y \neq j'$. Again, since j has at most one adjacent friend, after deleting at most one edge, we have that $u_j(\mathbf{v}) = 0$. Then, swapping j and y is profitable, since $u_j(\mathbf{v}) = 0 < u_j(\mathbf{v}^{j \leftrightarrow y})$ and $u_y(\mathbf{v}) < 1 = u_y(\mathbf{v}^{j \leftrightarrow y})$.

Since we have exhausted all possible cases, it follows that F_1 is fully occupied by agents from only one type T_t with $t \in \{1, 2\}$.

Induction step: Assume that it holds for some $i \in \mathbb{N}$, that all F_j with $j \leq i$ are occupied only by agents from T_t.
We now show that then F_{i+1} also needs to be occupied only by agents from T_t. We do so by showing that if there are agents from the other type $T_{t'}$ in F_{i+1}, then there exists a profitable swap after deleting at most two edges. Without loss of generality, assume that $|F_{i+1}| \geq 2$. If $|F_{i+1}| < 2$, then we have already reached a contradiction, since all other layers are occupied by agents from one type T_t only, but there are at least two agents from the other type $T_{t'}$ that have to be positioned.
First, suppose that there are only agents from $T_{t'}$ in F_{i+1}. Let $i \in T_{t'}$ be the agent in the bottom-left corner of F_{i+1}. Formally, let $i \in T_{t'}$ with $v_i = (a, b) \in F_{i+1}$ such that it holds for all other $(a', b') \in F_{i+1}$ that $a' \geq a$ and $b' \geq b$. Note that i is adjacent to two agents of the other type in F_i and has at most two adjacent friends. Thus, after deleting at most two edges, we have that $u_i(\mathbf{v}) = 0$. Now consider another agent $i' \in T_{t'}$ in F_{i+1}. Agent i' is adjacent to an agent $j \in T_t$ in F_i. It holds that $u_j(\mathbf{v}) < 1$. Then, swapping i and j is profitable, since $u_j(\mathbf{v}) < 1 = u_j(\mathbf{v}^{i \leftrightarrow j})$ and $u_i(\mathbf{v}) = 0 < u_i(\mathbf{v}^{i \leftrightarrow j})$.
Therefore, there have to be agents from T_t in F_{i+1}. Next, suppose that there are at least two agents from $T_{t'}$ in F_{i+1}. From this it follows that there exists an agent $i \in T_{t'}$ which is adjacent to an agent $j \in T_t$ in F_{i+1}, and that there exists a different agent $i' \in T_{t'}$ in F_{i+1}. Let $j' \in T_t$ be an agent in F_i adjacent to i'. We have that $u_{j'}(\mathbf{v}) < 1$. Note that agent i has at most two adjacent friends, since i is adjacent to two agents of the other type. If we delete all (at most two) edges to adjacent friends of i, then it holds that $u_i(\mathbf{v}) = 0$. Swapping i and j' is profitable, since $u_{j'}(\mathbf{v}) < 1 = u_{j'}(\mathbf{v}^{i \leftrightarrow j'})$ and $u_i(\mathbf{v}) = 0 < u_i(\mathbf{v}^{i \leftrightarrow j'})$.
Thus, there can be at most one agent $i \in T_{t'}$ in F_{i+1}. Observe that i has at most one friend, since all other agents in F_{i+1} and F_i are of the other type. Using the same argument as above and the fact that G is a grid, we obtain that there exists another agent $i' \in T_{t'}$ with $i' \neq i$ which is adjacent to an agent $j \in T_t$. If j is not adjacent to i, swapping i and j is profitable after deleting the edge to the adjacent friend of i

(if it exists). Otherwise, we have that j has at most two adjacent friends, since it is adjacent to both i and i'. Then, we again delete the edge to the adjacent friend of i (if it exists), and an edge between j and an adjacent friend of j. We have that $u_i(\mathbf{v}) = 0$ and $u_j(\mathbf{v}) \leq \frac{1}{3}$. Swapping i and j is profitable, since $u_i(\mathbf{v}) = 0 < u_i(\mathbf{v}^{i\leftrightarrow j})$ and $u_j(\mathbf{v}) \leq \frac{1}{3} < \frac{1}{2} \leq u_j(\mathbf{v}^{i\leftrightarrow j})$. Since we have exhausted all possibilities, it follows that F_{i+1} is also only occupied by agents from T_t.
Thus, the induction step follows and it follows that in case there exists a swap-equilibrium which is 2-robust all vertices need to be occupied by agents from the same type. As, however, we assume that $|T_1|, |T_2| \geq 2$, this leads to a contradiction and shows that the robustness of any swap-equilibrium on a grid is upper bounded by one. □

Next, we study Schelling games on grids with an equal number of agents for both types. We show that in such games, on any grid $(x \times y)$-grid with even $x \geq 4$, there exist swap-equilibria with robustness of one and zero. Hence, there exists an infinite class of Schelling games on grids with robustness-ratio two. First, we consider the swap-equilibrium with robustness one, which is shown in Figure 4.5.

Proposition 4.12. *In a Schelling games with $|T_1| = |T_2|$ on an $(x \times y)$-grid with even $x \geq 4$, there exists a swap-equilibrium with robustness one.*

Proof. We first construct an assignment $\mathbf{v}$ and prove that it is always a swap-equilibrium: The agents from T_1 occupy the first $\frac{x}{2}$ columns and the agents from T_2 are placed on the remaining columns. Observe that only the agents on columns $\frac{x}{2}$ and $\frac{x}{2}+1$ have a utility of less than 1. Therefore, only these agents can be involved in a profitable swap. Consider some agent i from column $\frac{x}{2}$. Since $\frac{x}{2} \geq 2$, we have $u_i(\mathbf{v}) = \frac{2}{3}$ if i occupies the vertex on the top or bottom of the column and $u_i(\mathbf{v}) = \frac{3}{4}$ otherwise. If i swaps with some agent j from column $\frac{x}{2}+1$, then $u_i(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}}) \leq \frac{1}{3}$ if v_j is the vertex at the top or bottom of the column and $u_i(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}}) \leq \frac{1}{4}$ otherwise. Hence, no profitable swap involving an agent from T_1 exists and thus $\mathbf{v}$ is a swap-equilibrium.
We now show that $\mathbf{v}$ has robustness one. First, we observe that $\mathbf{v}$ is not 2-robust. If we take the first vertex from column $\frac{x}{2}$ occupied by agent $i \in T_1$ and delete both edges to the two friends of i, agent i and agent $j \in T_2$ positioned on the vertex at the bottom of column $\frac{x}{2}+1$ want to swap:

$$u_i(\mathbf{v}) = 0 < \frac{1}{3} = u_i(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}}), \text{ since } i \text{ has no edges to friends.}$$
$$u_j(\mathbf{v}) = \frac{2}{3} < 1 = u_l(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}}), \text{ since the only neighbour of } v_j \text{ is now in } T_2.$$

To show that $\mathbf{v}$ is 1-robust, consider deleting an edge $e \in E(G)$. From Lemma 4.3, we know that deleting an edge between agents of different types can not make $\mathbf{v}$ unstable. Therefore, we only consider edges between agents of the same type. Hence, by symmetry, let e be an edge between two agents from T_1. Still, only the agents on columns $\frac{x}{2}$ and $\frac{x}{2}+1$ have utility of less than 1. If an agent $i \in T_1$ on vertex v_i incident to e is positioned on the top or bottom of column $\frac{x}{2}$, then $u_i(\mathbf{v}) = \frac{1}{2}$ and $u_i(\mathbf{v}) = \frac{2}{3}$ otherwise. By swapping with some agent $j \in T_2$, agent i can get at most $u_i(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}}) = \frac{1}{3}$, as argued above. Therefore, the swap can not be profitable and $\mathbf{v}$ is 1-robust and hence has robustness one. □

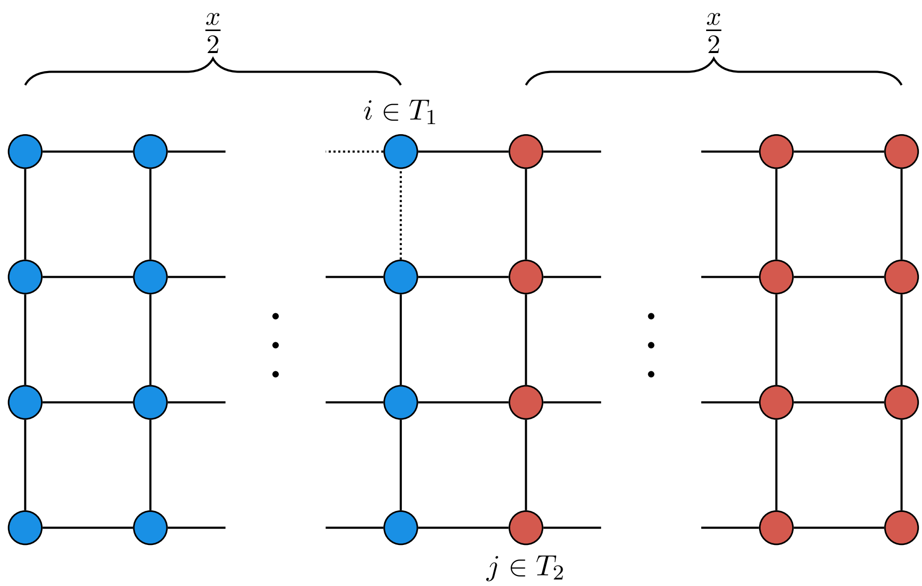


Figure 4.5: Equilibrium assignment **v** from Proposition 4.12. After deleting the edges to the two friends of $i \in T_1$, i and agent $j \in T_2$ want to swap.

However, by slightly modifying this swap-equilibrium, we can construct a swap-equilibrium that can be made unstable by deleting a single edge (i.e., that has robustness zero). This swap-equilibrium is depicted in Figure 4.6.

Proposition 4.13. *In a Schelling games with $|T_1| = |T_2|$ on an $(x \times y)$-grid with even $x \geq 4$, there exists a swap-equilibrium with robustness zero.*

Proof. We define an assignment $\mathbf{v}$ as follows: The first column is occupied by agents from T_1, the following columns up to column $\frac{x}{2} + 1$ are occupied by the agents from T_2 and the remaining agents from T_1 are placed on columns $\frac{x}{2} + 2$ to x.
We now show that $\mathbf{v}$ is a swap-equilibrium. Observe that it holds for all agents $i \in T_2$ that $u_i(\mathbf{v}) \geq \frac{1}{2}$. An agent $j \in T_1$ has at most one neighbor in T_2. Therefore, agent i could at best achieve $u_i(\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{j}}) = \frac{1}{2} \leq u_i(\mathbf{v})$ by swapping with an agent $j \in T_1$. Since no profitable swap exists, the assignment $\mathbf{v}$ is an equilibrium.
in $\mathbf{v}$, there exists an agent $i \in T_1$ on the vertex at the top of the first column with only one adjacent friend and agent $j \in T_2$ at the bottom of the second column with utility of less than one. Hence, we can apply Lemma 4.6 and it follows that $\mathbf{v}$ has robustness zero. □

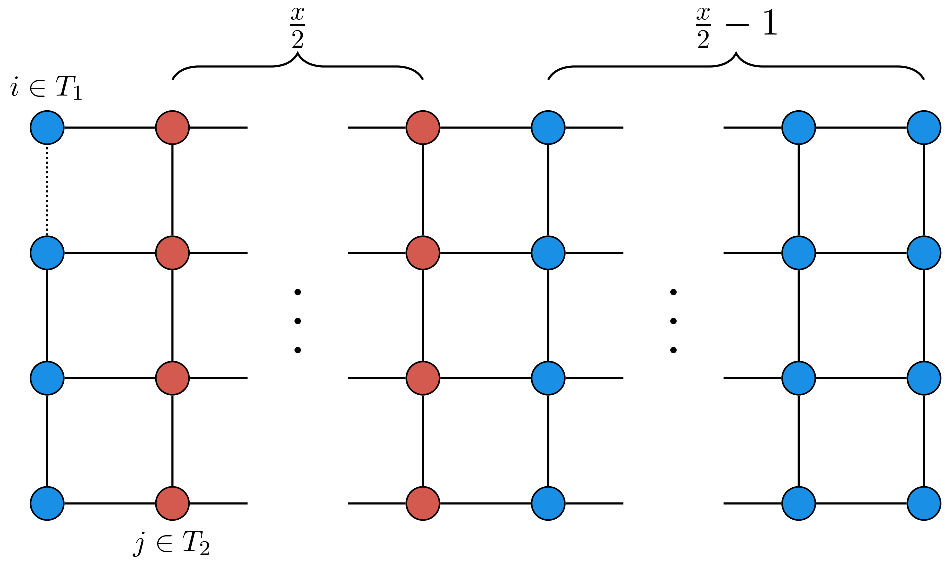


Figure 4.6: Equilibrium assignment $\mathbf{v}$ from Proposition 4.13. After deleting the edge to the only adjacent friend of $i \in T_1$, i and agent $j \in T_2$ want to swap.

Star-constellation Graphs

Recall that for all graph classes considered so far, we showed that there exists a swap-equilibrium with robustness zero on a graph from each one of these classes. We are thus interested in finding an infinite class of graphs such that any swap-equilibrium on a graph from this class has non-zero robustness. To this end, we consider α-star-constellation graphs. These graphs consist of stars, where the central vertices of the stars can be connected by edges such that every central vertex is adjacent to at least α more degree-one vertices than other central vertices. An example is given in Figure 4.7. Formally, it holds for all $v \in V$ with $\deg_G(v) > 1$ that $|\{w \in N_G(v) \mid \deg_G(w) = 1\}| \geq |\{w \in N_G(v) \mid \deg_G(w) > 1\}| + \alpha$. Moreover, such graphs seem quite natural in the context of social networks, as the central vertices could resemble organizations or groups with the members being the adjacent degree-one vertices, and the edges between the central vertices resembling interactions between these entities.
In this section, we show that every swap-equilibrium on an α-star-constellation graph has robustness at least α. We find that swap-equilibria may fail to exist on α-star-constellation graphs but that we can precisely characterize swap-equilibria assignments on such graphs. Moreover, we derive a polynomial-time algorithm for deciding the existence of swap-equilibria on α-star-constellation graphs using this characterization, and provide a subclass of α-star-constellation graphs where a swap-equilibrium is guaranteed to exist. We begin by showing that every swap-equilibrium on an α-star-constellation graph is α-robust.

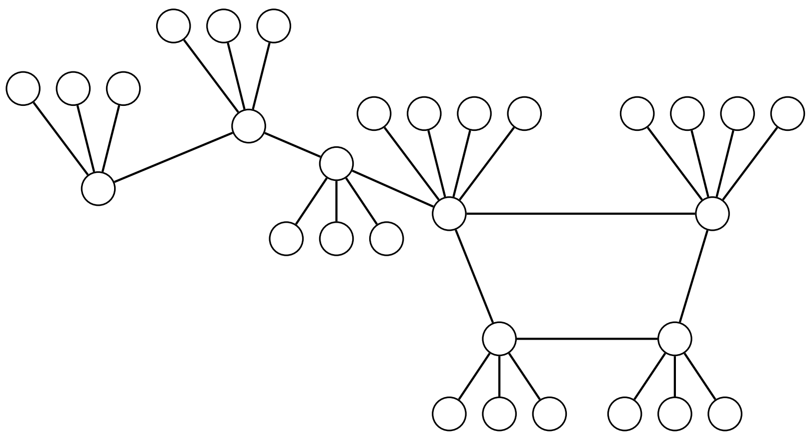


Figure 4.7: An example for a 1-star-constellation graph.

Theorem 4.14. *Let G be an α-star-constellation graph for some $\alpha \in \mathbb{N}_0$. Then, every swap-equilibrium for a Schelling game on G is α-robust.*

Proof. Let $\mathbf{v}$ be a swap-equilibrium on G. We distinguish the following two cases.

Case 1: There exists a vertex v with $\deg_G(v) = 1$ occupied by an agent i such that the only adjacent vertex w is occupied by an agent j of the other type. Assume without loss of generality that $i \in T_1$ and $j \in T_2$. We have $u_i(\mathbf{v}) = 0$. Then, every other agent $j' \in T_2$ with $j \neq j'$ has to have utility $u_{j'}(\mathbf{v}) = 1$. Otherwise, swapping i and j' is profitable: Since $u_{j'}(\mathbf{v}) < 1$, the vertex $v_{j'}$ is adjacent to at least one agent in T_1 and we have $u_i(\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{j}'}) > 0 = u_i(\mathbf{v})$. We also have $u_{j'}(\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{j}'}) = 1 > u_{j'}(\mathbf{v})$, because $j \in T_2$ is the only neighbor of agent i in $\mathbf{v}$. It follows that each agent $i' \in T_1$ with $v_{i'} \notin N_G(v_j)$ also has $u_{i'}(\mathbf{v}) = 1$, since no agent $j' \in T_2$ with $j' \neq j$ is adjacent to an agent in T_1. Summarizing, we have that the agent j is the only agent from T_2 that is adjacent to an agent from the other type.
Now, we show that $\mathbf{v}$ is α-robust. Consider deleting a set of edges $S \subseteq E$ with $|S| \leq \alpha$. On the topology $G - S$, for all $j' \in T_2$ with $j \neq j'$ we have $u_{j'}^{G-S}(\mathbf{v}) = 1$ if $v_{j'}$ has remaining neighbors or $v_{j'}$ is an isolated vertex. The same holds for all $i' \in T_1$ with $v_{i'} \notin N_G(v_j)$. Hence, only agent $j \in T_2$ and an agent $x \in T_1$ with $v_x \in N_G(v_j)$ can be involved in a profitable swap. We however have $u_j^{G-S}(\mathbf{v}^{\mathbf{x} \leftrightarrow \mathbf{j}}) = 0$, since $j \in T_2$ is only adjacent to agents from T_1 in $\mathbf{v}^{\mathbf{x} \leftrightarrow \mathbf{j}}$. Therefore, no profitable swap is possible.

Case 2: Now assume that it holds for all vertices v with $\deg(v) = 1$ that v and the only vertex adjacent to v are occupied by agents of the same type. In the following, we show that $\mathbf{v}$ is α-robust. Let $S \subseteq E$ be a set of edges with $|S| \leq \alpha$ and consider the game on $G - S$. Note that only agents $i \in T_1$ and $j \in T_2$ with $\deg_G(v_i) > 1$ and $\deg_G(v_j) > 1$ in the original topology can be involved in a profitable swap, since all other agents either occupy an isolated vertex or only have one neighbor of the same type. If v_i or v_j is an isolated vertex in $G - S$, swapping i and j can not be profitable. We therefore assume that both vertices have remaining neighbors. Recall that by definition of G, we have $|\{w \in N_G(v) \mid \deg_G(w) = 1\}| \geq |\{w \in N_G(v) \mid \deg_G(w) > 1\}| + \alpha$ for all

$v \in V$ with $\deg_G(v) > 1$. Since we delete at most α edges and it holds that $\deg_G(v_i) > 1$ and $\deg_G(v_j) > 1$, it follows that $|\{w \in N_{G-S}(v) \mid \deg_G(w) = 1\}| \geq |\{w \in N_{G-S}(v) \mid \deg_G(w) > 1\}|$ for $v \in \{v_i, v_j\}$. By our assumption, the agents on the vertices w with $\deg_G(w) = 1$ in the neighborhoods of v_i and v_j are friends of i and j. We hence have that $u_i^{G-S}(\mathbf{v}) \geq \frac{1}{2}$ and $u_i^{G-S}(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}}) \leq \frac{1}{2}$, the same holds for agent i. Thus, the swap is not profitable and $\mathbf{v}$ is α-robust. □

Note that the theorem above has no implications for the existence of swap-equilibria on α-star-constellation graphs. Indeed, we observe that on general α-star-constellation graphs, a swap-equilibrium may fail to exist (see Figure 4.8).

Proposition 4.15. *A Schelling game on an α-star-constellation graph G may fail to admit a swap-equilibrium.*

Proof. Consider the graph G (shown in Figure 4.8) that consists of three 3-stars with central vertices x, y, z, where the vertices x, y, z form a clique. Note that G is a 1-star-constellation graph. Next, we will show that the Schelling game with $|T_1| = 5$ and $|T_2| = 7$ does not admit a swap-equilibrium on G. Observe that all stars in G consist of four vertices and $|T_1| = 5$ and $|T_2| = 7$ are not divisible by four. Therefore, in any assignment $\mathbf{v}$, there exists a degree-one vertex occupied by an agent $i \in T_l$ such that the adjacent central vertex is occupied by an agent $j \in T_{l'}$ of the other type with $l \neq l'$. Let $v \neq v' \in \{x, y, z\}$ be the remaining two central vertices. We distinguish the following two cases.

Case 1: The agents on the degree-one vertices adjacent to v and v' have the same type as their respective neighbor on the central vertex. Since we have $|T_1| < 8$ and $|T_2| < 8$, the vertices v and v' can not be occupied by agents of the same type. Assume by symmetry that an agent $j' \in T_{l'}$ occupies vertex v and v' is occupied by $i' \in T_l$. Then, we have $u_{j'}(\mathbf{v}) < 1$ and swapping i and j' is profitable. It holds that $u_i(\mathbf{v}) = 0 < u_i(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}'})$ and $u_{j'}(\mathbf{v}) < 1 = u_{j'}(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}'})$.

Case 2: There exists an agent on a degree-one vertex that has a different type than the agent on the adjacent central vertex v or v'. One of the agents has to be of type T_l and the other agent has to be of type $T_{l'}$. In any case, there exists an agent $j' \neq j$ from T_2 with $u_{j'}(\mathbf{v}) < 1$. Then, similarly to the case above, swapping i and j' is profitable. It holds that $u_i(\mathbf{v}) = 0 < u_i(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}'})$ and $u_{j'}(\mathbf{v}) < 1 = u_{j'}(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}'})$. □

Note that the graph we used as a counterexample is a *split graph*, that is, the vertices can be partitioned into a clique and an independent set. Intuitively, in the proof above, we exploited that there always exists an agent i on a degree-one vertex without adjacent friends and an agent of the other type with utility less than 1 that is not adjacent to i and thus wants to swap with i. In the following theorem, we generalize this idea to derive a precise characterization of swap-equilibria on α-star-constellation graphs. We show that, in every swap-equilibrium on an α-star-constellation graph, all stars are each only occupied by agents from one type (i.e., there is no agent on a degree-one vertex with no adjacent friends) or for one of the types there exists only one agent that is adjacent to agents of the opposite type.

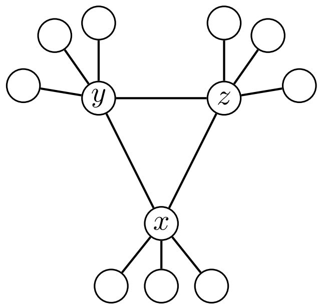


Figure 4.8: An example for a 1-star-constellation graph that does not admit an equilibrium for the Schelling game with $|T_1| = 5$ and $|T_2| = 7$.

Theorem 4.16. *Let G be an α-star-constellation graph $G = (V, E)$ with $\alpha \in \mathbb{N}_0$ and let $\mathbf{v}$ be an assignment for some Schelling game on G. The assignment $\mathbf{v}$ is a swap-equilibrium if and only if at least one of the following two conditions holds.*

1. *Every vertex $v \in V$ with $\deg_G(v) = 1$ is occupied by an agent from the same type as the agent on the only adjacent vertex.*
2. *There exists an agent $i \in T_l$ for one of the types $l \in \{1, 2\}$ such that all other agents $i' \in T_l \setminus \{i\}$ of type T_l are only adjacent to friends.*

Proof. First, we prove that any assignment that fulfills at least one of the two conditions is a swap-equilibrium. Let $\mathbf{v}$ be an assignment satisfying the first condition, that is, every $v \in V$ with $\deg_G(v) = 1$ is occupied by an agent from the same type as the agent on the only adjacent vertex. For all $i \in N$ with $\deg_G(v_i) = 1$, we have $u_i(\mathbf{v}) = 1$. Thus, no such agent i can be involved in a profitable swap. Recall that by definition of G, we have $|\{w \in N_G(v) \mid \deg_G(w) = 1\}| \geq |\{w \in N_G(v) \mid \deg_G(w) > 1\}| + \alpha$ for all $v \in V$ with $\deg_G(v) > 1$. Since all agents on vertices adjacent to v_j with degree one are friends, we have $u_j(\mathbf{v}) \geq \frac{1}{2}$ for all $j \in N$ with $\deg_G(v_j) > 1$. Now consider an agent $j \in T_l$ and an agent $j' \in T_{l'}$ of the other type $l' \neq l$ on vertices $v_j, v_{j'}$ with degree larger than one. If we swap j and j', we have $u_j(\mathbf{v}^{\mathbf{j} \leftrightarrow \mathbf{j}'}) \leq \frac{1}{2} \leq u_j(\mathbf{v})$. The same holds for j'. Thus, no profitable swap is possible and $\mathbf{v}$ is a swap-equilibrium.

Now, we consider an assignment that fulfills the second condition. Let $\mathbf{v}$ be an assignment such that there exists an agent $i \in T_l$ for one of the types $l \in \{1, 2\}$ such that all other agents $i' \in T_l \setminus \{i\}$ of type T_l are only adjacent to friends. Assume without loss of generality that $l = 1$. For all $i' \in T_1 \setminus \{i\}$, we have $u_{i'}(\mathbf{v}) = 1$. Similarly, for all $j \in T_2$ with $v_j \notin N_G(v_i)$, we also have $u_j(\mathbf{v}) = 1$. Hence, only agent $i \in T_1$ and an agent $j' \in T_2$ with $v_{j'} \in N_G(v_i)$ can have a profitable swap. However, after swapping i and j', agent $i \in T_1$ is only adjacent to agents from T_2 and has $u_i(\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{j}}) = 0$. Therefore, no profitable swap is possible.

Next, we will argue that any assignment $\mathbf{v}$ for which both conditions do not hold can not be a swap-equilibrium. Thus, in assignment $\mathbf{v}$, there exists an agent $i \in N$ with $\deg_G(v_i) = 1$ such that the only adjacent agent is of the other type. Assume without loss of generality that $i \in T_1$. Additionally, for both types there exist two agents $x, x' \in T_1$ with $x \neq x'$ and $y, y' \in T_2$ with $y \neq y'$ such that $\{v_x, v_y\} \in E$ and $\{v_{x'}, v_{y'}\} \in E$. We have $u_i(\mathbf{v}) = 0$. Furthermore, since $\deg_G(v_i) = 1$, at least one of the agents $y, y' \in T_2$ has to be positioned outside of the neighborhood of v_i. Assume without loss of generality that $v_y \notin N_G(v_i)$ and thus also $x \neq i$. Then, swapping i and y is profitable. We have $u_i(\mathbf{v}) = 0 < u_i(\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{y}})$, since i is adjacent to x in $\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{y}}$. It also holds that $u_y(\mathbf{v}) < 1 = u_y(\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{y}})$, since $y \in T_2$ is adjacent to $x \in T_1$ in $\mathbf{v}$ and the only neighbor of y in $\mathbf{v}^{\mathbf{i} \leftrightarrow \mathbf{y}}$ has the same type. Hence, the assignment $\mathbf{v}$ can not be a swap-equilibrium. □

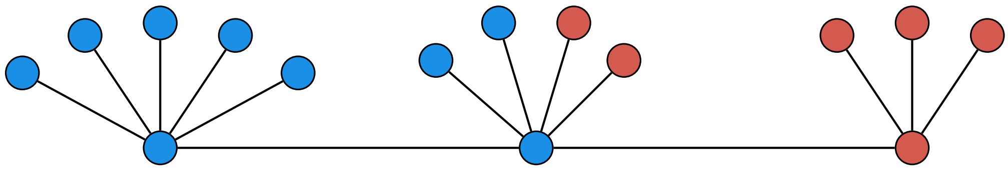
Next, we aim to use the theorem above to find a subclass of α-star-constellation graphs where a swap-equilibrium is guaranteed to exist. Especially for the second condition, it is clear that the existence of an assignment that satisfies this condition heavily depends on the structure of the underlying graph formed by the central vertices (i.e. the non-degree-one vertices). Thus, we capture this notion as the *core* of an α-star-constellation graph.

Definition 4.17. The *core* of an α-star-constellation graph G is the subgraph G' of G where all degree-one vertices are deleted. Formally, $G' = G[V(G) \setminus L]$ with $L = \{v \in V(G) \mid \deg_G(v) = 1\}$.

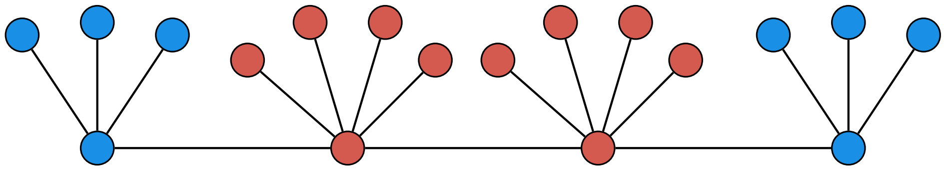
The core of the graph from Figure 4.8 is induced by the vertices x, y and z. In Proposition 4.15, we saw that a swap-equilibrium may fail to exist if the core is a clique or cycle. We now study graphs where the core is a path. These graphs are special *caterpillar* graphs. A caterpillar is an acyclic graph, where every vertex is adjacent to or on a central path. If the core is a path, one can easily construct an assignment that satisfies the second condition from the theorem above (an example is given in Figure 4.9a). Furthermore, in the following proposition, we show that this swap-equilibrium has robustness $|E(G)|$, and that for some games on such α-star-constellation graphs there also exists a swap-equilibrium with robustness only α (see Figure 4.9b). This implies that the upper bound for the robustness of swap-equilibria on α-star-constellation graphs differs from the lower bound of α (by Theorem 4.14), and for every $\alpha \in \mathbb{N}$ there exists a Schelling game on an α-star-constellation graph with robustness-ratio larger than one.

Proposition 4.18. *For every Schelling game on an α-star-constellation graph where the core is a path, there exists a swap-equilibrium with robustness $|E(G)|$. For every $\alpha \in \mathbb{N}_0$, there is a Schelling game on an α-star-constellation graph where the core is a path that admits a swap-equilibrium with robustness α.*

Proof. We start with the first part of the proposition. Consider a Schelling game on an α-star-constellation graph G with $w_1, \ldots, w_\ell$ being the non-degree-one vertices forming a path. It is easy to construct a swap-equilibrium $\mathbf{v}$ by assigning for each $i \in \{1, \ldots, \ell\}$ agents from T_1 first to w_i and then to the degree-one vertices adjacent to w_i, until there are no remaining unassigned agents from T_1. In this case the remaining vertices are filled with agents from T_2. By Theorem 4.16, $\mathbf{v}$ is a swap-equilibrium (as it satisfies the second condition). Note that in $\mathbf{v}$ there exists only one agent from T_1, say $i \in T_1$, who is adjacent to an agent from T_2. Let $S \subseteq E$ be a subset of edges of arbitrary size. We now argue that $\mathbf{v}$ is a swap-equilibrium on $G - S$. First, note that i is still the only agent from T_1 who is adjacent to an agent from T_2 in $\mathbf{v}$ on $G - S$. For all $i' \in T_1 \setminus \{i\}$, we have $u_{i'}^{G-S}(\mathbf{v}) = 1$ or $v_{i'}$ is an isolated vertex. Similarly, for all $j \in T_2$ with $v_j \notin N_{G-S}(v_i)$,



(a) An example for a swap-equilibrium on a 2-star-constellation graph where there core is a path for a Schelling game with $|T_1| = 9$ and $|T_2| = 6$. By Proposition 4.18, this swap-equilibrium has robustness $|E(G)|$.



(b) An example for a swap-equilibrium on a 2-star-constellation graph where there core is a path for a Schelling game with $|T_1| = 6$ and $|T_2| = 10$. As shown in Proposition 4.18, this swap-equilibrium has robustness only α.

Figure 4.9: Two swap-equilibria on α-star-constellation graphs with robustness α and $|E(G)|$, as constructed in Proposition 4.18.

we also have $u_j^{G-S}(\mathbf{v}) = 1$ or v_j is an isolated vertex. Hence, only agent $i \in T_1$ and an agent $j' \in T_2$ with $v_{j'} \in N_{G-S}(v_i)$ can have a profitable swap. However, after swapping i and j', agent $i \in T_1$ is only adjacent to agents from T_2 and has $u_i^{G-S}(\mathbf{v}^{i\leftrightarrow j}) = 0$. Therefore, no profitable swap is possible and $\mathbf{v}$ has robustness $|E(G)|$.

Let us now come to the second part of the proposition. For $\alpha \in \mathbb{N}_0$, let G be an α-star-constellation graph where the core is a path formed by the non-degree-one vertices $w_1, \ldots, w_4$. For $i \in \{1, 4\}$, w_i is adjacent to $\alpha + 1$ degree-one vertices, and for $i \in \{2, 3\}$, w_i is adjacent to $\alpha + 2$ degree-one vertices. We consider the Schelling game on G with $|T_1| = 2 \cdot (\alpha + 1)$ and $|T_2| = 2 \cdot (\alpha + 2)$. Let $\mathbf{v}$ be the assignment where agents from T_1 occupy the $2 \cdot (\alpha + 1)$ vertices from the stars with central vertices w_1, w_4 and the agents from T_2 occupy the $2 \cdot (\alpha + 2)$ vertices from the stars with central vertices w_2, w_3. Note that $\mathbf{v}$ fulfills the first condition from Theorem 4.16 and is thus a swap-equilibrium which is by Theorem 4.14 α-robust. To show that $\mathbf{v}$ has robustness α it remains to provide a set of $\alpha + 1$ edges whose deletion make $\mathbf{v}$ unstable (i.e., $\mathbf{v}$ is not $\alpha + 1$-robust). Let S be the set of $\alpha + 1$ edges containing all edges between w_1 and its degree-one neighbors. Then, on $G - S$, swapping agent $i \in T_1$ on w_1 and agent $j \in T_2$ on w_3 is profitable, as $u_i^{G-S}(\mathbf{v}) = 0 < u_i^{G-S}(\mathbf{v}^{i\leftrightarrow j})$ and $u_j^{G-S}(\mathbf{v}) < 1 = u_j^{G-S}(\mathbf{v}^{i\leftrightarrow j})$. □

Note that the swap-equilibria constructed above are somewhat similar to the swap-equilibria on paths with robustness zero and $|E(G)|$ from Propositions 4.9 and 4.10. Finally, we will sketch how we can efficiently decide whether an assignment that satisfies at least one of the conditions from Theorem 4.16 exists on an α-star-constellation graph. For this, we introduce the *star-partition* of an α-star-constellation graph G. Recall that α-star-constellation graph can be thought of as graphs consisting of stars with additional edges between the central vertices of the stars. The star-partition partitions G into the sets of vertices of these stars.

Definition 4.19. The *star-partition* of an α-star-constellation graph is the collection of sets $\{\{v\} \cup \{w \in N_G(v) \mid \deg_G(w) = 1\} \mid \deg_G(v) > 1\}$.

We note that the star-partition is unique, since every vertex w with $\deg_G(w) = 1$ is adjacent to only one vertex and can thus be in only one set. Additionally, it can easily be computed in polynomial-time by iterating over all vertices with degree of at least two. Next, we observe that an assignment $\mathbf{v}$ that meets the first condition from Theorem 4.16 exists if and only if the star-partition of G can be partitioned into two sets A and B such that $|\bigcup_{S \in A} S| = |T_1|$ and $|\bigcup_{S \in B} S| = |T_2|$. Whether such a partition exists reduces to solving an instance of subset sum with integers bounded by n, which can be solved in polynomial-time by using dynamic programming. To decide whether an assignment $\mathbf{v}$ exists that fulfills the second condition, the following naive approach yields a polynomial-time algorithm. For each vertex $v \in V$, compute the connected components $C_1, \dots, C_m$ of $G - v$. Since vertex v has to be occupied by an agent from either T_1 or T_2, we check whether the components can be partitioned into two subsets A and B such that $|\bigcup_{C \in A} V(C)| = |T_1| - 1$ and $|\bigcup_{C \in B} V(C)| = |T_2|$ or $|\bigcup_{C \in A} V(C)| = |T_1|$ and $|\bigcup_{C \in B} V(C)| = |T_2| - 1$. Again, this can be solved by using dynamic programming in polynomial-time. This allows us to conclude the following corollary.

Corollary 4.20. *For a Schelling game on an α-star-constellation graph, it can be decided in polynomial-time whether a swap-equilibrium exists.*

4.1.3 Influence of Locality

In this section, we turn to local swap-equilibria, where only adjacent agents are allowed to swap. We study whether restricting the game to local swaps influences the robustness of swap-equilibria. Since every swap-equilibrium is also a local swap-equilibrium, our lower bounds on the robustness of swap-equilibria from Section 4.1 also apply to local swap-equilibria. This raises the question whether we can improve these bounds for local swap-equilibria. In the following, we provide a simple case where this is indeed possible: In Proposition 4.8, we showed that the robustness of non-local swap-equilibria on a cycle is zero. In contrast to this, we show that every local swap-equilibrium on a topology with maximum degree of two has robustness $|E(G)|$.

Proposition 4.21. *If the topology G has $\Delta(G) \leq 2$, then every local swap-equilibrium $\mathbf{v}$ has robustness $|E(G)|$.*

Proof. Let $S \subseteq E(G)$ be any set of edges and consider the topology $G' = G - S$. Suppose for the sake of contradiction that a profitable local swap is possible, that is, there exist agents $i \in T_1$ and $j \in T_2$ that want to swap with $\{v_i, v_j\} \in E(G')$. It has to hold that $\deg_{G'}(v_i) = \deg_{G'}(v_j) = 2$: If $\deg_{G'}(v_i) = 1$, then after swapping to v_i agent j only has one neighbor i of the other type and $u_j(\mathbf{v}^{i \leftrightarrow j}) = 0$. Consequently, the swap cannot be profitable. The same holds if v_j has degree one.
Since G has $\Delta(G) \leq 2$, no edge incident to v_i or v_j is in S and has been deleted from G. Hence, the neighborhoods of i and j in G and G' are identical and i and j have the same utility in the topologies G and G'. This contradicts our assumption that $\mathbf{v}$ is a swap-equilibrium for the topology G, since swapping i and j would also be a profitable swap in G. □

4.2 Jump-Equilibria

In the following, we shortly study the robustness of jump-equilibria and highlight differences to swap-equilibria. We show that on a connected topology, contrary to swap-equilibria (see Proposition 4.9 for paths), the robustness of jump-equilibria is upper bounded by the maximum degree of the topology.
An important difference between jump- and swap-equilibria is the role of agents on isolated vertices. For swap-equilibria, no agent wants to swap with an agent on an isolated vertex, since swapping would result in a utility of 0. However, for jump-equilibria, any agent on an isolated vertex (or more generally, an agent without adjacent friends) wants to jump to an arbitrary unoccupied vertex where she has utility larger than 0.

Observation 4.22. *If an agent $i \in T_j$ with no adjacent friends and an unoccupied vertex w adjacent to another agent from T_j exist in an assignment $\mathbf{v}$, then $\mathbf{v}$ is not a jump-equilibrium.*

Proof. Observe that $u_i(\mathbf{v}) = 0$. Since w is adjacent to at least one agent from T_j, we have that $u_i(\mathbf{v}^{i \to w}) > 0$. Thus, the jump is profitable and $\mathbf{v}$ is not a jump-equilibrium. □

We now aim to exploit this observation in order to prove that the robustness of jump-equilibria on a connected topology is upper bounded by the maximum degree. It is easy to see that we can "isolate" an agent i by deleting all edges to friends. However, we also have to show that a suitable unoccupied vertex w exists, where i has utility larger than 0. To this end, we show the following lemma.

Lemma 4.23. *Let $\mathbf{v}$ be a jump-equilibrium assignment for some Schelling game where the topology G is connected, then one of the following two properties holds.*

1. *There exist agents $i \in T_1$ and $j \in T_2$ on vertices v_i and v_j that are both adjacent to unoccupied vertices.*
2. *There exists an agent $i \in N$ such that i is the only agent that is adjacent to unoccupied vertices.*

Proof. Let G be a connected graph and let $\mathbf{v}$ be a jump-equilibrium for some Schelling game on G. Assume for the sake of contradiction that $\mathbf{v}$ does not satisfy any one of the two conditions. Specifically, it holds without loss of generality that no unoccupied vertex is adjacent to an agent in T_2. Consider two agents $i \in T_1$ and $i' \in T_2$. Since G is connected, there exists a path between v_i and $v_{i'}$. By our assumption, there exists no edge between an unoccupied vertex and some vertex occupied by an agent from T_2. Therefore, there has to be some vertex v_j on the path with $j \in T_1$ that has an edge $\{v_j, v_{j'}\}$ to some agent $j' \in T_2$ on the path. We have $u_j(\mathbf{v}) < 1$, since j is adjacent to at least one agent from T_2. Now, j can increase its utility to 1 by jumping to any unoccupied vertex with occupied neighbors other than j which by our assumption are from T_1. Such a vertex exists, since G is connected, so at least one occupied vertex has to be adjacent to some unoccupied vertex. Furthermore, not all unoccupied vertices that are adjacent to agents can be adjacent only to j, since assignment $\mathbf{v}$ does not satisfy the second property (by our assumption). This contradicts that $\mathbf{v}$ is a jump-equilibrium and completes the proof. □

Using this lemma, we can now prove that the robustness of a jump-equilibrium on a connected topology is upper bounded by the maximum degree of the topology.

Proposition 4.24. *The robustness of a jump-equilibrium* $\mathbf{v}$ *on a connected topology* G *is upper bounded by* $\min a_i(\mathbf{v}) - 1 \leq \deg_G(v_i) - 1$ *for* $i \in N$ *such that* $i \in T_t$ *is not the only agent from* T_t *that is adjacent to an unoccupied vertex.*

Proof. Let G be a connected graph and let $\mathbf{v}$ be a jump-equilibrium for some Schelling game on G. Let $i \in T_t$ be an agent that minimizes $a_i(\mathbf{v})$ and that is not the only agent from T_t that is adjacent to unoccupied vertices. Such an agent always exists, since we have at least two agents from each type. Let $r := a_i(\mathbf{v})$. Without loss of generality, assume that $i \in T_1$. If we delete the r edges to all friends of i, then we have $u_i(\mathbf{v}) = 0$. According to Lemma 4.23 and our assumption that i is not the only agent from T_1 adjacent to some unoccupied vertex, there exists some unoccupied vertex w adjacent to at least one other agent in T_1, so i can increase its utility by jumping to w. Therefore, $\mathbf{v}$ is no longer a swap-equilibrium and has at most robustness $r - 1 = a_i(\mathbf{v})$. □

Note that Proposition 4.24 only holds if the topology is connected. Otherwise, we can not apply Lemma 4.23 which guarantees that an unoccupied vertex exists that agent i wants to jump to. For example, consider the case where all unoccupied vertices are isolated vertices. Finally, we apply the proposition above to paths and derive an upper bound for the robustness of jump-equilibria. This shows a difference between jump-equilibria and swap-equilibria, as we proved in Proposition 4.9 that there always exists a swap-equilibrium with robustness of $|E(G)|$ on a path.

Corollary 4.25. *The robustness of jump-equilibria on a path is upper bounded by* 1.

Proof. On paths, we have $\Delta(G) = 2$, hence it holds that $\deg_G(v_i) - 1 \leq 1$ for all $i \in N$. Therefore, the upper bound follows from Proposition 4.24 □

4.3 Computational Aspects

For paths, we observed that the robustness-ratio between the robustness of the most and least robust swap-equilibrium can be arbitrarily large. Since equilibria with high or even low robustness might be desirable in certain settings, this motivates studying the computational aspects of robustness. In this section, we address two questions: First, we show that we can efficiently check whether a given equilibrium is r-robust. However, we also proof that it is NP-complete to decide whether an equilibrium with a given robustness r exists.

4.3.1 Computing the Robustness of Swap-Equilibria

To begin, we address the first question and provide a polynomial-time algorithm to determine whether a given swap-equilibrium $\mathbf{v}$ has robustness $r \in \mathbb{N}_0$. Recall that if the assignment is not r-robust, then for some set of edges S with $|S| \leq r$, there exist two agents $i, j \in N$ that want two swap on $G - S$. Hence, our goal is to check whether such a pair of agents and set S exist. Here, we make the following observation: Whether a swap is profitable only depends on the utilities of the involved agents before and after the swap. These utilities again only depend on the neighborhoods of both agents. Deleting edges outside of their neighborhoods does not impact whether the swap is profitable.

Observation 4.26. *Let* $\mathbf{v}$ *be a (local) swap-equilibrium for a Schelling game on topology* G *and let* $S \subseteq E(G)$ *be a set of edges. A pair of agents* $i, j \in N$ *has a profitable swap on* $G - S$ *if and only if it holds that the swap is profitable on* $G - S'$ *with* $S' = \{e \in S \mid e \cap \{v_i, v_j\} \neq \emptyset\}$*.*

Moreover, combining this with the observation that no profitable swap can involve an agent on an isolated vertex, it follows that if a swap-equilibrium cannot be made unstable by deleting $2 \cdot (\Delta(G) - 1)$ edges, then it cannot be made unstable by deleting an arbitrary number of edges:

Observation 4.27. *Let* $\mathbf{v}$ *be a swap-equilibrium for a Schelling game on topology* G*. If* $\mathbf{v}$ *is* $2 \cdot (\Delta(G) - 1)$*-robust, then* $\mathbf{v}$ *has robustness* $|E(G)|$*.*

As whether a swap is profitable only depends on the neighborhoods of the involved agents (see Observation 4.26), we simply iterate over all pairs of agents i and j and check whether we can delete at most r edges between v_i and adjacent vertices occupied by friends of i and between v_j and adjacent vertices occupied by friends of j such that the swap of i and j becomes profitable (note that the stability of $\mathbf{v}$ only depends on the number of such deleted edges in the neighborhood of each agent, not the exact subset of edges).

Theorem 4.28. *For a given Schelling-game with* n *agents, a swap-equilibrium* $\mathbf{v}$ *and an integer* $r \in \mathbb{N}_0$*, we can decide whether* $\mathbf{v}$ *is* r*-robust in running time* $\mathcal{O}(n^2 \cdot r)$*.*

Proof. Recall the definition of $a_i(\mathbf{v})$ as the number of adjacent friends of agent i for an assignment $\mathbf{v}$ and $b_i(\mathbf{v})$ as the number of agents of a different type in the neighborhood of i. We define $\mathbb{1}_{i,j} = 1$ if the agents i and j are neighbors and $\mathbb{1}_{i,j} = 0$ otherwise.
We solve the problem using Algorithm 1 for which we prove the correctness and running time in the following: First, we prove that Algorithm 1 outputs *yes* if $\mathbf{v}$ is r-robust and *no* otherwise. Assume that $\mathbf{v}$ is r-robust. It therefore holds for all $i \in T_1$ and $j \in T_2$ and $S \subseteq E(G)$ with $|S| \leq r$ that swapping i and j is not profitable in $G - S$. Hence, it holds that the swap of i and j is not profitable if we delete x edges between v_i and vertices that are occupied by friends of i and y edges between v_j and vertices that are occupied by friends of j for all $x, y \in \mathbb{N}_0$ with $x + y \leq r$ and $x \leq a_i(\mathbf{v})$ and $y \leq a_j(\mathbf{v})$. After deleting these edges, the utilities of i and j are given by $u'_i(\mathbf{v}) = \frac{a_i(\mathbf{v})-x}{|N_i(\mathbf{v})|-x}$ and $u'_j(\mathbf{v}) = \frac{a_j(\mathbf{v})-y}{|N_j(\mathbf{v})|-y}$. Swapping positions results in utility $u'_i(\mathbf{v}^{i\leftrightarrow j}) = \frac{b_j(\mathbf{v})-\mathbb{1}_{i,j}}{|N_j(\mathbf{v})|-y}$ for i and $u'_j(\mathbf{v}^{i\leftrightarrow j}) = \frac{b_i(\mathbf{v})-\mathbb{1}_{i,j}}{|N_i(\mathbf{v})|-x}$ for j. Notice that we subtract $\mathbb{1}_{i,j} = 1$ in the numerator if i and j are adjacent, since the vertex previously occupied by i or j in $\mathbf{v}$ is occupied by the other agent of a different type in $\mathbf{v}^{i\leftrightarrow j}$. Since, by our assumption, $\mathbf{v}$ is r-robust, it has to hold that $u'_i(\mathbf{v}) \geq u'_i(\mathbf{v}^{i\leftrightarrow j})$ or $u'_j(\mathbf{v}) \geq u'_j(\mathbf{v}^{i\leftrightarrow j})$. Hence, Algorithm 1 outputs *yes*.
Assume that $\mathbf{v}$ is not r-robust. Then, there exists a pair of agents $i, j \in N$ and a set of edges $S \subseteq E(G)$ with $|S| \leq r$ such that the swap involving i and j is profitable on $G - S$. Note that, as argued before, we only have to consider deleting edges to friends in the

Algorithm 1 Robustness of a Swap-Equilibrium

Input: topology G, equilibrium $\mathbf{v}$, sets of agents T_1, T_2 and $r \in \mathbb{N}_0$
Output: yes if $\mathbf{v}$ is r-robust and no otherwise

function ROBUSTNESS($G, \mathbf{v}, r, T_1, T_2$)
  for each pair $i \in T_1$ and $j \in T_2$ **do** ▷ Iterate over all possible swaps
    for $x = \min\{r, a_i(\mathbf{v})\}$ **to** 0 **do** ▷ Number of edges we delete in $N_G(v_i)$
      $y \leftarrow \min\{r - x, a_j(\mathbf{v})\}$ ▷ Delete remaining edges in $N_G(v_j)$
      $u'_i(\mathbf{v}) \leftarrow \frac{a_i(\mathbf{v})-x}{|N_i(\mathbf{v})|-x}$ ▷ Utility of i after deleting edges to x friends
      $u'_i(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}}) \leftarrow \frac{b_j(\mathbf{v})-\mathbb{1}_{i,j}}{|N_j(\mathbf{v})|-y}$ ▷ Utility of i on v_j after deleting y edges in $N_G(v_j)$
      $u'_j(\mathbf{v}) \leftarrow \frac{a_j(\mathbf{v})-y}{|N_j(\mathbf{v})|-y}$
      $u'_j(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}}) \leftarrow \frac{b_i(\mathbf{v})-\mathbb{1}_{i,j}}{|N_i(\mathbf{v})|-x}$
      if $u'_i(\mathbf{v}) < u'_i(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}})$ **and** $u'_j(\mathbf{v}) < u'_j(\mathbf{v}^{\mathbf{i}\leftrightarrow\mathbf{j}})$ **then**
        return no ▷ Swapping i and j is profitable
      end if
    end for
  end for
  return yes ▷ No profitable swap is possible
end function

neighborhoods of i and j. Therefore, there exist $w, z \in \mathbb{N}_0$ with $w + z \leq r$, $w \leq a_i(\mathbf{v})$ and $z \leq a_j(\mathbf{v})$ such that swapping i and j is profitable after deleting w edges to adjacent friends of i and z edges to adjacent friends of j. As proven in Corollary 4.4, deleting additional edges between i and friends of i and j and friends of j cannot make $\mathbf{v}$ stable again. Thus, it holds for all $w', z' \in \mathbb{N}_0$ with $w \leq w' \leq a_i(\mathbf{v})$ and $z \leq z' \leq a_j(\mathbf{v})$ that swapping i and j is profitable after deleting w' edges to adjacent friends of i and z' edges to adjacent friends of j. Thus, in Algorithm 1 with $x = w$ and $y = \min\{r-x, a_j(\mathbf{v})\} \geq z$, we have $u'_i(\mathbf{v}) < u'_i(\mathbf{v}^{i\leftrightarrow j})$ and $u'_j(\mathbf{v}) < u'_j(\mathbf{v}^{i\leftrightarrow j})$ and therefore return *no*.
Next, we analyze the running time. We first iterate over all pairs of agents $i \in T_1$ and $j \in T_2$ that can potentially be involved in a profitable swap, the number of pairs is upper bounded by n^2. For each pair, we iterate over at most r possible values for x from $\min\{r, a\}$ to zero. All other operations are simple arithmetic operations that can be computed in constant time (assuming we precomputed all $|N_i(\mathbf{v})|$ and $a_i(\mathbf{v})$ in linear time), hence our algorithm runs in $\mathcal{O}(n^2 \cdot r)$ time. □

Note that a very similar algorithm is also possible for jump-equilibria. Instead of iterating over pairs of vertices, we iterate over all pairs of agents and unoccupied vertices.

4.3.2 NP-Hardness of r-Robust Equilibrium Existence

Next, we answer the second question and prove that it is NP-hard to decide whether a Schelling game admits a swap- or jump-equilibrium with robustness at least r. The decision problem for swap-equilibria is formally defined below.

Robust Swap-Equilibrium Existence (ROB-SWAP-EQ)
Input: A topology G, a set $N = [n]$ of agents with $|V(G)| = n$ partitioned into types T_1 and T_2, and an integer $r \in \mathbb{N}_0$.
Question: Does the Schelling game on G admit a swap-equilibrium with robustness at least r?

The decision problem for jump-equilibria is defined analogously, except that we require that $|V(G)| > n$ such that there are unoccupied vertices.

Robust Jump-Equilibrium Existence (ROB-JUMP-EQ)
Input: A topology G, a set $N = [n]$ of agents with $|V(G)| > n$ partitioned into types T_1 and T_2, and an integer $r \in \mathbb{N}_0$.
Question: Does the Schelling game on G admit a jump-equilibrium with robustness at least r?

We first prove the hardness of ROB-SWAP-EQ, as the hardness of ROB-JUMP-EQ follows analogously. In order to prove that ROB-SWAP-EQ is NP-hard, we observe that for $r = 0$, ROB-SWAP-EQ reduces to deciding whether the given game admits any swap-equilibrium (as every swap-equilibrium has at least robustness zero).

Observation 4.29. *For $r = 0$,* ROB-SWAP-EQ *and* SWAP-EQ *are equivalent.*

From this, it is easy to follow that ROB-SWAP-EQ is NP-complete.

Corollary 4.30. ROB-SWAP-EQ *is NP-complete.*

Proof. With Observation 4.29, the NP-hardness of ROB-SWAP-EQ follows directly from Theorem 3.2. Observe that ROB-SWAP-EQ is in NP since we can verify in polynomial-time that an assignment $\mathbf{v}$ is a swap-equilibrium: For any pair of agents, check if the agents want to swap by calculating their utilities before and after swapping. Additionally, we can check in polynomial-time if a swap-equilibrium $\mathbf{v}$ is r-robust by Theorem 4.28. □

The NP-hardness of ROB-JUMP-EQ follows analogously from the NP-hardness of JUMP-EQ by Theorem 3.4. Thus, we conclude with the following corollary.

Corollary 4.31. ROB-JUMP-EQ *is NP-complete.*

Chapter 5

Multimodal Schelling Games

In this chapter, we analyze multimodal Schelling games on multilayer graphs. A multilayer graph is a graph with multiple sets of edges over a fixed set of vertices. We refer to the graph given by the fixed vertex set and one of the edge sets as a layer. In an urban setting, the different layers could represent different means of transportation, some of which might only be accessible to certain agents. As defined in Chapter 2, we say that an assignment is a multimodal swap-equilibrium if it is an equilibrium on every one of the layers. An example for a swap-equilibrium in a 2-modal game is given in Figure 5.1. We study the existence of multimodal swap-equilibria and find that a multimodal swap-equilibrium may fail to exist even on very simple multilayer graphs. Furthermore, we show that deciding whether a multimodal swap- or jump-equilibrium exists is NP-complete.

5.1 Existence of Equilibria

We begin by analyzing the existence of swap-equilibria in multimodal Schelling games. A first naive approach would be to analyze the structure of the different layers independently and to try to identify conditions for the existence of multimodal swap-equilibria based on the relationship between the structure of the different layers. However, if we only consider the structure of the individual layers, then we do not take into account which vertices correspond to each other in the different layers. By showing that a multimodal swap-equilibrium for a 2-modal game may fail to exist even when both layers are isomorphic and paths, the following result suggests that only considering the structure of the layers independently seems to be insufficient.

Theorem 5.1. *A multimodal swap-equilibrium for a 2-modal Schelling game may fail to exist, even when G_1 and G_2 are isomorphic and paths.*

Proof. Consider a 2-modal Schelling game with $T_1 = \{i_1, i_2\}$ and $T_2 = \{i_3, i_4\}$. The multilayer graph with vertices $V = \{w_1, \ldots, w_4\}$ is given by two isomorphic paths $G_1 = (V, \{\{w_1, w_2\}, \{w_2, w_3\}, \{w_3, w_4\}\})$ and $G_2 = (V, \{\{w_1, w_4\}, \{w_4, w_3\}, \{w_3, w_2\}\})$ as in Figure 5.2. Now consider a swap-equilibrium $\mathbf{v}$ on G_1. It holds that the vertices w_1 and w_2 have to be occupied by agents of the same type: Assume without loss of generality for the sake of contradiction that agent $i_1 \in T_1$ is positioned on w_1 and agent $i_3 \in T_2$

Figure 5.1: An example of a swap-equilibrium on a 2-layer graph with layers G_1 and G_2. Note that the depicted assignment is stable on both layers (i.e., there exists no profitable swap).

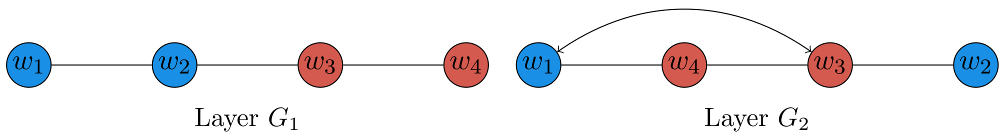


Figure 5.2: The only swap-equilibrium on G_1. On G_2, the agents on vertices w_1 and w_3 want to swap.

occupies w_2. We have $u_{i_1}(\mathbf{v}) = 0$. Now, since the remaining agents $i_2 \in T_1$ and $i_4 \in T_2$ are neighbors, we have $u_{i_4}(\mathbf{v}) < 1$. Therefore, swapping i_1 and i_4 is profitable: $u_{i_4}(\mathbf{v}^{\mathbf{i_1 \leftrightarrow i_4}}) = 1 > u_{i_4}(\mathbf{v})$ and $u_{i_1}(\mathbf{v}^{\mathbf{i_1 \leftrightarrow i_4}}) \geq \frac{1}{2} > 0 = u_{i_1}(\mathbf{v})$. This contradicts our assumption that $\mathbf{v}$ is an equilibrium.
Hence, it holds without loss of generality that in any equilibrium $\mathbf{v}$ on G_1, the vertices w_1 and w_2 are occupied by the agents $i_1, i_2 \in T_1$ and the remaining two agents $i_3, i_4 \in T_2$ occupy w_3 and w_4. However, on topology G_2, we have $u_{i_1}(\mathbf{v}) = 0$ and $u_{i_3}(\mathbf{v}) = \frac{1}{2}$. Therefore, the swap of i_1 and i_3 is profitable: $u_{i_1}(\mathbf{v}^{\mathbf{i_1 \leftrightarrow i_3}}) = \frac{1}{2} > 0 = u_{i_1}(\mathbf{v})$ and $u_{i_3}(\mathbf{v}^{\mathbf{i_1 \leftrightarrow i_3}}) = 1 > \frac{1}{2} = u_{i_3}(\mathbf{v})$. Hence, assignment $\mathbf{v}$ is not a swap-equilibrium on G_2 and no swap-equilibrium for the multimodal game exists. □

In the theorem above, we exploited that we can "reorder" the vertices on the path to obtain an isomorphic graph, where no swap-equilibrium for the first layer is stable. It therefore seems reasonable that to derive useful conditions for multimodal swap-equilibrium existence, we have to also incorporate which vertices correspond to each other in the layers (i.e., in this case, how the vertices are ordered). This motivates the following definition, which we later use to show a positive result.

Definition 5.2. Let $\mathcal{G} = (V, \{E_1, \ldots, E_l\})$ be a multilayer graph. We call a layer $G_i = (V, E_i)$ with $i \in \{1, \ldots, l\}$ a *top-layer* of $\mathcal{G}$, if it holds that $E_j \subseteq E_i$ for all $j \in \{1, \ldots, l\}$. A multilayer graph that has a top-layer is called *top-layered*.

Observe that the multilayer graph in Figure 5.2 is an example for a multilayer graph without a top-layer. Informally, we can think of the top-layer as the graph out of which all other layers can be built by deleting edges. This intuitive view is somewhat similar to the perspective of robustness, where we analyzed whether equilibria remain stable when edges are removed. We now show that some results from our analysis of robustness translate to multimodal games on top-layered multilayer graphs. Recall that we proved in Section 4.1.2 that for topologies from some graph classes, there always exists a swap-equilibrium with robustness $|E(G)|$. If the top-layer is from such a graph class, then we know that a swap-equilibrium with robustness $|E(G)|$ is guaranteed to exist on the top-layer. As a swap-equilibrium with robustness $|E(G)|$ remains stable after deleting any set of edges, this swap-equilibrium is therefore also stable on any other layer (since all other layers contain a subset of the edges of the top-layer). Thus, if the top-layer of a top-layered multilayer graph is from a graph class where a swap-equilibrium with robustness $|E(G)|$ is guaranteed to exist, then there exist a multimodal swap-equilibrium on the multilayer graph. Analogously, the same holds for other types of equilibria.

Proposition 5.3. *Let $\mathcal{G} = (V, \{E_1, \dots, E_l\})$ be a multilayer graph with top-layer $G_i = (V, E_i)$ with $i \in \{1, \dots, l\}$. It there exists a (local) swap- or jump-equilibrium with robustness $|E(G_i)|$ for the induced game on G_i, then the multimodal game on $\mathcal{G}$ admits a multimodal (local) swap- or jump-equilibrium.*

Proof. Let $\mathbf{v}$ be the (local) swap- or jump-equilibrium with robustness $|E(G_i)|$ on the top-layer G_i. We have to show that $\mathbf{v}$ is also a (local) swap- or jump-equilibrium for the induced game on all G_j for $j \in \{1, \dots, l\}$ with $j \neq i$. Consider a layer $G_j = (V, E_j)$ with $j \in \{1, \dots, l\}$ and $j \neq i$. Since G_i is the top-layer of $\mathcal{G}$, it holds that $E_j \subseteq E_i$. That is, $G_j = G_i - S$ for some $S \subseteq E(G_i)$. Since $\mathbf{v}$ is $|E(G_i)|$-robust by definition, $\mathbf{v}$ is also a (local) swap- or jump-equilibrium on G_j. □

This allows us to use results from Section 4.1.2. For example, we showed that on any path there exists a swap-equilibrium with robustness $|E(G)|$ (see Proposition 4.9). Thus, we can conclude the corollary below.

Corollary 5.4. *For multimodal games on top-layered multilayer graphs where the top-layer is a path, there always exists a swap-equilibrium.*

It is easy to see that the condition that the top-layer G_i admits a swap-equilibrium with robustness $|E(G_i)|$ is not a necessary condition for the existence of a multimodal swap-equilibrium. Consider a 2-modal game on a 2-layer graph $\mathcal{G}$ where the first layer G_1 is a clique and the second layer G_2 is any arbitrary topology that admits a swap-equilibrium. Note that $\mathcal{G}$ is a top-layered multilayer graph with top-layer G_1. As any assignment is a swap-equilibrium on a clique, any swap-equilibrium on G_2 is also a swap-equilibrium for the 2-modal game. However, the clique top-layer G_1 is not $|E(G_1)|$-robust (see Proposition 4.7).
Thus, one could study further conditions for the existence of multimodal swap-equilibria on top-layered multilayer graphs. Moreover, the analysis of multimodal equilibrium existence could be extended to multilayer graphs without a top-layer.

5.2 Computational Complexity

Next, we show that deciding whether a multimodal Schelling game admits a jump- or swap-equilibrium is NP-complete. We first prove this statement for the decision problem for swap-equilibria, as defined below.

l-MODAL SWAP-EQUILIBRIUM EXISTENCE (l-MODAL SWAP-EQ)

Input: A multilayer graph $\mathcal{G} = (V, \{E_1, \ldots, E_l\})$ and a set $N = [n]$ of agents with $|V(G)| = n$ partitioned into types T_1 and T_2.

Question: Does the l-modal Schelling game on $\mathcal{G}$ admit a multimodal swap-equilibrium?

We first observe that l-MODAL SWAP-EQ is in NP: To verify that an assignment $\mathbf{v}$ is a swap-equilibrium, for all layers G_1 to G_l, we check if any pair of agents wants to swap by calculating their utilities before and after swapping.
The NP-Hardness follows directly from the NP-Hardness of SWAP-EQ by Theorem 3.2. Given a Schelling game on topology $G' = (V', E')$ with types T_1' and T_2', we construct an l-modal Schelling game with the same types on $\mathcal{G} = (V, \{E_1, \ldots, E_l\})$. We set $V = V'$ and $E_1 = E'$, that is, it holds that $G_1 = G$. For the remaining layers, set $E_j = \emptyset$ for all $j \in \{2, \ldots, l\}$. Note that every assignment is a swap-equilibrium on a topology which only consist of isolated vertices. Therefore, the constructed l-modal game admits a swap-equilibrium if and only if the given Schelling game on G admits a swap-equilibrium. We summarize our results in the following observation.

Observation 5.5. l-MODAL SWAP-EQ *is NP-complete for all* $l \geq 2$.

The decision problem for multimodal jump-equilibria is analogously defined as follows.

l-MODAL JUMP-EQUILIBRIUM EXISTENCE (l-MODAL JUMP-EQ)

Input: A multilayer graph $\mathcal{G} = (V, \{E_1, \ldots, E_l\})$ and a set $N = [n]$ of agents with $|V(G)| > n$ partitioned into types T_1 and T_2.

Question: Does the l-modal Schelling game on $\mathcal{G}$ admit a multimodal jump-equilibrium?

Analogously to the argument for swap-equilibria, we can reduce an instance of JUMP-EQ with topology G to l-MODAL JUMP-EQ by constructing a multilayer graph where the first layer is equal to G and all other layers only consist of isolated vertices. This allows us to conclude the following.

Observation 5.6. l-MODAL JUMP-EQ *is NP-complete for all* $l \geq 2$.

Chapter 6

Conclusion

Answering an open question by Elkind et al. [Elk+19], we showed that deciding the existence of swap- and jump-equilibria remains NP-hard in our simpler variant of their model, where all agents are strategic. Regarding the robustness of equilibria, we found that the robustness of swap-equilibria in Schelling games heavily depends on the structure of the underlying topology along the following two dimensions. First, we showed that the minimum and the maximum robustness of swap-equilibria on topologies from various graph classes vary significantly. For most of the analyzed graph classes, we found that there exist swap-equilibria that can be made unstable by deleting a single edge (i.e. the robustness is zero). However, with α-star-constellation graphs for any $\alpha \in \mathbb{N}_0$ we provided a graph class where each swap-equilibrium has robustness at least α. Second, the structure of the topology influences the robustness-ratio between the robustness of the most and least robust swap-equilibrium on a given topology. More precisely, we showed that on topologies from some graph classes any swap-equilibrium has the same robustness (e.g., cycles), while the robustness-ratio can be arbitrarily large on topologies from other graph classes (e.g., paths). From a practical perspective, one may thus be interested in finding more robust equilibria. However, we proved that deciding the existence of equilibria with at least a given robustness is NP-hard. On the positive side, we showed that one can efficiently determine the robustness of an equilibrium. Moreover, we studied the robustness of jump- and local swap-equilibria and observed differences between these models.

Turning to multimodal games, we showed that a multimodal swap-equilibrium may fail to exist even on a simple 2-layer graph where both layers are isomorphic and paths. In this multilayer graph, we exploited that we can reorder the vertices in the second path layer to obtain an isomorphic graph where no swap-equilibrium for the first layer is stable. This result shows that the existence of multimodal equilibria does not only depend on the structure of the individual layers, but also on the correspondence between the vertices of the different layers (in our example above, how the vertices are ordered in the different layers). Motivated by this observation, we defined top-layered multilayer graphs. Using this notion, we were then able to prove that on a top-layered multilayer graph where the top-layer is a path a swap-equilibrium is guaranteed to exist. Interestingly, this also allowed us to translate some results from our analysis of robustness to multimodal Schelling games.

For future research, an immediate direction based on our analysis of the robustness of swap-equilibria on different graph classes is to study more thoroughly why swap-equilibria are more or less robust on certain graph classes and to try to identify the underlying structural properties that influence the robustness of swap-equilibria. It would also be interesting to investigate the robustness of equilibria with regard to different kinds of changes to the topology, such as adding edges or deleting vertices. Furthermore, one could consider other notions of robustness. For example, given a topology that admits an equilibrium, one natural question would be to ask how many edges have to be deleted such that the resulting topology no longer admits an equilibrium. We conjecture that this notion of robustness also heavily depends on the structure of the topology: It is easy to construct examples where deleting a single edge is sufficient, on the other hand, a graph from a graph class for which equilibrium existence is guaranteed and that is closed under edge deletion (e.g., the class of graphs with maximum degree of two) would be arbitrarily robust with regard to the existence of equilibria. Moreover, given a topology that does not admit an equilibrium, one could consider the distance to an equilibrium, that is, the minimum number of changes such that the resulting topology admits an equilibrium. A first approach to prove upper bounds for the distance to an equilibrium could be to consider simple graph classes where equilibrium existence is guaranteed and then to upper bound the number of changes that have to be made to a given topology to obtain a graph from this class.

In this work, we have taken an adversarial perspective and considered a worst-case approach to robustness. It would also be interesting to investigate which kinds of changes can be safely made such that an equilibrium remains stable. For example, we proved that only the deletion of edges between agents of the same type can make a swap-equilibrium unstable.

With regard to multimodal Schelling games, the analysis of multimodal equilibria existence can be extended in two directions. First, one could study additional conditions for the existence of equilibria on top-layered multilayer graphs. Second, the analysis can be extended to non-top-layered multilayer graphs.

To conclude, in our analysis of multimodality and the robustness of equilibria in Schelling games, we found that both aspects are strongly influenced by the underlying topology. Our results paint a contrasted picture of the robustness of swap-equilibria based on the structure of the underlying topology. As described above, there is an abundance of directions to extend and to better understand this picture, and there are many alternative notions of robustness to consider.

Literature

[Aga+20] Aishwarya Agarwal, Edith Elkind, Jiarui Gan, and Alexandros A. Voudouris. *Swap Stability in Schelling Games on Graphs*. In: *Proceedings of The Thirty-Fourth AAAI Conference on Artificial Intelligence (AAAI 2020)*. AAAI Press, 2020, pp. 1758–1765 (cit. on pp. 5, 10–13, 21, 22).

[BEL14] George Barmpalias, Richard Elwes, and Andy Lewis-Pye. *Digital Morphogenesis via Schelling Segregation*. In: *Proceedings of the 55th Annual Symposium on Foundations of Computer Science (FOCS 2014)*. IEEE Computer Society. 2014, pp. 156–165 (cit. on p. 9).

[Bil+20] Davide Bilò, Vittorio Bilò, Pascal Lenzner, and Louise Molitor. *Topological Influence and Locality in Swap Schelling Games*. In: *Proceedings of the 45th International Symposium on Mathematical Foundations of Computer Science (MFCS 2020)*. Vol. 170. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020, 15:1–15:15 (cit. on pp. 10–14, 47).

[BMR14] Prateek Bhakta, Sarah Miracle, and Dana Randall. *Clustering and Mixing Times for Segregation Models on* $\mathbb{Z}^2$. In: *Proceedings of the 25th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2014)*. Society for Industrial and Applied Mathematics, 2014, pp. 327–340 (cit. on p. 9).

[Bra+12] Christina Brandt, Nicole Immorlica, Gautam Kamath, and Robert Kleinberg. *An analysis of one-dimensional Schelling segregation*. In: *Proceedings of the 44th annual ACM symposium on Theory of computing (STOC 2012)*. Association for Computing Machinery, 2012, pp. 789–804 (cit. on p. 9).

[BW07] Stephen Benard and Robb Willer. *A wealth and status-based model of residential segregation*. In: *The Journal of Mathematical Sociology* 31.2 (2007), pp. 149–174 (cit. on p. 9).

[Cab13] Dustin A. Cable. *The racial dot map*. 2013. URL: `https://demographics.coopercenter.org/Racial-Dot-Map/` (visited on 05/29/2021) (cit. on p. 10).

[CF08] William AV Clark and Mark Fossett. *Understanding the social context of the Schelling segregation model*. In: *Proceedings of the National Academy of Sciences* 105.11 (2008), pp. 4109–4114 (cit. on p. 9).

[Cha03] Camille Zubrinsky Charles. *The dynamics of racial residential segregation*. In: *Annual Review of Sociology* 29.1 (2003), pp. 167–207 (cit. on p. 9).

[CLM18] Ankit Chauhan, Pascal Lenzner, and Louise Molitor. *Schelling Segregation with Strategic Agents*. In: *Proceedings of the 11th Symposium on Algorithmic Game Theory (SAGT 2018)*. Vol. 11059. Lecture Notes in Computer Science. Springer, 2018, pp. 137–149 (cit. on pp. 10–12).

[CT18] Alex Carver and Paolo Turrini. *Intolerance does not Necessarily Lead to Segregation: A Computer-aided Analysis of the Schelling Segregation Model*. In: *Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems (AAMAS 2018)*. International Foundation for Autonomous Agents and Multiagent Systems, 2018, pp. 1889–1890 (cit. on p. 9).

[Ech+19] Hagen Echzell, Tobias Friedrich, Pascal Lenzner, Louise Molitor, Marcus Pappik, Friedrich Schöne, Fabian Sommer, and David Stangl. *Convergence and Hardness of Strategic Schelling Segregation*. In: *Proceedings of the 15th Conference on Web and Internet Economics (WINE 2019)*. Springer, 2019, pp. 156–170 (cit. on pp. 10–12).

[Elk+19] Edith Elkind, Jiarui Gan, Ayumi Igarashi, Warut Suksompong, and Alexandros A. Voudouris. *Schelling Games on Graphs*. In: *Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence (IJCAI 2019)*. International Joint Conferences on Artificial Intelligence Organization, 2019, pp. 266–272 (cit. on pp. 5, 10–14, 21, 29–31, 69).

[Imm+17] Nicole Immorlica, Robert Kleinberg, Brendan Lucier, and Morteza Zadomighaddam. *Exponential Segregation in a Two-Dimensional Schelling Model with Tolerant Individuals*. In: *Proceedings of the 28th annual ACM-SIAM symposium on discrete algorithms (SODA 2017)*. Society for Industrial and Applied Mathematics, 2017, pp. 984–993 (cit. on p. 9).

[KKV20] Panagiotis Kanellopoulos, Maria Kyropoulou, and Alexandros A. Voudouris. *Modified Schelling Games*. In: *Proceedings of the 13th Symposium on Algorithmic Game Theory (SAGT 2020)*. Vol. 12283. Lecture Notes in Computer Science. Springer, 2020, pp. 241–256 (cit. on pp. 10, 12).

[KKV21] Panagiotis Kanellopoulos, Maria Kyropoulou, and Alexandros A. Voudouris. *Not all Strangers are the Same: The Impact of Tolerance in Schelling Games*. 2021. arXiv: `2105.02699 [cs.GT]` (cit. on pp. 10, 12).

[Man20] Alexander V Mantzaris. *Incorporating a monetary variable into the Schelling model addresses the issue of a decreasing entropy trace*. In: *Scientific Reports* 10.1 (2020), pp. 1–12 (cit. on p. 9).

[MD88] Douglas S Massey and Nancy A Denton. *The dimensions of residential segregation*. In: *Social Forces* 67.2 (1988), pp. 281–315 (cit. on p. 9).

[Sch69] Thomas C. Schelling. *Models of segregation*. In: *The American Economic Review* 59.2 (1969), pp. 488–493 (cit. on p. 9).

[Sch71] Thomas C. Schelling. *Dynamic Models of Segregation*. In: *The Journal of Mathematical Sociology* 1.2 (1971), pp. 143–186 (cit. on p. 9).

[WC01] David R. Williams and Chiquita Collins. *Racial Residential Segregation: A Fundamental Cause of Racial Disparities in Health.* In: *Public Health Reports* 116.5 (2001), pp. 404–416 (cit. on p. 9).

[WHW12] Kellee White, Jennifer S. Haas, and David R. Williams. *Elucidating the Role of Place in Health Care Disparities: The Example of Racial/Ethnic Residential Segregation*. In: *Health Services Research* 47 (2012), pp. 1278–1299 (cit. on p. 9).